Conditioned Weiner Processes as Nonlinearities: A Rigorous Probabilistic Analysis of Dynamics

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Abstract

We study a Weiner process that is conditioned to pass through a finite set of points and consider the dynamics generated by iterating a sample path from this process. Using topological techniques we characterize the global dynamics and deduce the existence, structure and approximate location of invariant sets. Most importantly, we compute the probability that this characterization is correct. This work is probabilistic in nature and intended to provide a theoretical foundation for the statistical analysis of dynamical systems which can only be queried via finite samples.

Probabilistic Assumptions and a Lemma

Our Gaussian process $f = \{f(x)\}_{x \in X}$ is a sequence of Brownian bridges with variance parameter σ^2 conditioned to interpolate the set $\mathcal{T} = \{(x_n, y_n)\}_{n=0}^N$; here $X := [x_0, x_N]$. We denote the mean of this process by μ.

We will need to compute the Probability of the events

$$S_n(\alpha, \beta) := \{f(x) \in (\alpha, \beta) \mid x \in [x_{n-1}, x_n]\}.$$

Lemma: Assume that $\alpha < \min(y_{n-1}, y_n) \le \max(y_{n-1}, y_n) < \beta$. Then

$$\mathbb{P}(S_n(\alpha, \beta)) = 1 - \pi\left(\frac{(\beta - y_n)}{\sigma\sqrt{x_n - x_{n-1}}}, \frac{(\beta - y_{n-1})}{\sigma\sqrt{x_n - x_{n-1}}}, \frac{(y_n - \alpha)}{\sigma\sqrt{x_n - x_{n-1}}}, \frac{(y_{n-1} - \alpha)}{\sigma\sqrt{x_n - x_{n-1}}}\right)$$

where $\pi : \mathbb{R}^4 \to \mathbb{R}$ is defined explicitly in [7]; this lemma follows easily from a result in [3].

Characterization of Dynamics

Let $g : Y \to Y$ be a continuous function on a compact metric space Y and Q a finite partially ordered set with partial order $\preceq$. A **Morse tiling** of Y for g is a decomposition of Y into a collection

$$\mathcal{M} = \{M(q) \mid q \in \mathsf{Q}\}$$

of regular closed sets with disjoint interiors with the following property. Let $y \in Y$. If $g^n(y) \in M(p)$, $g^m(y) \in M(q)$, and $n < m$, then $q \preceq p$.

The sets $M(q)$ are **Morse tiles** for g and the partially ordered set Q is a **Morse graph** for g.

An algebraic topological invariant, *the Conley index*, can be assigned to each Morse Tile. The Conley index can be used to deduce the existence of interesting invariant sets for g such as fixed points, periodic orbits, heteroclinic orbits, bistability, and chaotic dynamics [6].

Attractor Blocks and Morse Tilings

A closed set $K \subset Y$ is an **attractor block** for a continuous function $g : Y \to Y$ if

$$g(K) \subset \operatorname{int}(K)$$

where $\operatorname{int}(K)$ denotes the interior of K. The set of all attractor blocks for g forms a bounded and distributive lattice [4] and is denoted by $\mathsf{ABlock}(g)$.

We denote the set of join-irreducible elements of a lattice L by $\mathsf{J}(\mathsf{L})$; for each $c \in \mathsf{J}(\mathsf{L})$ we let $\overleftarrow{c} \in \mathsf{L}$ denote the unique immediate predecessor of c.

A lattice of attractor blocks determines a Morse tiling of the phase space. Let K be a sublattice of $\mathsf{ABlock}(g)$. For each $K \in \mathsf{J}(\mathsf{K})$ define

$$M(K) := \operatorname{cl}(K \setminus \overleftarrow{K}). \quad (1)$$

Then

$$\mathcal{M}(\mathsf{K}) := \{M(K) \mid K \in \mathsf{J}(\mathsf{K})\} \quad (2)$$

is a Morse tiling of Y for g.

Computation of Dynamics

We decompose X as a simplicial complex $\mathcal{X}$ and use a **combinatorial multivalued map** $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$. $\mathcal{F}$ is an outer approximation of $g : X \to X$ if $g(\xi) \subset \operatorname{int}(\mathcal{F}(\xi))$ for each $\xi \in \mathcal{X}^{(1)}$. In this case $\mathsf{Invset}^+(\mathcal{F}) := \{S \subset \mathcal{X} \mid \mathcal{F}(S) \subset S\}$ is a sublattice of $\mathsf{ABlock}(g)$ [5].

We define an outer approximation $\mathcal{F}_\mu$ of the known mean μ of f and use this to identify a sublattice K of $\mathsf{ABlock}(\mu)$. We then construct another CMVM $\mathcal{F}_{\mathsf{K}}$ and compute the probability that $\mathcal{F}_{\mathsf{K}}$ is an outer approximation of a sample path of f; this probability is equivalent to the probability that K is a lattice of attractor blocks for this sample path.

We use $\mathcal{F}_{\mathsf{K}}$ in order to define the Conley index $\mathrm{Con}_*(M(K))$; details are found in [7]. This index is shift equivalent to the classic homology Conley index $\mathrm{Con}_*(\mathrm{Inv}(M(K), g)$ where $g : X \to X$ is any function having $\mathcal{F}_{\mathsf{K}}$ as an outer approximation and $\mathrm{Inv}(N, g)$ denotes the maximal invariant set contained in a set $N \subset Y$.

In computing the probability that $\mathcal{M}$ is a Morse tiling for f we use functions $\alpha, \beta : \{1, 2, \cdots, N\} \to \mathbb{R}$, where $\alpha(n)$ (resp. $\beta(n)$) is the minimum (resp. maximum) of $\mathcal{F}_{\mathsf{K}}([x_{n-1}, x_n])$.

Main Theorem

Let $\mathsf{K} \hookrightarrow \mathsf{Invset}^+(\mathcal{F})$. With probability

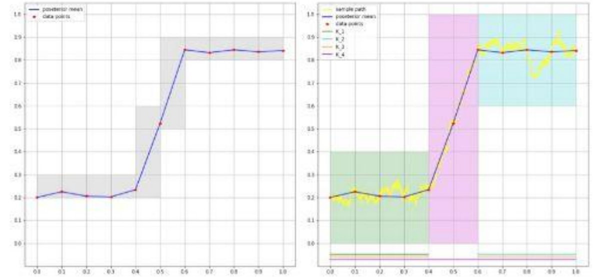
$$\prod_{n \in \mathcal{I}} \mathbb{P}(S_n(\alpha(n), \beta(n)))$$

$\mathcal{M}(\mathsf{K}) = \{M(K) \mid K \in \mathsf{J}(\mathsf{K})\}$ is a Morse tiling of X for f and $\mathrm{Con}_*(M(K))$ is the Conley index of each Morse tile.

Remark: A similar approach of combining Gaussian process surrogate modeling with Conley theory has been explored in [1] for different Gaussian processes, but rigorous bounds on the relevant probabilities for finite data sets are not available in that case.

Example: Detecting Bistability

We consider the set $\mathcal{T} = \{(x_n, y_n) \mid x_n = n/10\}_{n=0}^{10}$, shown in the figure below, and let f be Brownian motion with variance parameter $\sigma^2 = 1/16$ conditioned to pass through $\mathcal{T}$. We identify the lattice $\mathsf{K} = \{K_i\}_{i=0}^4$ shown below ($K_0 = \emptyset$).



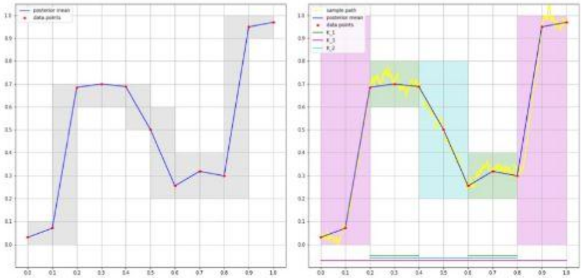
The lattice K gives the Morse tiling

$$\mathcal{M}(\mathsf{K}) = \{M(K_1) = [0, .4], M(K_2) = [.6, 1], M(K_4) = [.4, .6] \mid M(K_4) > M(K_1), M(K_4) > M(K_2)\}.$$

This Morse tiling indicates that the dynamical system is bistable, with attractors in K_1 and K_2. Moreover, there is a non-trivial invariant set in (x_4, x_6) with the Conley index of a repelling fixed point. All of this information is valid for f with probability 0.9989.

Example: Detecting a Periodic Orbit

Consider the set $\mathcal{T} = \{(x_n, y_n) \mid x_n = n/10\}_{n=0}^{10}$ shown below and let f be Brownian motion on $X = [0, 1]$ with variance parameter $\sigma^2 = 1/16$ conditioned to interpolate $\mathcal{T}$. Then $\mathsf{K} = \{K_0 = \emptyset, K_1 = [0.2, 0.4] \cup [0.6, 0.8], K_2 = [0, 2, 0.8], K_3 = X\} \hookrightarrow \mathsf{ABlock}(\mu)$. $\mathcal{M}(\mathsf{K})$ is unlikely to be a Morse tiling for f; the probability of this event is 0.1199.



However, while we are unable to draw strong conclusions about the global dynamics in this case, we can still find important local information with relatively high probability. If we restrict to the domain $X' := K_2$ then we can use the lattice $\mathsf{K}' = \{K_0 = \emptyset, K_1, K_2\}$ in order to obtain a Morse tiling of X':

$$\mathcal{M}(\mathsf{K}') = \{M(K_1) = K_1, M(K_2) = \operatorname{cl}(K_2 \setminus K_1) \mid M(K_2) > M(K_1)\}.$$

This is a Morse tiling for $f|_{K_2}$ with probability 0.6283. Using this tiling and the Conley index we see that $f|_{K_2}$ has a periodic orbit with this same probability.

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