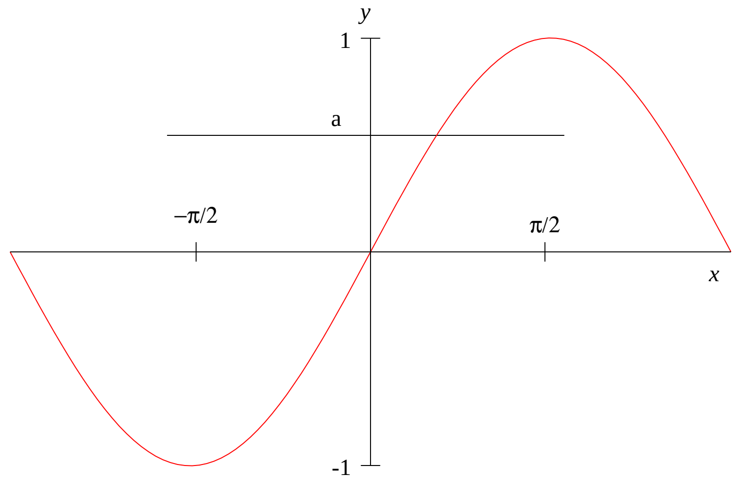
The probability density function of a sinusoid (1)

Let x be uniformly distributed over

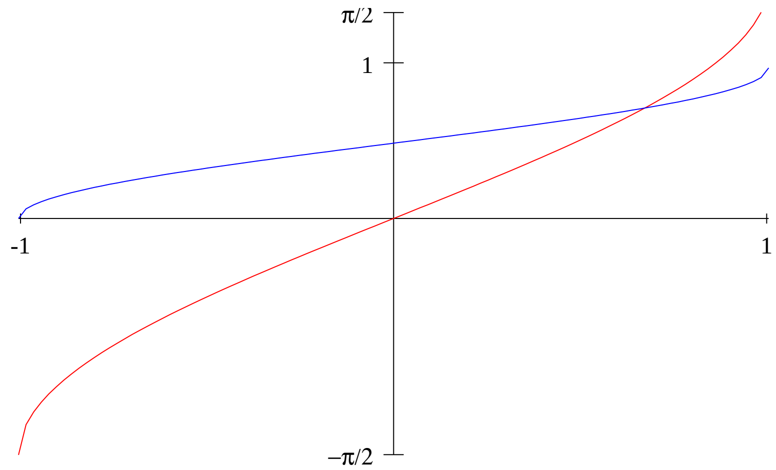
so y is in (-1,1)



The cumulative probability $P(y \leq a) = P(\sin x \leq a)$, so, $P(x \leq \operatorname{asin} a)$

Now normalize P so that $P(-1) = 0$ and $P(1) = 1$:

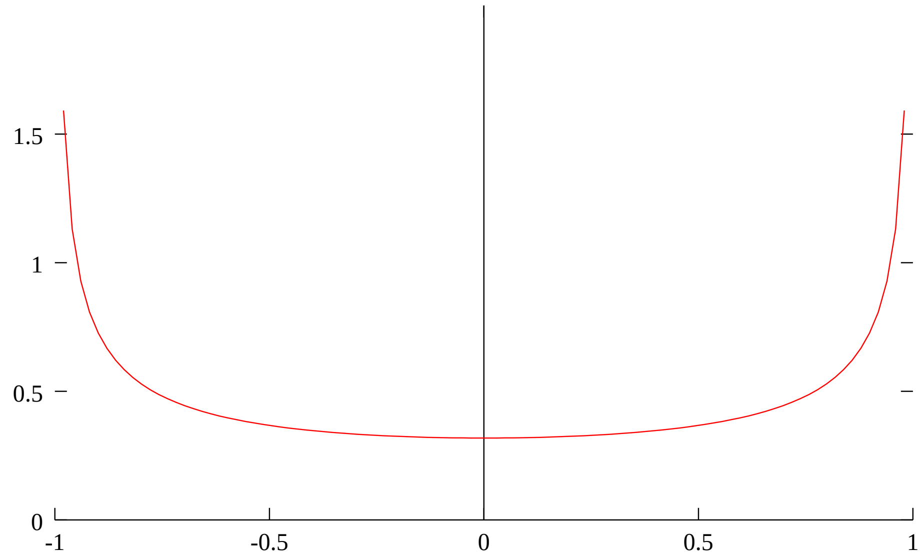
$$P\left(x\, lte \operatorname{asin} a\right) = \frac{\operatorname{asin} a + \frac{\pi}{2}}{\pi} = \frac{\operatorname{asin} a}{\pi} + \frac{1}{2}$$



The probability density function of a sinusoid (2)

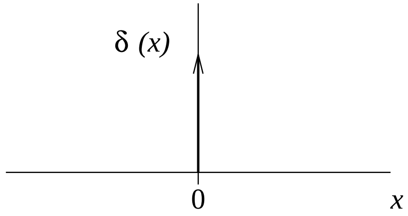
Differentiate $P(a)$ to obtain the probability density function $p(a)$:

$$p(a)=\frac{dP(a)}{da}=\frac{1}{\pi\sqrt{(1-x^2)}}$$



The Dirac δ

$\delta(x)=0$ everywhere except at $x=0$



$$\int_{-\infty}^{\infty} \delta(x)\,dx = 1$$

$\delta(x)$ has unit surface

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

It is a sampling function

$$\int_{-\infty}^{\infty} \delta(x)\,e^{-j2\pi fx\,dx} = 1$$

Fourier transform of $\delta(x)$ is unity

$$\int_{-\infty}^{\infty} 1\cdot e^{j2\pi f x\,df} = \delta(x)$$

Inverse Fourier transform of unity is $\delta(x)$

Convolution

$$g(t)*h(t)=\mathscr{F}^{-1}\{G(f)\cdot H(f)\}$$

$$G(f)=\mathscr{F}\{g(t)\}=\int_{-\infty}^{\infty} g(t)e^{-j2\pi f t}dt$$ (Replace *t* by *x*)

$$H(f)=\mathscr{F}\{h(t)\}=\int_{-\infty}^{\infty} h(t)e^{-j2\pi f t}dt$$ (Replace *t* by *y*)

$$g(t)*h(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)e^{j2\pi f(t-x-y)}dx\,dy\,df$$ Integrate over *f*

$$g(t)*h(t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)h(y)\delta(t-x-y)dx\,dy$$ Integrate over *x* or *y*

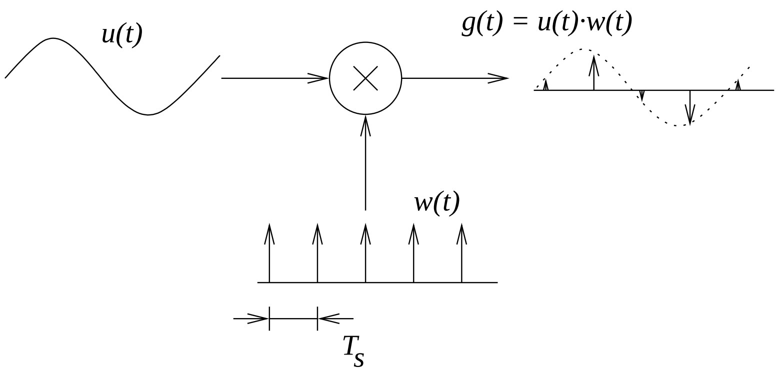
$$g(t)*h(t)=\int_{-\infty}^{\infty} g(t-y)h(y)dy$$

$$g(t)*h(t)=\int_{-\infty}^{\infty} g(x)h(t-x)dx$$

The convolution integral

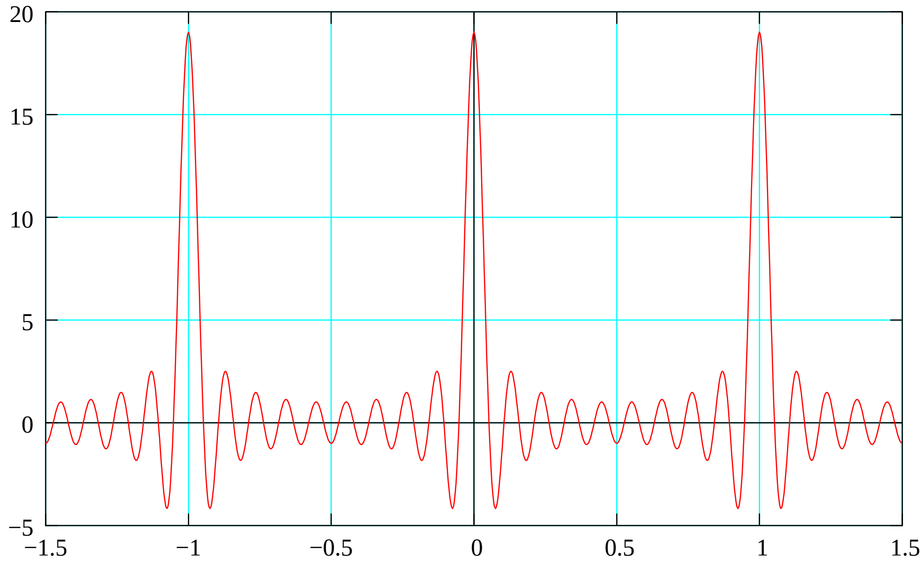
Sampling

$$g(t)=u(t)\cdot w(t)$$



First the Fourier transform of $w(t)$:

$$W(f)=\mathscr{F}\{w(t)\}=\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(t-nT_s)e^{-j2\pi ft}dt$$



$$W(f)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(t-nT_s)e^{-j2\pi ft}dt$$

(First 19 terms)

$$W(f)=\sum_{n=-\infty}^{\infty}e^{-j2\pi f nT_s}=1+2\sum_{n=1}^{\infty}\overbrace{\cos 2\pi f nT_s}=\sum_{n=-\infty}^{\infty}\delta(f-nF_s)$$

Sampling -2

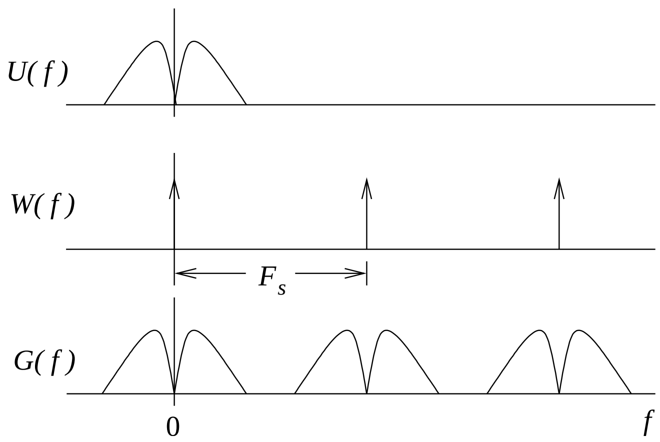
Convolve spectrum of sampling waveform with spectrum of input signal:

$$G(f)=U(f)*W(f)=\int_{-\infty}^{\infty} U(\phi)W(f-\phi)\phi$$

$$G(f)=\int_{-\infty}^{\infty} U(\phi)\sum_{n=-\infty}^{\infty}\delta(f-nF_s-\phi)d\phi$$

$$G(f)=\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty} U(\phi)\delta(f-nF_s-\phi)d\phi$$

$$G(f)=\sum_{n=-\infty}^{\infty} U(f-nF_s)$$



Clock signal: $V(t)=\sin(2\pi F_s+\varphi(t))$

Significant instants: $2\pi F_s t_1+\varphi(t_1)=0$ and $2\pi F_s t_2+\varphi(t_2)=2\pi N$

$$2\pi F_s(t_2-t_1)+\varphi(t_2)-\varphi(t_1)=2\pi N$$

The time between these instants is an integer number of periods plus some jitter:

$$t_2-t_1=\frac{N}{F_s}+\Delta t$$

$$2\pi F_s(\frac{N}{F_s}+\Delta t)+\varphi(t_2)-\varphi(t_1)=2\pi N$$

$$\Delta t=\frac{1}{2\pi F_s}(\varphi(t_1)-\varphi(t_2))$$

Expected value of variance:

$$\langle \Delta t^2 \rangle = \frac{1}{4\pi^2 F_s^2} \left(\langle \varphi(t_1)^2 \rangle - 2\langle \varphi(t_1)\varphi(t_2) \rangle + \langle \varphi(t_2)^2 \rangle \right)$$

Jitter does not depend on chosen instant:

$$\langle \varphi(t_1)^2 \rangle = \langle \varphi(t_2)^2 \rangle = \langle \varphi(t)^2 \rangle = \int_0^\infty S_\varphi(f)\, df$$

Variance of middle term is a cross covariance: Use cosine transform

$$\langle \varphi(t_1)\varphi(t_2) \rangle = \int_0^\infty S_\varphi(f) \cos(2\pi f \tau)\, df \quad \text{with} \quad \tau = t_1 - t_2$$

Jitter - 3

$$\langle \Delta t^2 \rangle = \frac{1}{2\pi^2 F_s^2} \int_0^\infty S_\varphi(f)(1-\cos(2\pi f \tau))\, df$$

$$\langle \Delta t^2 \rangle = \frac{1}{\pi^2 F_s^2} \int_0^\infty S_\varphi(f)(\sin^2(\pi f \tau))\, df$$

RMS jitter

$$\langle \Delta t \rangle = \frac{1}{\pi F_s} \sqrt{\int_0^\infty S_\varphi(f)(\sin^2(\pi f \tau))\, df}$$