Multiscale Basis Dictionaries on Higher-Order Networks

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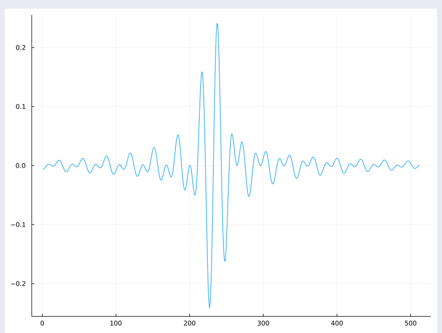
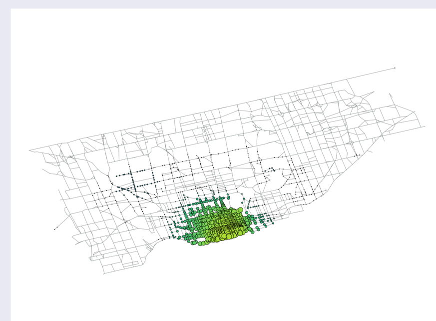
Motivation: Lifting Multiscale Basis Dictionaries to Graphs

- For conventional digital signals and images sampled on regular lattices, *Multiscale Basis Dictionaries* including *wavelet packet dictionaries* (which in turn include *wavelet bases*) and *local cosine dictionaries* have a proven track record of success, e.g.: *JPEG 2000* Image Compression Standard; *Modified Discrete Cosine Transform* (MDCT) in MP3; Discriminant feature extraction for signal classification; …
- Want to lift/generalize these dictionaries to the graph setting for *graph signal processing* and *graph data analysis*



Shannon wavelet on $\mathbb{R}$



Graph wavelet packet vector

Roadmap So Far

- We have developed the graph versions of the *local cosine and wavelet packet dictionaries* for analysis of graph signals *sampled at nodes*.
- All these are based on the *hierarchical partitioning* of either a primary graph G or the so-called *dual graph* $G^\star$.
- Let Ω be a domain to be hierarchically partitioned. Then, we have the correspondence:

Classical Basis Dict.	Ω	Graph Basis Dict.	Ω
Hier. Block DCT	time axis	*HGLET*	G
LCT	time axis	*LP-HGLET*	G
Haar-Walsh WPs	time/freq. axes	*GHWT/eGHWT*	G
Cmpt-Supp. WPs	frequency axis	*LP-NGWPs*	$G^\star$
Shannon WPs	frequency axis	*NGWPs*	$G^\star$

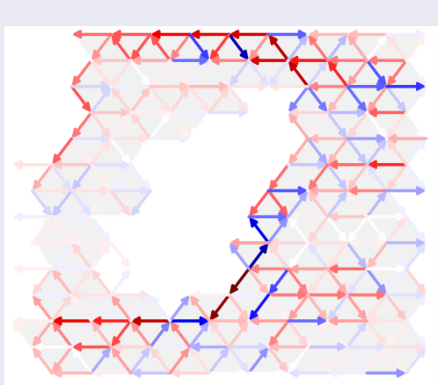
HGLET := Hierarchical Graph Laplacian Eigen Transform [Irion-Saito (2014)];
GHWT := Generalized Haar-Walsh Transform [Irion-Saito (2014)];
eGHWT := extended GHWT [Saito-Shao (2022)];
NGWPs := Natural Graph Wavelet Packets [Cloninger-Li-Saito (2021)];
LP-HGLET/NGWPs := Lapped-HGLET/NGWPs [Li (2021)]

Underlying Philosophy/Basso Continuo:
Split $\Longrightarrow$ "Organize" $\Longrightarrow$ Merge

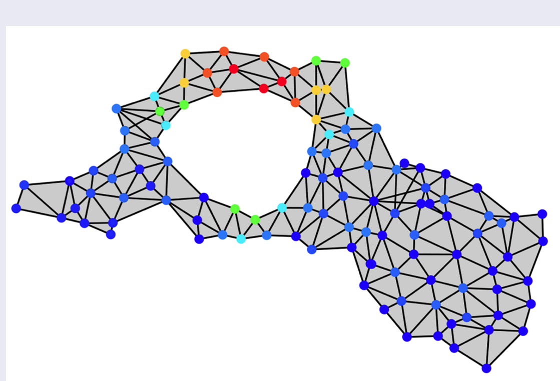
Higher-Order Graph Signals

Recently there has been great interest in analyzing *higher-order graph signals*.

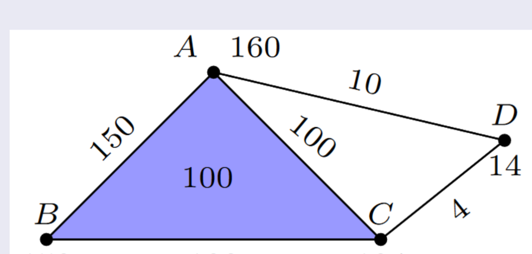
- Data are sampled over oriented *k-simplices* of a graph for some $k \in \mathbb{N}$
  - For $k = 0$, these signals take values over nodes of a graph as usual
  - For $k = 1$, these signals take values over oriented edges of a graph
  - For $k = 2$, these signals take values over oriented triangles of a graph
- Examples: regional weather data, molecular chemistry, neuronal networks, social networks, discrete exterior calculus/geometry, …



Buoys drifting around Madagascar



Gene expression correlations [Govek et al. 2019]

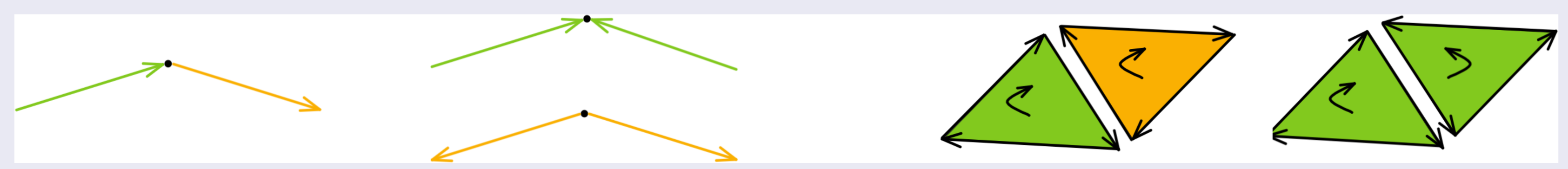


Authorship Graph [Ebli et al. 2022]

Representing Higher-Order Graphs

- A *simplicial complex* represents a combinatorial description of a topological space that can be represented and handled by a computer.
- The k-simplices in a simplicial complex are typically captured by *boundary matrices* B_{k-1}, B_k expressing adjacency and relative orientation of each k-simplex σ with each $(k-1)$-simplex α or $(k+1)$-simplex β respectively.

$$[B_{k-1}]_{\alpha\sigma} = \begin{cases} 1 & \alpha, \sigma \text{ have consistent orientation} \\ -1 & \alpha, \sigma \text{ have inconsistent orientation} \\ 0 & \text{otherwise} \end{cases} \qquad [B_k]_{\sigma\beta} = \begin{cases} 1 & \sigma, \beta \text{ have consistent orientation} \\ -1 & \sigma, \beta \text{ have inconsistent orientation} \\ 0 & \text{otherwise} \end{cases}$$

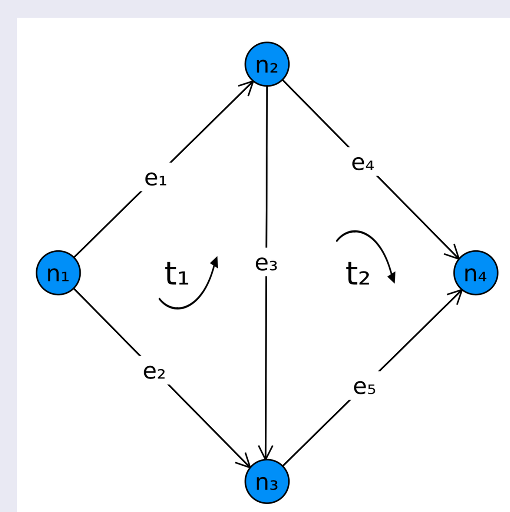


Hodge Laplacian

- The *Hodge Laplacian* [e.g., L.-H. Lim: *SIAM Review* (2020); M. T. Schaub et al.: *Signal Process.* (2021)] provides a spectral decomposition for a signal measured on k-simplices in a given simplicial complex.
- Since the k-Laplacian has both "upper" and "lower" parts, we need a new notion of 'neighbors'. Two k-simplices are 'adjacent' if either:
  - they have a $(k-1)$-simplex in common as a facet; or
  - they are both facets of some $(k+1)$-simplex in the complex.
- Hodge Laplacian via Boundary Matrices:

$$L_k := B_{k-1}^\intercal B_{k-1} + B_k B_k^\intercal; \quad D_k := \operatorname{diag}(L_k)$$

Example Simplicial Complex



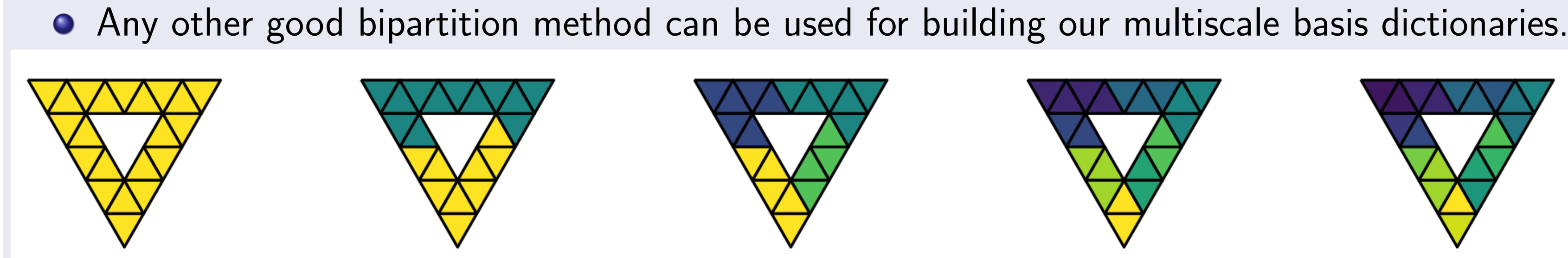
$$B_0 = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad L_0 = \begin{bmatrix} 2 & -1 & -1 & 0 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ 0 & -1 & -1 & 2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -1 & 0 \\ 1 & 0 \\ -1 & -1 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 3 & 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 & -1 \\ 0 & 0 & 4 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 \\ 0 & -1 & 0 & 0 & 3 \end{bmatrix}$$

$$B_2 = O \qquad L_2 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Partitioning Simplicial Complexes

- The *random-walk normalized* Hodge Laplacian $L_k^{\mathrm{rw}} := D_k^{-1} L_k$ also admits a *Fielder vector*, whose sign provides a partition on k-simplices minimizing a relaxed version of *Normalized Cut*.
- L_k induces a *signed graph* on the k-simplices, with $[L_k]_{\sigma\tau} < 0$ when σ, τ have consistent orientations, and $[L_k]_{\sigma\tau} > 0$ when σ, τ have inconsistent orientations.
- Unlike L_0^{rw}, the components of $\boldsymbol{\phi}_0$ of L_k^{rw} may change their signs in general; hence $\boldsymbol{\phi}_1 \odot \operatorname{sign} \boldsymbol{\phi}_0$ provides the Fielder vector.
- While the Hodge Laplacian optimizes for encoding topological information, modification such as the *signed Laplacian* is more closely connected to the appropriate Cut objective.
- Any other good bipartition method can be used for building our multiscale basis dictionaries.

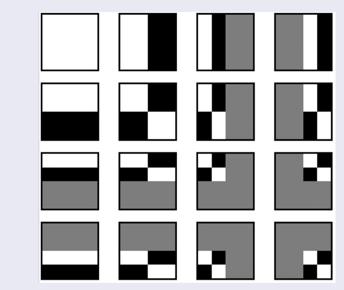


A hierarchical partitioning of a triangle graph with $k = 2$

k-Haar Basis

We can use the partitioning induced by the Fielder vector to develop a *top-down, piecewise constant, and locally concentrated* basis with good approximation properties. However, there are some challenges:

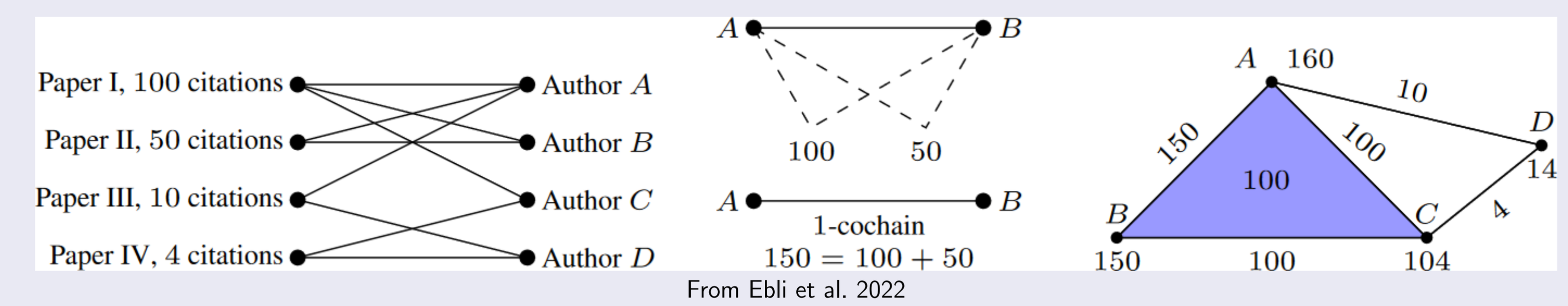
- Since the division is not symmetrically dyadic, we need to compute the scaling factor for each atom separately.
- The presence of both upper and lower boundary terms means that the *discrete nodal domain theorem* does not always apply.



The 0-Haar basis on 4×4 grids

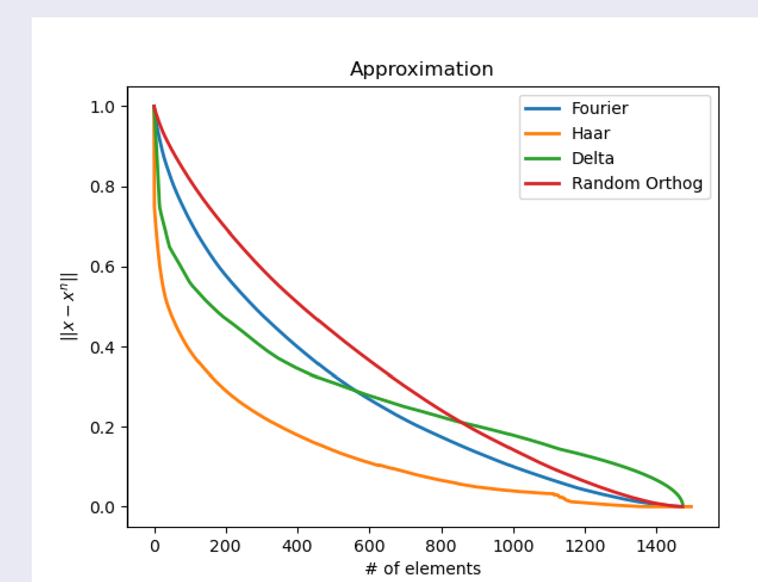
The 2-Haar basis on a triangle graph

Haar Approximation of the Citation Complex

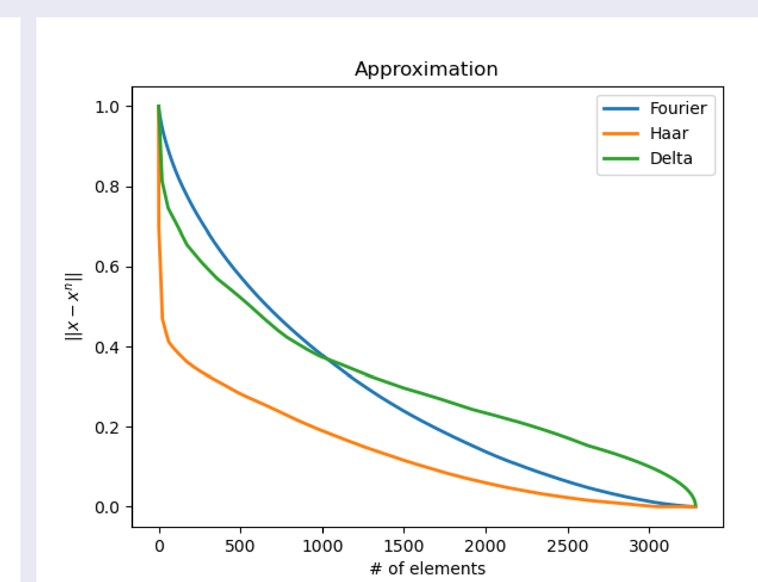


- The citation complex [Patania et al. 2017] can be created by linking papers, authors, and co-authors from the CORA citation network. Specifically, we use the subgraph suggested by [Elbi et al. 2022].
- Each Paper node has a citation value corresponding to the number of citations of the paper.
- Each Author node has a citation value corresponding to the total publications of the author.
- Each $(k+1)$-simplex value is equal to the sum of the k-neighbors (see above).

Dimension	0	1	2	3	4	5	6	7	8	9	10
CC1	352	1474	3285	5019	5559	4547	2732	1175	343	61	5



Edge Signal Approximation



Face Signal Approximation

Summary

- Proposed a hierarchical partitioning method for simplicial complexes using *Hodge Laplacians*
- Developed the *k-Haar transform* for signals on simplicial complexes, which is a part of our *multiscale higher-order graph signal basis dictionaries for graph signals on simplicial complexes*
- Will develop *tools to visualize and interpret important basis vectors* for signals on simplicial complexes including *graph embedding methods*
- Will extend the k-Haar transform to the *k-Haar/Walsh wavelet packet dictionary*
- Will develop the *k-HGLET dictionary* using the eigenvectors of the Hodge Laplacians
- Will lift *Best Basis* [Coifman-Wickerhauser (1992)], *Local Discriminant Basis*, *Local Regression Basis* [Saito et al. (1995; 1997; 2002; …)] for signals on simplicial complexes