EW and QCD physics at the muon collider

Yang Ma

INFN Bologna

The 1st Collaboration Meeting of the Muon Collider Study
October 12, 2022

Our goal and the dream machine

A lot of particle physics is missing in the Standard Model

- Why Electroweak Symmetry Breaking occurs?
  What is the history of the Electroweak Phase Transition?
- The reason for the Hierarchy in Fermion Masses and their Flavor Structure
- The Nature of Dark Matter
- The origin of the Matter-Antimatter Asymmetry
- The generation of Neutrino Masses
- The cause of the Universe's accelerated expansion - Dark Energy
- What are the quantum properties of Gravity?
- What caused Cosmic Inflation after the Big Bang?

The SM is silent about all above, BSM physics is at the core of it all

The colliders

Our goal is to **“Address the Big Questions”** and to **“Explore the unknown”**

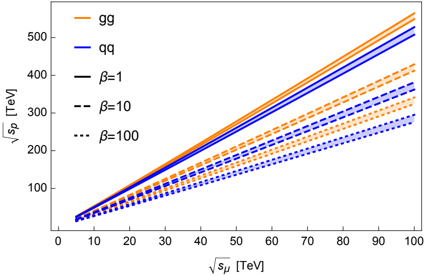
- Study known phenomena at high energies looking for indirect evidence of BSM physics
  Higgs Factories ⇒ Probe TeV scale via precision measurements
- Search for direct evidence of BSM physics at the energy frontier
  Directly reach the multi-TeV scale

Current Colliders

- Hadron colliders collide **composite particles** ⇒ To reach high energies
  Generate large QCD backgrounds and you use a fraction of the energy of beam for physics
- Lepton colliders collide **fundamental particles** ⇒ To reach high precisions
  Exploit the full energy and avoid large QCD backgrounds

Dream machine: A multi-TeV level lepton collider

Get use of the full machine energy



Discovery reach: $M \sim \frac{\sqrt{s}}{2}$

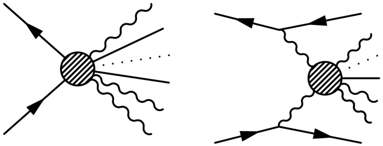
[2103.14043 “Muon Smasher’s Guide”]

muC$@10$ TeV $\sim pp@70$ TeV

10 TeV is not the limit

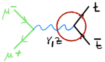
More than lepton collisions:

Two mechanisms: Annihilation VS Fusion



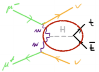
⇒ VBF collider:

$\sqrt{s} \lesssim 1\text{-}5\,\mathrm{TeV}$



$\sigma_s \sim \frac{1}{s}$ ⟷ $\sigma_s \sim \frac{1}{M^2} \log^n \frac{s}{M}$

$\sqrt{s} \gtrsim 1\text{-}5\,\mathrm{TeV}$



Need to resume the large Logs

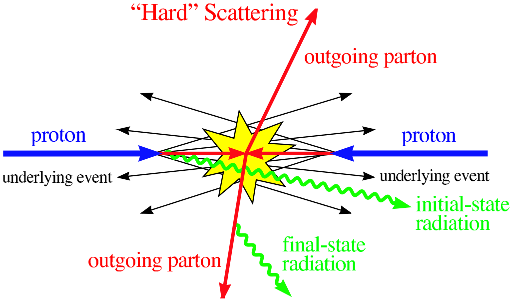
⇒ **The partonic picture is needed**

The partonic picture

[T. Han, YM, K. Xie, 2007.14300, 2103.09844]

Hadron colliders and the Parton Distribution Function (PDF)

- **Recall the hadron colliders: the $\mathrm{Sp\bar{p}S}$, the Tevatron, or the LHC**



- **Hadrons are composite**
  a, b are the "partons" from the beam particles A and B.
- **PDFs**
  $f_{a/A}, f_{b/B}$ are the probabilities to find a parton a (b) from the beam particle A (B) with a momentum fraction x_a (x_b).

- **Factorization formalism : PDFs $\otimes$ partonic cross sections**

$$\sigma(AB \to X) = \sum_{a,b} \int \mathrm{d}x_a \mathrm{d}x_b f_{a/A}(x_a, Q) f_{b/B}(x_b, Q) \hat{\sigma}(ab \to X)$$

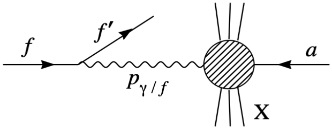
“Parton” of a lepton

Leptons are elementary particles ⇒ **“Equivalent photon approximation (EPA)”**

- Treat photon as a parton constituent in the electron

$$\sigma(\ell^- + a \to \ell^- + X) = \int \mathrm{d}x\, f_{\gamma/\ell}\hat{\sigma}(\gamma a \to X)$$

$$f_{\gamma/\ell,\mathrm{EPA}}(x_\gamma, Q^2) = \frac{\alpha}{2\pi}\frac{1+(1-x_\gamma)^2}{x_\gamma}\ln\frac{Q^2}{m_\ell^2}$$



[C. F. von Weizsacker, Z. Phys. 88, 612 (1934)]

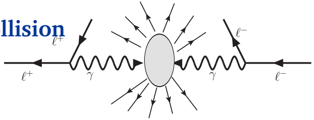
[E. J. Williams, Phys. Rev. 45, 729 (1934)]

- At lepton colliders

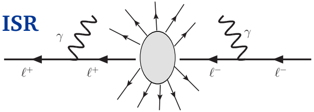
$$\sigma(\ell^+\ell^- \to F + X) = \int_{\tau_0}^{1} d\tau \sum_{ij} \frac{d\mathcal{L}_{ij}}{d\tau}\,\hat{\sigma}(ij \to F),\ \tau = \hat{s}/s$$

$$\frac{d\mathcal{L}_{ij}}{d\tau} = \frac{1}{1+\delta_{ij}}\int_{\tau}^{1}\frac{d\xi}{\xi}\left[f_i(\xi,Q^2)f_j\left(\frac{\tau}{\xi},Q^2\right) + (i \leftrightarrow j)\right]$$

$\gamma\gamma$ collision



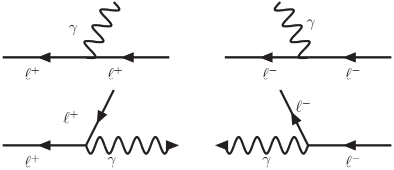
ISR



People have been doing:

- $\ell^+\ell^-$ annihilation

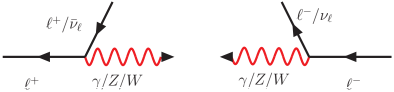
- EPA and ISR



- "Effective W Approx." (EWA)

[G. Kane, W. Repko, and W. Rolnick, PLB 148 (1984) 367]

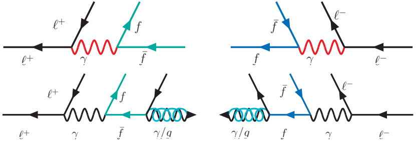
[S. Dawson, NPB 249 (1985) 42]



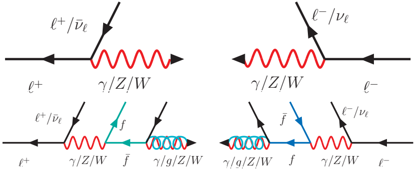
We will add:

[T. Han, Y. Ma, K.Xie 2007.14300, 2103.09844]

- Above μ_{QCD}: QED⊗QCD
  q/g emerge



- Above $\mu_{\mathrm{EW}} = M_Z$: EW⊗QCD
  EW partons / corrections to the above



In the end, everything is parton, i.e. need the full SM PDFs.

The PDF evolution: DGLAP

- The DGLAP equations

$$\frac{\mathrm{d} f_i}{\mathrm{d}\log Q^2} = \sum_I \frac{\alpha_I}{2\pi} \sum_j P_{ij}^I \otimes f_j$$

- The initial conditions

$$f_{\ell/\ell}(x, m_\ell^2) = \delta(1-x)$$

- Three regions and two matchings
  - $m_\ell < Q < \mu_{\mathrm{QCD}}$: QED
  - $Q = \mu_{\mathrm{QCD}} \lesssim 1$ GeV: $f_q \propto P_{q\gamma} \otimes f_\gamma, f_g = 0$
  - $\mu_{\mathrm{QCD}} < Q < \mu_{\mathrm{EW}}$: QED$\otimes$QCD
  - $Q = \mu_{\mathrm{EW}} = M_Z$: $f_\nu = f_t = f_W = f_Z = f_{\gamma Z} = 0$
  - $\mu_{\mathrm{EW}} < Q$: EW$\otimes$QCD.

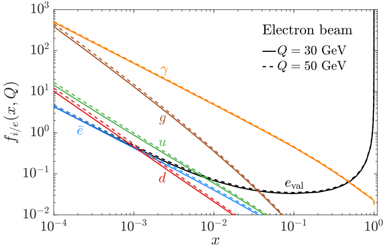
$$\begin{pmatrix} f_B \\ f_{W^3} \\ f_{BW^3} \end{pmatrix} = \begin{pmatrix} c_W^2 & s_W^2 & -2c_W s_W \\ s_W^2 & c_W^2 & 2c_W s_W \\ c_W s_W & -c_W s_W & c_W^2 - s_W^2 \end{pmatrix} \begin{pmatrix} f_\gamma \\ f_Z \\ f_{\gamma Z} \end{pmatrix}$$

- We work in the (B, W) basis. The technical details can be referred to the backup slides.

The QED⊗QCD PDFs for lepton colliders

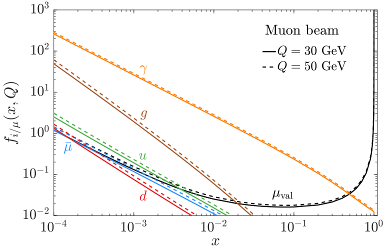
- **Electron PDFs**: $f_{e_{\text{val}}},\ f_{\gamma},\ f_{\ell_{\text{sea}}},\ f_{q},\ f_{g}$
- Scale uncertainty: 10% for $f_{g/e}$
- The averaged momentum fractions $\langle x_i \rangle = \int x f_i(x)\mathrm{d}x$

$Q(e^{\pm})$	e_{val}	γ	ℓsea	q	g
30 GeV	96.6	3.20	0.069	0.080	0.023
50 GeV	96.5	3.34	0.077	0.087	0.026
M_Z	96.3	3.51	0.085	0.097	0.028



- **Muon PDFs**: $f_{\mu_{\text{val}}},\ f_{\gamma},\ f_{\ell_{\text{sea}}},\ f_{q},\ f_{g}$
- Scale uncertainty: 20% for $f_{g/\mu}$
- The averaged momentum fractions $\langle x_i \rangle = \int x f_i(x)\mathrm{d}x$

$Q(\mu^{\pm})$	μ_{val}	γ	ℓsea	q	g
30 GeV	98.2	1.72	0.019	0.024	0.0043
50 GeV	98.0	1.87	0.023	0.029	0.0051
M_Z	97.9	2.06	0.028	0.035	0.0062



The PDFs of a lepton beyond the EW scale

- **All SM particles are partons**

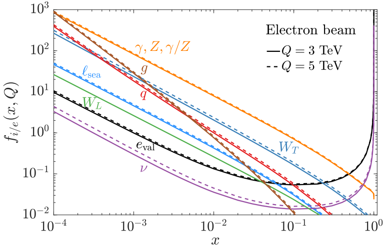
  [T. Han, Y. Ma, K.Xie 2007.14300, 2103.09844]
  - The sea leptonic and quark PDFs show up

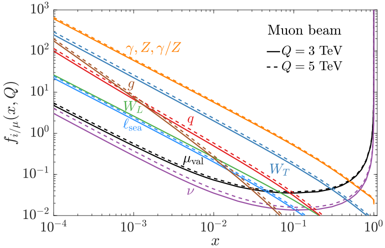
    $\nu = \sum_i(\nu_i + \bar{\nu}_i)$,

    $\ell\text{sea} = \bar{\mu} + \sum_{i\neq\mu}(\ell_i + \bar{\ell}_i)$,

    $q = \sum_{i=d}^{t}(q_i + \bar{q}_i)$

    There is even neutrino due to the EW sector
- W_L does not evolve at the leading order.
- The EW correction is not small: $\sim 50\%$ (100%) for $f_{d/e}$ $(f_{d/\mu})$ due to the relatively **large SU(2) gauge coupling**. [T. Han, Y. Ma, K.Xie 2103.09844]
- Scale uncertainty: $\sim 15\%$ (20%) between $Q = 3$ TeV and $Q = 5$ TeV





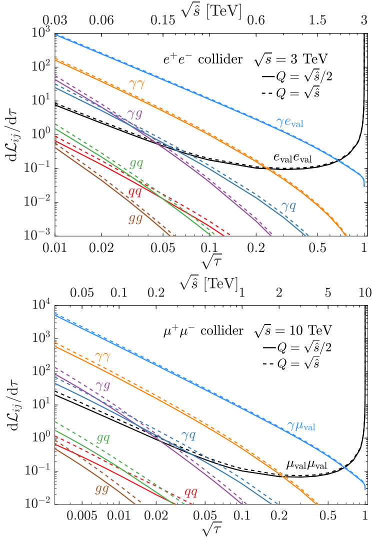
Parton luminosities at high-energy lepton colliders

A 3 TeV e^+e^- machine and a 10 TeV $\mu^+\mu^-$ machine

- Partonic luminosities for

$$\ell^+\ell^-,\ \gamma\ell,\ \gamma\gamma,\ qq,\ \gamma q,\ \gamma g,\ gq,\ \text{and}\ gg$$

- $\gamma\gamma$ gives the largest partonic luminosity
- The luminosity of $\gamma g + \gamma q$ is $\sim 50\%$ (20%) of $\gamma\gamma$
- The luminosities of qq, gq, and gg are $\sim 2\%$ (0.5%) of $\gamma\gamma$
- Given the stronger QCD coupling, **sizable QCD cross sections are expected**.
- Scale uncertainty is $\sim 20\%$ (50%) for photon (gluon) initiated processes.



The SM expectation

[T. Han, YM, K. Xie, 2007.14300, 2103.09844]

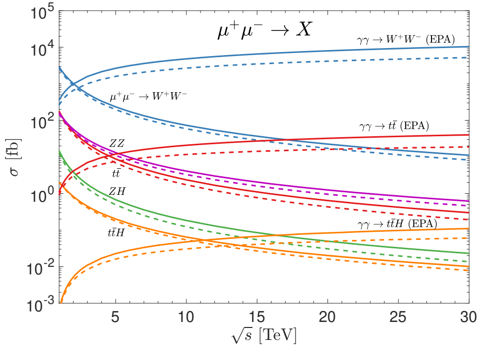
What do we get if the machine is turned on?

- What is the SM physics picture?
- What is the largest background signal?
- Where can we see the possible BSM physics?

Apply EPA at high-energy lepton colliders

A high-energy muon collider at first glance

What do people expect from a high-energy lepton (muon) collider?



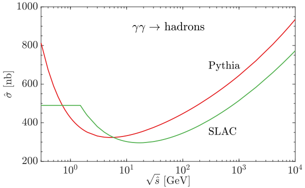
[T. Han, YM, K.Xie 2007.14300]

Some "commonsense":

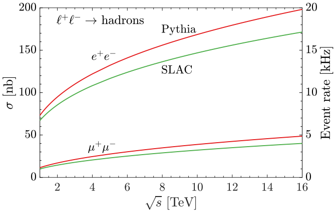
- The annihilations decrease as $1/s$.
- ISR needs to be considered, which can give over 10% enhancement.
- The fusions increase as $\ln^p(s)$, which take over at high energies.
- The large collinear logarithm $\ln(s/m_\ell^2)$ needs to be resummed, set $Q=\sqrt{\hat{s}}/2$,
- $\gamma\gamma \to W^+W^-$ production has the largest cross section.

Photon induced hadronic production at high-energy lepton colliders

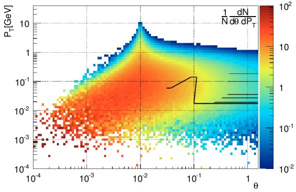
- **Model-dependent** $\hat{\sigma}_{\gamma\gamma}$



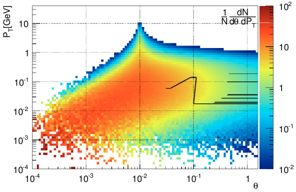
- $\sigma_{\ell\ell}$ may reach nano-barns



- The events populate at low p_T regime



(a) Pythia sample

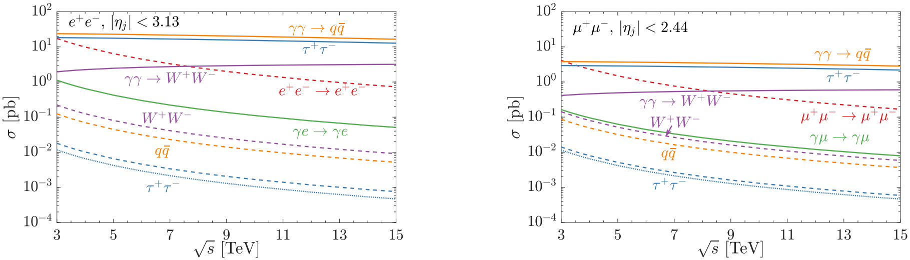


(b) SLAC sample

[T. Barklow, D. Dannheim, M. O. Sahin, and D. Schulte, LCD-2011-020]

What are the dominant processes in the high p_T range?

- Detector angle & Threshold: $\theta_{\rm cut} = 5^\circ\ (10^\circ) \Longleftrightarrow |\eta| < 3.13(2.44)$, $m_{ij} > 20$ GeV
- To separate from the nonperturbative hadronic production: $p_T > \left(4 + \frac{\sqrt{s}}{3\,\mathrm{TeV}}\right)\ \mathrm{GeV}$
  - Leading-order: $\ell^+\ell^- \to \ell^+\ell^-,\ \tau^+\tau^-,\ q\bar{q},\ W^+W^-$, and $\gamma\ell \to \gamma\ell$
  - $\gamma\gamma$ scatterings: $\gamma\gamma \to \tau^+\tau^-,\ q\bar{q},\ W^+W^-$

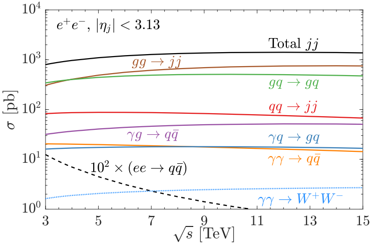


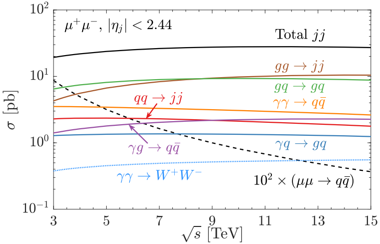
The full background: Di-Jet production at high-energy lepton colliders

- Consider all the "partons"
  $\Rightarrow$ **perturbatively computable processes**

$$\gamma\gamma \to q\bar{q},\ \gamma g \to q\bar{q},\ \gamma q \to gq,$$
$$qq \to qq\,(gg),\ gq \to gq \text{ and } gg \to gg\,(q\bar{q}).$$

- Large $\alpha_s \ln\left(Q^2\right)$ brings a $6\% \sim 15\%$ ($30\% \sim 40\%$) enhancement if $Q = 2Q$
- The QCD contributions result in total cross section.
- gg initiated cross sections are large for the **multiplicity**
- gq initiated cross sections are large for the **luminosity**.
- $\gamma\gamma$ gives smaller cross sections than the EPA does.

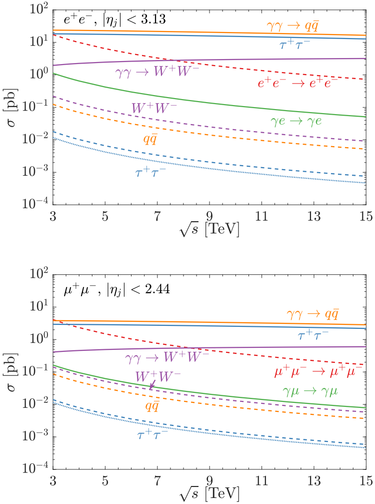




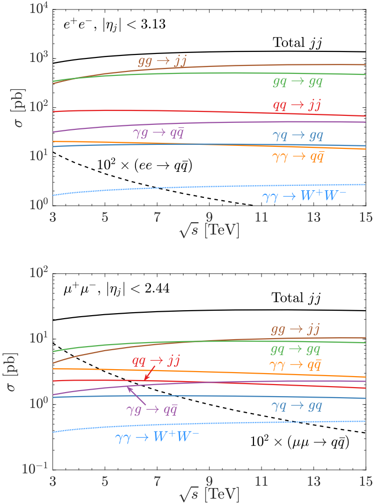
Compare the new with the old

Quark/gluon initiated jet production dominates

Before:



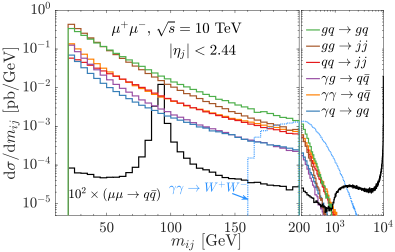
After:

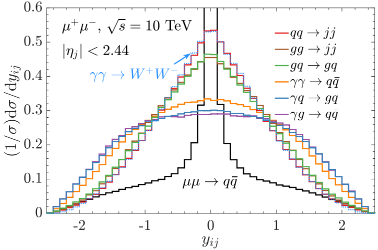


Di-jet distributions at a 10 TeV muon collider

Rather a conservative set up: $\theta = 10^\circ$

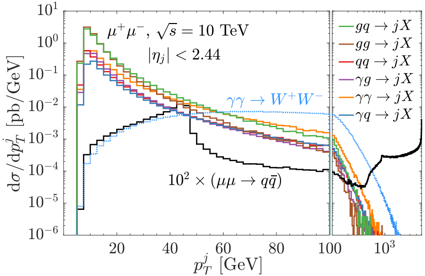
- Some physics:
  Two different mechanisms: $\mu^+\mu^-$ **annihilation** VS **Fusion processes**
  - Annihilation is more than 2 orders of magnitude smaller than fusion process.
  - Annihilation peaks at $m_{ij} \sim \sqrt{s}$;
  - Fusion processes peak near m_{ij} threshold.
  - Annihilation is very central, spread out due to ISR;
  - Fusion processes spread out, especially for γq and γg initiated ones.

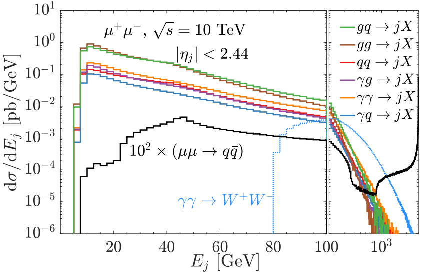




Inclusive jet distributions at a 10 TeV muon collider

Important guidelines for future analysis





We expect

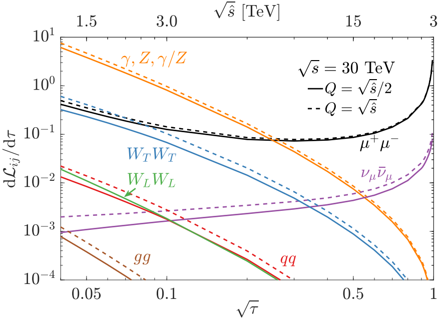
- Jet production dominates over WW production until $p_T > 60$ GeV;
- WW production takes over around energy ~ 200 GeV.

The SM EW sector, as well as any possible BSM, can only be seen in the high p_T (E_j) range.

The full picture a multi-TeV lepton collider: An EW version of LHC

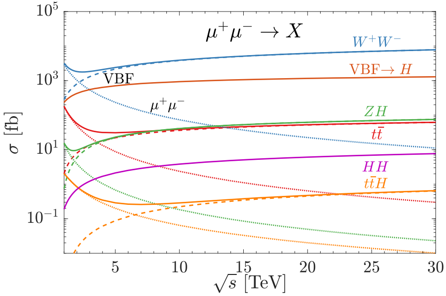
- All SM particles are partons
- We are allowed to determine the partons with their different polarizations

The EW parton luminosities of a 30 TeV muon collider



Just like in hadronic collisions:

$\mu^+\mu^- \to$ exclusive particles + remnants



Compare the “EW LHC” with LHC

pp VS $\mu\mu$

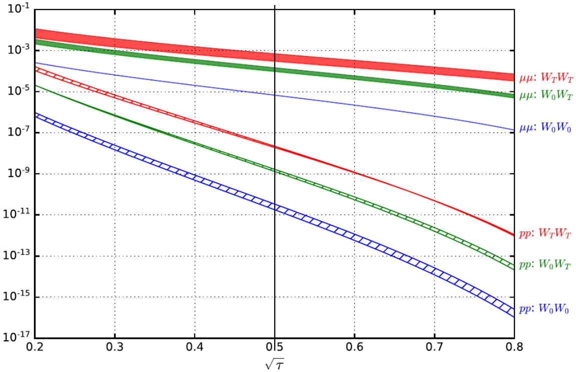
$$\mathcal{L}_{W^+_{\lambda_1} W^-_{\lambda_2}} = \int_\tau^1 \frac{\mathrm{d}\xi}{\xi} f_{W_{\lambda_1}}(\xi, \mu_f) f_{W_{\lambda_1}}(\frac{\tau}{\xi}, \mu_f)$$

Consider the two colliders in the same ring

$$\sqrt{s}_{\mu\mu} = \sqrt{s}_{pp}$$

For $2 \to 1$ processes, take a benchmark

$$\sqrt{\tau} = \frac{M}{\sqrt{s}} = \frac{1}{2}$$

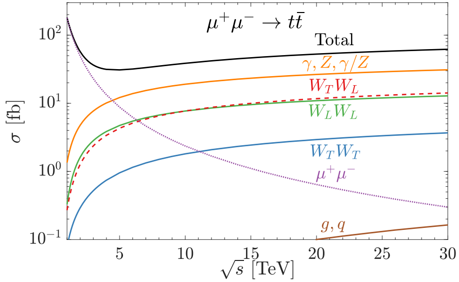


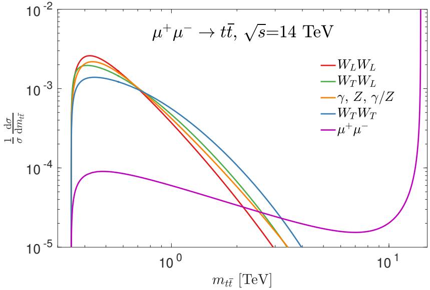
[2005.10289]

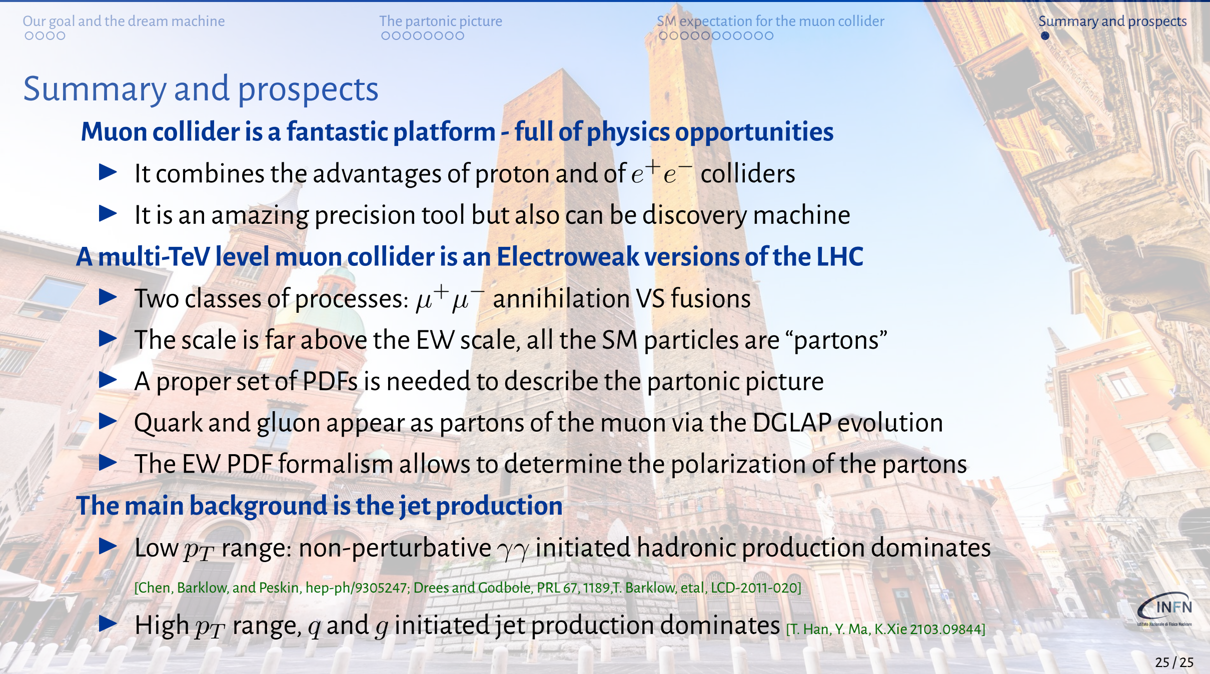
The ratio $\mu\mu/pp$ is larger than 10^4!

One example: $t\bar{t}$ production at a future muon collider

- Two different mechanisms: **Annihilation** and **Fusion**
- The VBF processes exceed the $\mu^+\mu^-$ annihilation at high energies
- The EW PDF formalism allows to determine different partons/polarizations
- The resummation effects lie in the tails.







Summary and prospects

Muon collider is a fantastic platform - full of physics opportunities

- It combines the advantages of proton and of e^+e^- colliders
- It is an amazing precision tool but also can be discovery machine

A multi-TeV level muon collider is an Electroweak versions of the LHC

- Two classes of processes: $\mu^+\mu^-$ annihilation VS fusions
- The scale is far above the EW scale, all the SM particles are “partons”
- A proper set of PDFs is needed to describe the partonic picture
- Quark and gluon appear as partons of the muon via the DGLAP evolution
- The EW PDF formalism allows to determine the polarization of the partons

The main background is the jet production

- Low p_T range: non-perturbative $\gamma\gamma$ initiated hadronic production dominates

  [Chen, Barklow, and Peskin, hep-ph/9305247; Drees and Godbole, PRL 67, 1189,T. Barklow, etal, LCD-2011-020]
- High p_T range, q and g initiated jet production dominates [T. Han, Y. Ma, K.Xie 2103.09844]

One example in precision physics: The Muon-Higgs Coupling

[T. Han, W. Kilian, N. Kreher, YM, T. Striegl, J. Reuter, and K. Xie, 2108.05362]

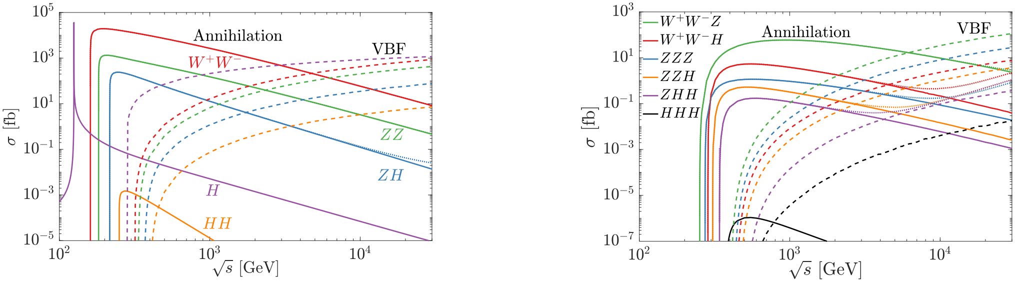
[E. Celada, T. Han, W. Kilian, N. Kreher, YM, F. Maltoni, D. Pagani, T. Striegl, J. Reuter, and K. Xie, in progress]

- Physics: We actually do not know whether the SM mass-generation mechanism applies just to the heavy particles, or also to the 1st/2nd generations.
- Logical possibility: Muon mass not (only) generated by SM Higgs.
  $\Rightarrow$ **Why not have an arbitrary Yukawa coupling?**

Multi-boson final states and the Muon-Higgs coupling

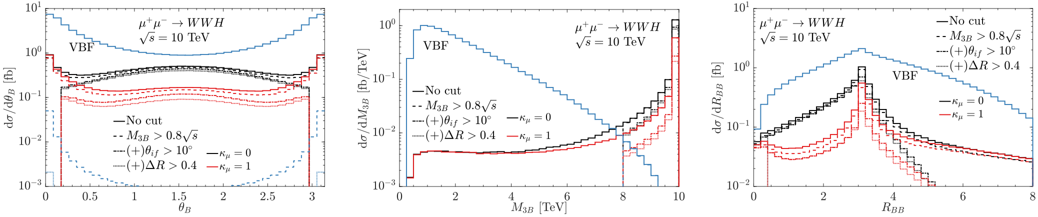
- **SM**: $\lambda(\text{Muon} - \text{Higgs}) \sim y_\mu^{\text{SM}} = \sqrt{2} m_\mu^{\text{SM}}/v$
- **Possible BSM physics**: $m_\mu = m_\mu^{\text{SM}}$, $\lambda(\text{Muon} - \text{Higgs}) \sim \kappa_\mu y_\mu^{\text{SM}}$, e.g. $\kappa_\mu = 0$

Two-boson final states **Three-boson final states**



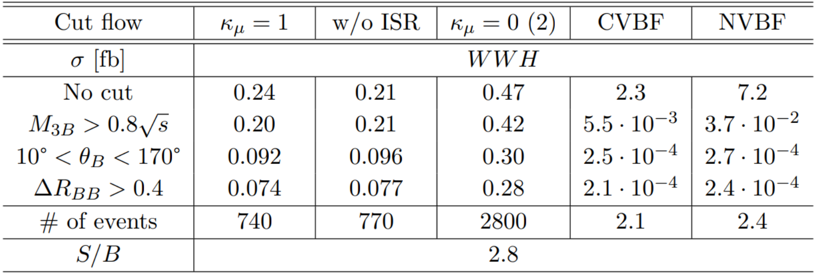
New physics signal shows up in the high energy region

WWH at a 10 TeV muon collider: Kinematics



- Background (VBF) is much larger than signal (annihilation)
- VBF events accumulate around threshold, and mostly forward
- Annihilation in the rest frame (central, and $M \sim \sqrt{s}$ spread by ISR)
- Annihilation also has forward dominance, due to the gauge splitting $W \to WH$

WWH at a 10 TeV muon collider: Cuts

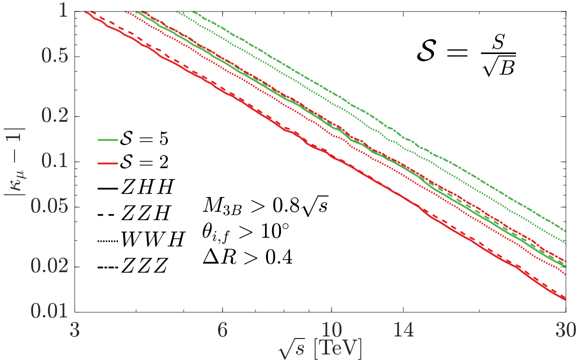


Cut flow	$\kappa_\mu = 1$	w/o ISR	$\kappa_\mu = 0\ (2)$	CVBF	NVBF
σ [fb]	WWH				
No cut	0.24	0.21	0.47	2.3	7.2
$M_{3B} > 0.8\sqrt{s}$	0.20	0.21	0.42	$5.5 \cdot 10^{-3}$	$3.7 \cdot 10^{-2}$
$10^\circ < \theta_B < 170^\circ$	0.092	0.096	0.30	$2.5 \cdot 10^{-4}$	$2.7 \cdot 10^{-4}$
$\Delta R_{BB} > 0.4$	0.074	0.077	0.28	$2.1 \cdot 10^{-4}$	$2.4 \cdot 10^{-4}$
# of events	740	770	2800	2.1	2.4
S/B	2.8				

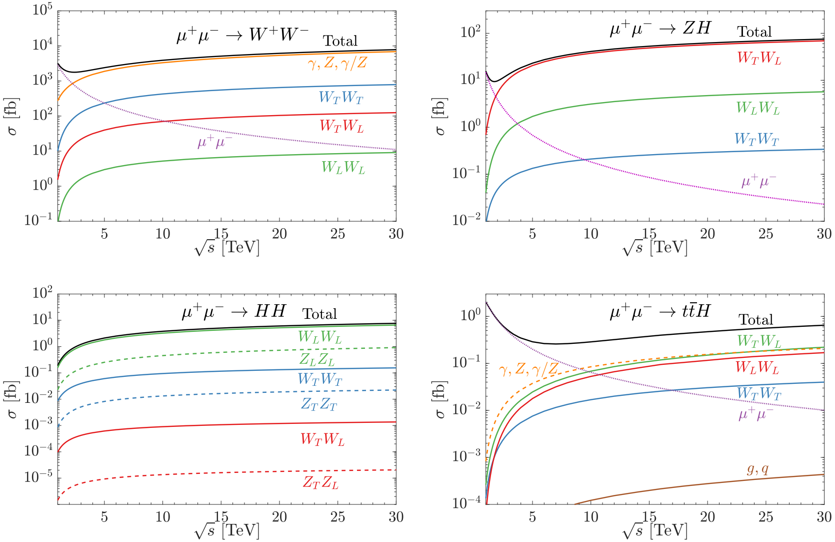
- Integrated luminosity $\mathcal{L} = (\sqrt{s}/10\ \text{TeV})^2 \cdot 10\,\text{ab}^{-1}$ [1901.06150]
- $S = N_{\kappa_\mu} - N_{\kappa_\mu=1},\ B = N_{\kappa_\mu=1} + N_{\text{VBF}}.$
- VBF and ISR are mostly excluded by invariant mass cut.
- Angular cut also weaken VBF further.

Test the muon Yukawa: statistical sensitivity

- The most sensitive channels are ZHH and ZZH, similar probes due to GBET.
- Taking $S = 2$ criterion, we can test the muon-Higgs coupling up to 10% (1%) precision at a 10 (30) TeV muon collider, corresponding to new physics scale $\Lambda_{\mathrm{NP}} \sim 30 - 100$ TeV.



Other processes: $W^+W^-, ZH, HH, t\bar{t}H$



The DGLAP

[T. Han, YM, K. Xie, 2007.14300, 2103.09844]

Solving the DGLAP: Singlet and Non-singlet PDFs

The singlets

$$f_L = \sum_{i=e,\mu,\tau} (f_{\ell_i} + f_{\bar{\ell}_i}),\ f_U = \sum_{i=u,c} (f_{u_i} + f_{\bar{u}_i}),\ f_D = \sum_{i=d,s,b} (f_{d_i} + f_{\bar{d}_i})$$

The non-singlets

- The only non-trivial singlet $f_{e,NS} = f_e - f_{\bar{e}}$
- the leptons $f_{\ell_i,NS} = f_{\ell_i} - f_{\bar{\ell}_i}(i = 2, 3),\ f_{\ell,12} = f_{\bar{e}} - f_{\bar{\mu}},\ f_{\ell,13} = f_{\bar{e}} - f_{\bar{\tau}};$
- the up-type quarks $f_{u_i,NS} = f_{u_i} - f_{\bar{u}_i},\ f_{u,12} = f_u - f_c;$
- and the down-type quarks $f_{d_i,NS} = f_{d_i} - f_{\bar{d}_i},\ f_{d,12} = f_d - f_s,\ f_{d,13} = f_d - f_b.$

Reconstruction:

$$f_e = \frac{f_L + (2N_\ell - 1)f_{e,NS}}{2N_\ell},\ f_{\bar{e}} = f_\mu = f_{\bar{\mu}} = f_\tau = f_{\bar{\tau}} = \frac{f_L - f_{e,NS}}{2N_\ell}.$$

$$f_u = f_{\bar{u}} = f_c = f_{\bar{c}} = \frac{f_U}{2N_u},\ f_d = f_{\bar{d}} = f_s = f_{\bar{s}} = f_b = f_{\bar{b}} = \frac{f_D}{2N_d}.$$

The QED⊗QCD case

- The singlets and gauge bosons

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2}\begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix} = \begin{pmatrix} P_{\ell\ell} & 0 & 0 & 2N_\ell P_{\ell\gamma} & 0 \\ 0 & P_{uu} & 0 & 2N_u P_{u\gamma} & 2N_u P_{ug} \\ 0 & 0 & P_{dd} & 2N_d P_{d\gamma} & 2N_d P_{dg} \\ P_{\gamma\ell} & P_{\gamma u} & P_{\gamma d} & P_{\gamma\gamma} & 0 \\ 0 & P_{gu} & P_{gd} & 0 & P_{gg} \end{pmatrix} \otimes \begin{pmatrix} f_L \\ f_U \\ f_D \\ f_\gamma \\ f_g \end{pmatrix}$$

- The non-singlets

$$\frac{\mathrm{d}}{\mathrm{d}\log Q^2} f_{NS} = P_{ff} \otimes f_{NS}.$$

- The averaged momentum fractions of the PDFs: $f_{\ell_{\mathrm{val}}},\ f_\gamma,\ f_{\ell_{\mathrm{sea}}},\ f_q,\ f_g$

$$\langle x_i \rangle = \int x f_i(x)\mathrm{d}x,\ \sum_i \langle x_i \rangle = 1$$

$$\frac{\langle x_q \rangle}{\langle x_{\ell \mathrm{sea}} \rangle} \lesssim \frac{N_c \left[\sum_i (e^2_{u_i} + e^2_{\bar{u}_i}) + \sum_i (e^2_{d_i} + e^2_{\bar{d}_i})\right]}{e^2_{\ell_{\mathrm{val}}} + \sum_{i \neq \ell \mathrm{val}} (e^2_{\ell_i} + e^2_{\bar{\ell}_i})} = \frac{22/3}{5}$$

The DGLAP for the full SM

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{0\pm} & 0 & 0 & 0 & 0 & P_{LB}^{0\pm} & P_{LW}^{0\pm} & 0 \\ 0 & P_{QQ}^{0\pm} & 0 & 0 & 0 & P_{QB}^{0\pm} & P_{QW}^{0\pm} & P_{Qg}^{0\pm} \\ 0 & 0 & P_{EE}^{0\pm} & 0 & 0 & P_{EB}^{0\pm} & 0 & 0 \\ 0 & 0 & 0 & P_{UU}^{0\pm} & 0 & P_{UB}^{0\pm} & 0 & P_{Ug}^{0\pm} \\ 0 & 0 & 0 & 0 & P_{DD}^{0\pm} & P_{DB}^{0\pm} & 0 & P_{Dg}^{0\pm} \\ P_{BL}^{0\pm} & P_{BQ}^{0\pm} & P_{BE}^{0\pm} & P_{BU}^{0\pm} & P_{BD}^{0\pm} & P_{BB}^{0\pm} & 0 & 0 \\ P_{WL}^{0\pm} & P_{WQ}^{0\pm} & 0 & 0 & 0 & 0 & P_{WW}^{0\pm} & 0 \\ 0 & P_{gQ}^{0\pm} & 0 & P_{gU}^{0\pm} & P_{gD}^{0\pm} & 0 & 0 & P_{gg}^{0\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{0\pm} \\ f_Q^{0\pm} \\ f_E^{0\pm} \\ f_U^{0\pm} \\ f_D^{0\pm} \\ f_B^{0\pm} \\ f_W^{0\pm} \\ f_g^{0\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L}\begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix} = \begin{pmatrix} P_{LL}^{1\pm} & 0 & P_{LW}^{1\pm} & P_{LM}^{1\pm} \\ 0 & P_{QQ}^{1\pm} & P_{QW}^{1\pm} & P_{QM}^{1\pm} \\ P_{WL}^{1\pm} & P_{WQ}^{1\pm} & P_{WW}^{1\pm} & 0 \\ P_{ML}^{1\pm} & P_{MQ}^{1\pm} & 0 & P_{MM}^{1\pm} \end{pmatrix} \otimes \begin{pmatrix} f_L^{1\pm} \\ f_Q^{1\pm} \\ f_W^{1\pm} \\ f_{BW}^{1\pm} \end{pmatrix}$$

$$\frac{\mathrm{d}}{\mathrm{d}L} f_W^{2\pm} = P_{WW}^{2\pm} \otimes f_{WW}^{2\pm}$$

The splitting functions can be found in [Chen et al. 1611.00788, Bauer et al. 1703.08562,1808.08831]