ASPECTS OF CONFINEMENT WITHIN NON-ABELIAN GAUGE THEORIES
– Corrections to the Problems –

Urko Reinosa

4-17 June 2023

QCD MASTER CLASS

Saint-Jacut-de-la-Mer, France

Chapter 1

Polyakov Loop and Center Symmetry

Problem 1.1: That the periodicity conditions are preserved by the transformation $U(\tau, \vec{x})$ means that, for any $A_\mu(\tau, \vec{x})$ such that $A_\mu(\tau + \beta, \vec{x}) = A_\mu(\tau, \vec{x})$, we should also find $A^U_\mu(\tau + \beta, \vec{x}) = A^U_\mu(\tau, \vec{x})$. Making this latter condition more explicit, we find

$$\begin{aligned} & U(\tau + \beta, \vec{x}) A_\mu(\tau + \beta, \vec{x}) U^\dagger(\tau + \beta, \vec{x}) + iU(\tau + \beta, \vec{x}) \partial_\mu U^\dagger(\tau + \beta, \vec{x}) \\ & \quad = U(\tau, \vec{x}) A_\mu(\tau, \vec{x}) U^\dagger(\tau, \vec{x}) + iU(\tau, \vec{x}) \partial_\mu U^\dagger(\tau, \vec{x}) \,. \end{aligned}$$

Using the assumed periodicity of $A_\mu(\tau, \vec{x})$ together with the unitarity of $U(\tau + \beta, \vec{x})$ which implies $U(\tau+\beta, \vec{x})\partial_\mu U^\dagger(\tau+\beta, \vec{x}) = -(\partial_\mu U(\tau+\beta, \vec{x}))U^\dagger(\tau+\beta, \vec{x})$, we rewrite this conveniently as

$$\begin{aligned} & U(\tau + \beta, \vec{x}) A_\mu(\tau, \vec{x}) U^\dagger(\tau + \beta, \vec{x}) - i(\partial_\mu U(\tau + \beta, \vec{x})) U^\dagger(\tau + \beta, \vec{x}) \\ & \quad = U(\tau, \vec{x}) A_\mu(\tau, \vec{x}) U^\dagger(\tau, \vec{x}) + iU(\tau, \vec{x}) \partial_\mu U^\dagger(\tau, \vec{x}) \,. \end{aligned}$$

Then, upon left multiplication by $U^\dagger(\tau, \vec{x})$ and right multiplication by $U(\tau + \beta, \vec{x})$, this rewrites

$$\begin{aligned} & Z(\tau, \vec{x}) A_\mu(\tau, \vec{x}) - iU^\dagger(\tau, \vec{x}) \partial_\mu U(\tau + \beta, \vec{x}) \\ & \quad = A_\mu(\tau, \vec{x}) Z(\tau, \vec{x}) + i(\partial_\mu U^\dagger(\tau, \vec{x})) U(\tau + \beta, \vec{x}) \,, \end{aligned}$$

where $Z(\tau, \vec{x}) \equiv U^\dagger(\tau, \vec{x}) U(\tau + \beta, \vec{x})$. In fact, this identity rewrites solely in terms of $Z(\tau, \vec{x})$:

$$\partial_\mu Z(\tau, \vec{x}) - i[A_\mu(\tau, \vec{x}), Z(\tau, \vec{x})] = 0 \,.$$

Now, since the latter identity is valid for any periodic gauge field, it should be valid for $A_\mu = 0$. From this we deduce that Z is constant and that $[A_\mu, Z] = 0$ for any periodic gauge field configuration. In particular this means that Z should commute with any generator t^a of the algebra, and, therefore, with any element of the SU(N) group. The only possibility for Z is then to be of the form $e^{i\phi}\mathbb{1}$ with $\phi = 2\pi k/N$.

Problem 1.2: Since

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

we have

$$U(\beta) = \exp\left\{i\theta\frac{\sigma_3}{2}\right\} = \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

For $\theta = \pm 2\pi$, we find indeed $U(\beta) = -\mathbb{1}$, while for $\theta = \pm 4\pi$, we find $U(\beta) = \mathbb{1}$.

Problem 1.3: Owing to the path ordering, we have

$$\frac{d}{d\beta}L[A^U] = iA_0^U(\beta,\vec{x})L[A^U]\,.$$

On the other hand

$$\begin{aligned}
&\frac{d}{d\beta}U(\beta)L[A]U^\dagger(0)\\
&\quad= \frac{dU(\beta)}{d\beta}L[A]U^\dagger(0) + iU(\beta)A_0(\beta,\vec{x})L[A]U^\dagger(0)\\
&\quad= \left[\frac{dU(\beta)}{d\beta}U^\dagger(\beta) + iU(\beta)A_0(\beta,\vec{x})U^\dagger(\beta)\right]U(\beta)L[A]U^\dagger(0)\\
&\quad= i\left[U(\beta)A_0(\beta,\vec{x})U^\dagger(\beta) + iU(\beta)\frac{dU^\dagger(\beta)}{d\beta}\right]U(\beta)L[A]U^\dagger(0)\\
&\quad= iA_0^U(\beta,\vec{x})U(\beta)L[A]U^\dagger(0)\,.
\end{aligned}$$

This shows that $L[A^U]$ and $U(\beta)L[A]U^\dagger(0)$ obey the same first order differential equation with respect to β. Since they are both equal to $\mathbb{1}$ for $\beta = 0$, we deduce that $L[A^U] = U(\beta)L[A]U^\dagger(0)$.

Problem 1.4: Consider a change of variables in the form a k-twisted gauge transformation $A_\mu \to A_\mu^U$. We find

$$\begin{aligned}\ell_{\rho,\theta} &= \frac{1}{Z}\int_{p.b.c.} \mathcal{D}A^U\, e^{-S[A^U]+\rho e^{i\theta}\int_{\vec{x}}\Phi[A^U](\vec{x})}\,\Phi[A^U] \\ &= \frac{e^{i2\pi k/N}}{Z}\int_{p.b.c.} \mathcal{D}A\, e^{-S[A]+\rho e^{i\theta}e^{i2\pi k/N}\int_{\vec{x}}\Phi[A](\vec{x})}\,\Phi[A]\,,\end{aligned}$$

where we have used the invariance of the measure $\mathcal{D}A$ and the action $S[A]$, together with the transformation rule for $\Phi[A]$. It follows that

$$\ell_{\rho,\theta} = e^{i2\pi k/N}\ell_{\rho,\theta+2\pi k/N}\,.$$

In the case where center-symmetry is realized explicitly, the limit of $\ell_{\rho,\theta}$ as $\rho \to 0$ does not depend on θ: $\lim_{\rho\to 0}\ell_{\rho,\theta} = \ell$. The above identity becomes a constraint for ℓ which forces it to vanish. In contrast, in the case where center-symmetry is broken in the Nambu-Goldstone sense, the limit of $\ell_{\rho,\theta}$ as $\rho \to 0$ still depends on θ: $\lim_{\rho\to 0}\ell_{\rho,\theta} = \ell_\theta$. In this case, the above identity does not impose that the various possible limits ℓ_θ should vanish, but connects them to each other as

$$e^{-i2\pi k/N}\ell_\theta = \ell_{\theta+2\pi k/N}\,.$$

Chapter 2

SU(2) Confining Configurations

Problem 2.1: We have to compute the rotational of $\vec{A} = (\vec{r} \times \vec{B})/2 = B(y\vec{e}_x - x\vec{e}_y)/2$. We find

$$\begin{aligned}
(\vec{\nabla} \times \vec{A})_x &= \frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y} = 0\,, \\
(\vec{\nabla} \times \vec{A})_y &= \frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} = 0\,, \\
(\vec{\nabla} \times \vec{A})_z &= \frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x} = B/2 - (-B/2) = B\,.
\end{aligned}$$

Problem 2.2: Since the configuration is constant, the path-ordering can be ignored and we have

$$\begin{aligned}
\Phi[A] &= \frac{1}{2}\mathrm{tr}\, e^{i\beta A_0} = \frac{1}{2}\mathrm{tr}\, e^{ir\frac{\sigma_3}{2}} \\
&= \frac{1}{2}\mathrm{tr} \begin{pmatrix} e^{ir/2} & 0 \\ 0 & e^{-ir_3/2} \end{pmatrix} = \cos(r/2)\,.
\end{aligned}$$

Problem 2.3: The looked for transformation $U(\tau, \vec{x})$ should be such that for any configuration of the form $A_\mu(\tau, \vec{x}) = T\delta_{\mu 0}\, r\sigma_3/2$, the transformed configuration $A^U_\mu(\tau, \vec{x})$ should be also of this form, that is

$$A^U_\mu(\tau, \vec{x}) = T\delta_{\mu 0}\, \theta \frac{\sigma_3}{2}\,,$$

for some θ. Making the left-hand side explicit, this rewrites

$$T\delta_{\mu 0}\, rU(\tau,\vec{x})\frac{\sigma_3}{2}U^\dagger(\tau,\vec{x}) + iU(\tau,\vec{x})\partial_\mu U^\dagger(\tau,\vec{x}) = T\delta_{\mu 0}\,\theta\frac{\sigma_3}{2}\,,$$

which should be true for any value of μ and anu value of r, with θ depending on r. In particular, for $\mu = j \neq 0$, the condition reads $iU(\tau,\vec{x})\partial_j U^\dagger(\tau,\vec{x}) = 0$ which basically means that $U(\tau,\vec{x})$ depends only on τ and we denote it more simply as $U(\tau)$. Taking now $\mu = 0$ in the equation above, this transformation $U(\tau)$ is seen to obey the equation

$$T\, rU(\tau)\frac{\sigma_3}{2}U^\dagger(\tau) + iU(\tau)\frac{d}{d\tau}U^\dagger(\tau) = T\,\theta\frac{\sigma_3}{2}\,,$$

which rewrites as

$$\frac{d}{d\tau}U(\tau) = iT\left[\theta\frac{\sigma_3}{2}U(\tau) - U(\tau)r\frac{\sigma_3}{2}\right].$$

This equation should be valid for any value of r with θ depending on r. In particular, for $r = 0$, we deduce that there is some θ such that

$$\frac{d}{d\tau}U(\tau) = iT\,\theta\frac{\sigma_3}{2}U(\tau)\,.$$

This equation is solved as

$$U(\tau) = We^{i\theta\frac{\tau}{\beta}\frac{\sigma_3}{2}}\,,$$

where W is a color rotation. Since both $U(\tau)$ and $e^{i\theta\frac{\tau}{\beta}\frac{\sigma_3}{2}}$ preserve direction 3, we should have $W\sigma_3 W^\dagger = \sigma_3$ which implies that W is either the identity or the Weyl transformation discussed in the main text.

Problem 2.4: Recall that $\Phi[A] = \cos(r/2)$. Thus, for $r = \pi + 2\pi n$, we find indeed

$$\cos\left(\frac{\pi}{2} + n\pi\right) = (-1)^n\cos\left(\frac{\pi}{2}\right) = 0.$$

Problem 2.5: For the considered configurations, we have

$$A^C_\mu(x) = T\delta_{\mu 0}\,(-r)\frac{\sigma_3}{2}\,,$$

and thus

$$\Phi[A^C] = \cos((-r)/2) = \cos(r/2) = \Phi[A]\,.$$

This is expected since YM theory being charge conjugation invariant, there should be no way for a thermal bath of gluons to distinguish between a quark and an antiquark.

Chapter 3

Thermal Gluon Average

Problem 3.1: To linear order in θ^a, we can write

$$\begin{aligned} A_\mu^{U_0} &= (\mathbb{1} + i\theta^a t^a) A_\mu (\mathbb{1} - i\theta^a t^a) + i(\mathbb{1} + i\theta^a t^a)\partial_\mu(\mathbb{1} - i\theta^a t^a) \\ &= A_\mu + i(\theta^a t^a) A_\mu - i A_\mu(\theta^a t^a) + \partial_\mu(\theta^a t^a) \\ &= A_\mu + (\partial_\mu - i[A_\mu,\ ])(\theta^a t^a) = A_\mu + D_\mu[A](\theta^a t^a)\,. \end{aligned}$$

This rewrites

$$\delta A_\mu^{U_0} = -i D_\mu[A]\delta U_0\,,$$

which implies

$$\delta \partial_\mu A_\mu^{U_0} = -i\partial_\mu(D_\mu[A]\delta U_0)\,,$$

and from which one reads $\delta\partial_\mu A_\mu^{U_0}/\delta U_0\big|_{U_0=\mathbb{1}}$ and deduces that, up to an non-relevant multiplication constant, $J[A] = \det \partial_\mu D_\mu[A]$.

Problem 3.2: The Faddeev-Popov gauge-fixing part reads now

$$J_{\bar{A}}[A]\,\delta(D_\mu[\bar{A}](A - \bar{A})) \tag{3.1}$$

with

$$J_{\bar{A}}[A] = \det \left.\frac{\delta D_\mu[\bar{A}](A^{U_0} - \bar{A})}{\delta U_0}\right|_{U_0=\mathbb{1}}\,.$$

The functional δ is again taken into account by introducing a Nakanishi-Lautrup field:

$$\delta(D_\mu[\bar{A}](A_\mu-\bar{A}_\mu)) \propto \int \mathcal{D}h\, e^{i\int_x h^a(x)D_\mu[\bar{A}](A^a_\mu(x)-\bar{A}^a_\mu(x))}\,.$$

As for the Faddeev-Popov determinant, proceeding as in the previous problem, we find

$$\delta D_\mu[\bar{A}](A^{U_0}_\mu-\bar{A}_\mu) = -iD_\mu[\bar{A}](D_\mu[A]\delta U_0)\,,$$

from which one reads $\delta D_\mu[\bar{A}](A^{U_0}_\mu-\bar{A}_\mu)/\delta U_0\big|_{U_0=\mathbb{1}}$ and deduces that, up to an non-relevant multiplication constant, $J_{\bar{A}}[A]=\det D_\mu[\bar{A}]D_\mu[A]$. The, introducing ghost and antighost fields as usual, one arrives at

$$J_{\bar{A}}[A]=\det D_\mu[\bar{A}]D_\mu[A]=\int \mathcal{D}c\mathcal{D}\bar{c}\, e^{\int_x \bar{c}^a(x)D^{ab}_\mu[\bar{A}]D^{bc}_\mu[A]c^c(x)}\,. \tag{3.2}$$

Problem 3.3: We have

$$\begin{aligned}
D_\mu[A^U](UXU^\dagger) &= \partial_\mu(UXU^\dagger)-i[UA_\mu U^\dagger+iU\partial_\mu U^\dagger, UXU^\dagger]\\
&= \partial_\mu(UXU^\dagger)-iU[A_\mu,X]U^\dagger+U(\partial_\mu U^\dagger)UXU^\dagger-UX\partial_\mu U^\dagger\\
&= \partial_\mu(UXU^\dagger)-iU[A_\mu,X]U^\dagger-(\partial_\mu U)XU^\dagger-UX\partial_\mu U^\dagger\\
&= U(\partial_\mu X)U^\dagger-iU[A_\mu,X]U^\dagger\\
&= U(\partial_\mu-i[A_\mu,X])U^\dagger\,.
\end{aligned}$$

In the intermediate steps, we have used that $U(\partial_\mu U^\dagger)=-(\partial_\mu U)U^\dagger$ as follows from the unitarity of U.

Problem 3.4: That $\langle A_\mu\rangle_{\bar{A}}$ is constant and temporal for the considered backgrounds follows from transation invariance and rotation invariance. Now, apply the background gauge symmetry identity to constant color rotations of the form $e^{i\theta\sigma_3/2}$. Since the considered background is clearly invariant under these transformations, the identity turns into

$$e^{i\theta\sigma_3/2}\langle A_\mu\rangle_{\bar{A}}e^{-i\theta\sigma_3/2}=\langle A_\mu\rangle_{\bar{A}}\,,$$

which is an actual constraint on $\langle A_\mu\rangle_{\bar{A}}$. More precisely, since the action of $U=e^{i\theta\sigma_3/2}$ on the algebra is a rotation by an angle θ around direction 3, the only possibility for $\langle A_\mu \rangle_{\bar{A}}$ is to be aligned with that direction. In other words, it takes a similar form as the background

$$\langle A_\mu(x) \rangle_{\bar{A}} = T\delta_{\mu 0}\, r \frac{\sigma_3}{2} \,.$$

Chapter 4

Effective Action

Problem 4.1: We have

$$\frac{\delta S}{\delta\varphi^a(x)} = -\partial^2\varphi^a(x) + m^2\varphi^a(x) + \frac{\lambda}{6}(\varphi^b(x)\varphi^b(x))\varphi^a(x) \tag{4.1}$$

and thus

$$\begin{aligned}\frac{\delta^2 S}{\delta\varphi^a(x)\delta\varphi^b(y)} &= (-\partial^2 + m^2)\delta^{ab}\delta(x-y) \\ &+ \frac{\lambda}{6}(\varphi^b(x)\varphi^b(x))\delta^{ab}\delta(x-y) + \frac{\lambda}{3}\varphi^a(x)\varphi^b(y)\delta(x-y)\,.\end{aligned} \tag{4.2}$$

Problem 4.2: We have For $\varphi(x) = \phi$, the previous result writes

$$\begin{aligned}\frac{\delta^2 S}{\delta\varphi^a(x)\delta\varphi^b(y)} &= (-\partial^2 + m^2)\delta^{ab}\delta(x-y) \\ &+ \frac{\lambda}{6}(\varphi^b\varphi^b)\delta^{ab}\delta(x-y) + \frac{\lambda}{3}\varphi^a\varphi^b\delta(x-y)\,.\end{aligned} \tag{4.3}$$

whose Fourier transform is

$$\left(Q^2 + m^2 + \frac{\lambda}{6}\phi^a\phi^a\right)\delta^{ab} + \frac{\lambda}{3}\phi^a\phi^b\,. \tag{4.4}$$

Problem 4.3: We have

$$\begin{aligned}
F^a_{\mu\nu}[A+a] &= \partial_\mu(A^a_\nu + a^a_\nu) - \partial_\nu(A^a_\mu + a^a_\mu) + gf^{abc}(A^b_\mu + a^b_\mu)(A^c_\nu + a^c_\nu) \\
&= \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu \\
&+ \partial_\mu a^a_\nu + gf^{abc}A^b_\mu a^c_\nu \\
&- \partial_\nu a^a_\mu + gf^{abc}a^b_\mu A^c_\nu \\
&+ gf^{abc}a^b_\mu a^c_\nu \\
&= F^a_{\mu\nu}[A] + D^{ab}_\mu[A]a^b_\nu - D^{ab}_\nu[A]a^b_\mu + gf^{abc}a^b_\mu a^c_\nu \,.
\end{aligned}$$

Problem 4.4: We write

$$\begin{aligned}
\partial_\mu(X^a(x)Y^a(x)) &= \partial_\mu(X^a(x))Y^a(x) + X^a(x)\partial_\mu(Y^a(x)) \\
&= (\partial_\mu X^a(x) + gf^{acb}A^c_\mu X^b(x))Y^a(x) \\
&+ X^a(x)\partial_\mu(Y^a(x)) - gf^{acb}A^c_\mu X^b(x)Y^a(x) \\
&= (\partial_\mu X^a(x) + gf^{acb}A^c_\mu X^b(x))Y^a(x) \\
&+ X^a(x)\partial_\mu(Y^a(x)) + gf^{acb}A^c_\mu X^a(x)Y^b(x) \\
&= (\partial_\mu X^a(x) + gf^{acb}A^c_\mu X^b(x))Y^a(x) \\
&+ X^a(x)(\partial_\mu Y^a(x)) + gf^{acb}A^c_\mu Y^b(x)).
\end{aligned}$$

Problem 4.5: The matrix relation is trivially checked and then one uses that the determinant of a square block-diagonal matrix is the product of the determinants of the diagonal blocks.

Problem 4.6: Under the Weyl transformation $r \to -r$, we have $Q_\kappa = (\omega_n + r\kappa T, q) \to (\omega_n - r\kappa T, q)$. Since Q_κ always appears squared, we can absorb the change of sign of r by the change of variables $n \to -n$ in the Matsubara sums. Under $r \to r + 2\pi$, we have $Q_\kappa = (\omega_n + r\kappa T, q) \to (\omega_n + 2\pi T + r\kappa T, q)$. For the mode $\kappa = 0$ there is no change. For $\kappa = \pm 1$, the change can be absorbed by a change of variables $n \to n \mp 1$.

Chapter 5

Matsubara Sum-Integrals

Problem 5.1: We have

$$\frac{1}{Q_\kappa^2+m^2} = \frac{1}{-(i\omega_n + iT\,r\kappa)^2+\varepsilon_q^2}\,,$$

and thus

$$f(z) = \frac{1}{-(z+iT\,r\kappa)^2+\varepsilon_q^2} = \frac{1}{2\varepsilon_q}\left[\frac{-1}{z+iT\,r\kappa-\varepsilon_q}+\frac{1}{z+iT\,r\kappa+\varepsilon_q}\right].$$

This gives

$$\begin{aligned} T\sum_{n\in\mathbb{Z}}\frac{1}{Q_\kappa^2+m^2} &= -\left[-\frac{1}{2\varepsilon_q}n(\varepsilon_q-iT\,r\kappa)+\frac{1}{2\varepsilon_q}n(-\varepsilon_q-iT\,r\kappa)\right] \\ &= \frac{1+n(\varepsilon_q-iT\,r\kappa)+n(\varepsilon_q+iT\,r\kappa)}{2\varepsilon_q} \\ &= \frac{1}{2\varepsilon_q}+\mathrm{Re}\,\frac{n(\varepsilon_q-iT\,r\kappa)}{\varepsilon_q}\,. \end{aligned}$$

Problem 5.2: We have

$$\begin{aligned} (Q_\kappa\cdot\bar{Q}_\kappa)^2+m^2\bar{Q}_\kappa^2 &= (\omega_n^\kappa\bar{\omega}_n^\kappa+q^2)^2+m^2((\bar{\omega}_n^\kappa)^2+q^2) \\ &= q^4+(2\omega_n^\kappa\bar{\omega}_n^\kappa+m^2)q^2+((\omega_n^\kappa)^2+m^2)(\bar{\omega}_n^\kappa)^2\,. \end{aligned}$$

The square masses $M_\pm^2$ are minus the roots of this quadratic polynomial in q^2.

Appendix A

Extension to SU(3)

Problem A.1 The idea is to combine t^3 and t^8 such that

$$at^3 + bt^8 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad ct^3 + dt^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

since the first matrix generates the SU(2) algebra together with t^4 and t^5, while the second matrix commutes with these generators. This gives the systems

$$\begin{aligned} a + b/\sqrt{3} &= 1\,, \\ -a + b/\sqrt{3} &= 0\,, \end{aligned}$$

as well as

$$\begin{aligned} c + d/\sqrt{3} &= 1/\sqrt{3}\,, \\ -c + d/\sqrt{3} &= -2/\sqrt{3}\,, \end{aligned}$$

which are solved as $a = 1/2$, $b = \sqrt{3}/2$, $c = \sqrt{3}/2$ and $d = -1/2$. To proceed similarly with t^6 and t^7, we need to consider combinations such that

$$\begin{aligned} a + b/\sqrt{3} &= 0\,, \\ -a + b/\sqrt{3} &= 1\,, \end{aligned}$$

as well as

$$\begin{aligned} c + d/\sqrt{3} &= -2/\sqrt{3}\,, \\ -c + d/\sqrt{3} &= 1/\sqrt{3}\,, \end{aligned}$$

which are solved as $a = -1/2$, $b = \sqrt{3}/2$, $c = -\sqrt{3}/2$ and $d = -1/2$.

Problem A.2 From Problem A.1, we know that by introducing the combinations $t^{\pm} \equiv (t^4 \pm it^5)/\sqrt{2}$, we have

$$[at^3 + bt^8, t^{\pm}] = \pm t^{\pm} \quad \text{and} \quad [ct^3 + dt^8, t^{\pm}] = 0\,,$$

where $a = 1/2$, $b = \sqrt{3}/2$, $c = \sqrt{3}/2$ and $d = -1/2$. It follows that

$$[t^3, t^{\pm}] = \pm\frac{1}{2}t^{\pm} \quad \text{and} \quad [t^8, t^{\pm}] = \pm\frac{\sqrt{3}}{2}t^{\pm}\,,$$

from which we find the two roots $\pm(1/2, \sqrt{3}/2)$. Similarly, introducing the combinations $t^{\pm} \equiv (t^6 \pm it^7)/\sqrt{2}$, we know from Problem 6.2 that

$$[at^3 + bt^8, t^{\pm}] = \pm t^{\pm} \quad \text{and} \quad [ct^3 + dt^8, t^{\pm}] = 0\,,$$

where $a = -1/2$, $b = \sqrt{3}/2$, $c = -\sqrt{3}/2$ and $d = -1/2$. It follows that

$$[t^3, t^{\pm}] = \mp\frac{1}{2}t^{\pm} \quad \text{and} \quad [t^8, t^{\pm}] = \pm\frac{\sqrt{3}}{2}t^{\pm}\,,$$

from which we find the two roots $\pm(-1/2, \sqrt{3}/2)$.

Problem A.3 By construction, any point on one of the reflection axes of the lattice connects to the origin by a vector s such that $s \cdot \alpha \in \mathbb{Z}/2$ where the root α is the one orthogonal to the reflection axis. The vertices of the lattice are the intersetions of all these reflection axes and, therefore, obey the same condition but for all the possible roots.

Problem A.4: We have

$$\Phi[A] = \frac{1 + 2\cos(2\pi/3)}{3} = \frac{1 + 2(-1/2)}{3} = 0\,.$$

Problem A.5: We have

$$\begin{aligned} \Phi[A^C] &= \frac{1}{3}\left[e^{-i\frac{-4\pi x_8}{\sqrt{3}}} + 2e^{i\frac{-2\pi x_8}{\sqrt{3}}}\cos(-2\pi x_3)\right] \\ &= \frac{1}{3}\left[e^{-i\frac{-4\pi x_8}{\sqrt{3}}} + 2e^{i\frac{-2\pi x_8}{\sqrt{3}}}\cos(2\pi x_3)\right], \end{aligned}$$

which differs from $\Phi[A]$ only due to the sign in multiplying x_8. It follows that $\Phi[A^C] = \Phi[A]$ in the case where $x_8 = 0$.