Low and moderate x gluon contributions to DDVCS, DVCS and TCS

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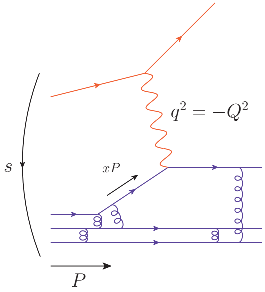
3DPartons Workshop

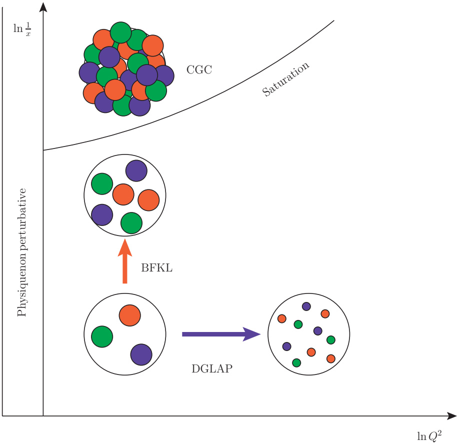
CPHT Centre de Physique Théorique · CNRS · l'X École Polytechnique · Institut Polytechnique de Paris

In collaboration with Y. Mehtar-Tani

Accessing the partonic content of hadrons with an electromagnetic probe

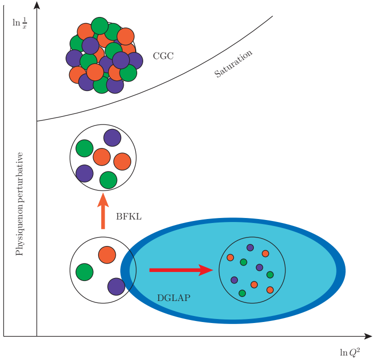


Electron-proton collision (parton model)



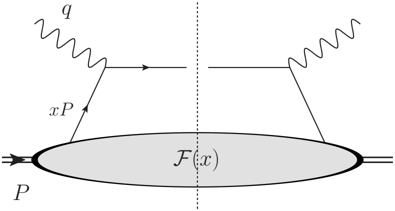
QCD at moderate $x_{\mathrm{Bj}} \sim Q^2/s$

Bjorken limit: $Q^2 \sim s$



QCD factorization

processes with a hard scale $Q \gg \Lambda_{QCD}$



$$\sigma = \mathcal{F}(x, \mu) \otimes \mathcal{H}(x, \mu)$$

At a scale μ, the process is factorized into:

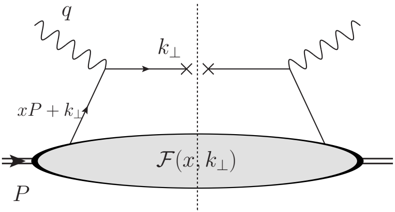
- A hard scattering subamplitude $\mathcal{H}(x, \mu)$
- A Parton Distribution Function (PDF) $\mathcal{F}(x, \mu)$

μ independence: DGLAP renormalization equation for $\mathcal{F}$

Transverse Momentum Dependent (TMD) factorization:

semi-inclusive processes with one hard and one semihard scale

$$Q \sim \sqrt{s} \gg k_\perp$$



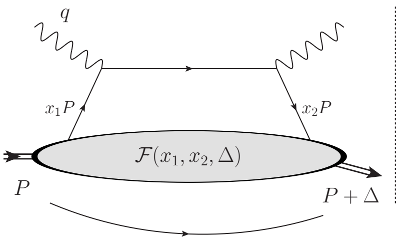
$$\sigma = \mathcal{F}(x, k_\perp, \zeta, \mu) \otimes \mathcal{H}(\mu) \otimes \hat{\mathcal{F}}(\hat{x}, \hat{k}_\perp, \hat{\zeta}, \mu)$$

At a scale μ, the process is factorized into:

- A hard scattering subamplitude $\mathcal{H}(\mu)$
- A TMD PDF $\mathcal{F}(x, k_\perp, \zeta, \mu)$
- A TMD FF $\hat{\mathcal{F}}(\hat{x}, \hat{k}_\perp, \hat{\zeta}, \mu)$

$\mu, \zeta, \hat{\zeta}$ independence: TMD evolution for $\mathcal{F}, \hat{\mathcal{F}}$

Factorization with Generalized Parton Distributions (GPD): exclusive processes with one hard scale $Q \sim \sqrt{s}$



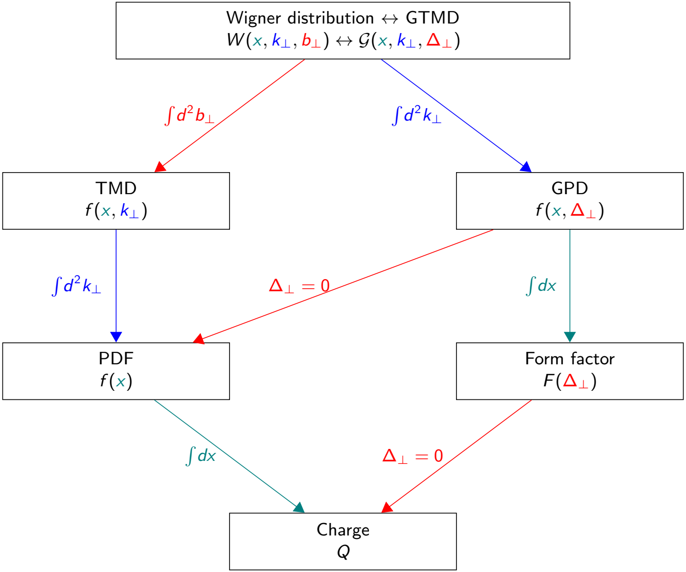
$$\sigma = \mathcal{F}(x_1, x_2, |\Delta_\perp|, \mu) \otimes \mathcal{H}(x_1, x_2, \mu)$$

At a scale μ, the process is factorized into:

- A hard scattering subamplitude $\mathcal{H}(x_1, x_2, \mu)$
- A Generalized Parton Distribution (GPD) $\mathcal{F}(x_1, x_2, |\Delta_\perp|, \mu)$

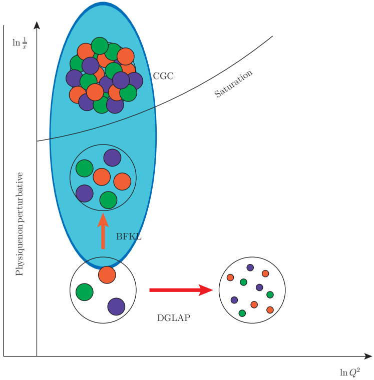
μ independence: DGLAP/ERBL renormalization equation for $\mathcal{F}$

The family tree of parton distributions

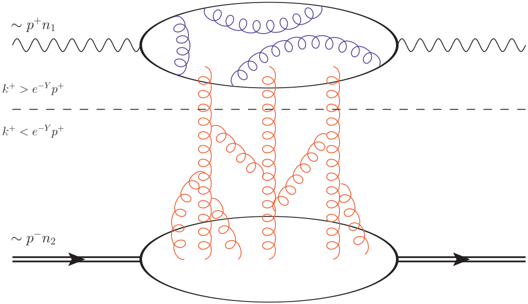


QCD at small $x_{\mathrm{Bj}} \sim Q^2/s$

Regge limit: $Q^2 \ll s$



Rapidity separation

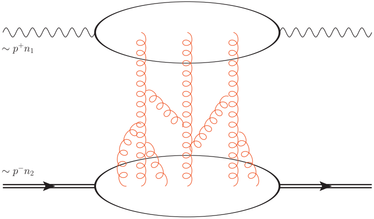


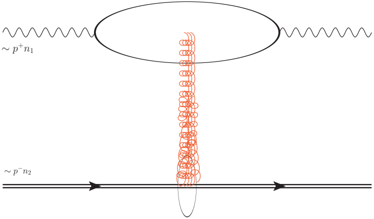
Let us split the gluonic field between "fast" and "slow" gluons

$$\mathcal{A}^{\mu a}(k^+, k^-, \boldsymbol{k}) = A^{\mu a}\left(|k^+| > e^{-Y} p^+, k^-, \boldsymbol{k}\right) + A^{\mu a}_{\text{cl}}(|k^+| < e^{-Y} p^+, k^-, \boldsymbol{k})$$

$e^{-Y} \ll 1$

Large longitudinal boost to the projectile frame





$$A^+_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad\qquad \frac{1}{\Lambda} A^+_{\rm cl}\left(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x}\right)$$

$$A^-_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \longrightarrow \qquad \Lambda A^-_{\rm cl}\left(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x}\right)$$

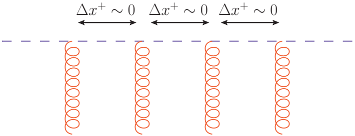
$$A^i_{\rm cl}(x^+, x^-, \boldsymbol{x}) \qquad \Lambda \sim \sqrt{\frac{s}{m_t^2}} \qquad A^i_{\rm cl}\left(\Lambda x^+, \frac{x^-}{\Lambda}, \boldsymbol{x}\right)$$

$$A^\mu_{\rm cl}(x) \to A^-_{\rm cl}(x)\, n_2^\mu = \delta(x^+)\, \mathbf{A}(\boldsymbol{x})\, n_2^\mu + O\left(\sqrt{\frac{m_t^2}{s}}\right)$$

Shock wave approximation

Effective Feynman rules in the slow background field

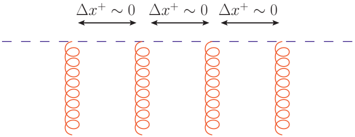
Effective fermion propagator in the external classical field



- $A^i_{\rm cl} = 0, \ A^+_{\rm cl} = 0$: the Dirac structure factorizes
- $A_{\rm cl}$ does not depend on x^-: conservation of $+$ momentum
- $A_{\rm cl}$ is peaked around $x^+ = 0$:
  - Most external propagators get factorized out
  - Gaussians $\sim \delta$ functions: conservation of transverse position
  - Possibility to extend Wilson lines to infinity $[x^+, y^+]_x = [\infty^+, -\infty^+]_x \equiv U_x$

Effective Feynman rules in the slow background field

The interactions with the background field can be exponentiated

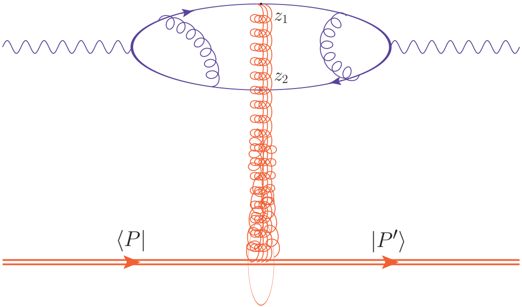


$$D_F(x_2, x_0)|_{x_2^+ > 0, x_0^+ < 0} = \int \mathrm{d}^D x_1 \, \delta(x_1^+) \, D_0(x_2, x_1) \, \gamma^+ U_{\boldsymbol{x}_1} D_0(x_1, x_0)$$

Each fast parton is dressed by an infinite Wilson line

$$U_{\boldsymbol{x}} \equiv \mathcal{P} \exp \left[ig \int_{-\infty}^{+\infty} \mathrm{d}x \cdot A_{\mathrm{cl}}(x) \right]$$

Factorized picture



Factorized amplitude

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{x}_1 \mathrm{d}\boldsymbol{x}_2\, \Phi^Y(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \langle P'|[\mathrm{Tr}(U^Y_{x_1} U^{Y\dagger}_{x_2}) - N_c]|P\rangle$$

Written similarly for any number of Wilson lines in any color representation!

Y independence: B-JIMWLK, BK equations. Resums logarithms of s

The seemingly incompatible nature of the distributions

Two different kinds of gluon distributions

Moderate x distributions

TMD, PDF...

$$\langle P | F^{-i} W F^{-j} W | P \rangle$$

Low x distributions

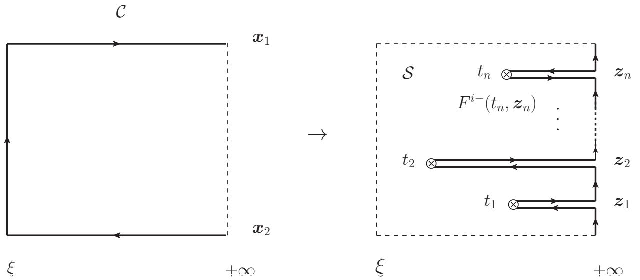
Dipole scattering amplitude

$$\langle P | \mathrm{tr}(U_1 U_2^\dagger) | P \rangle$$

The Wilson line $\leftrightarrow$ parton distribution equivalence

Most general equivalence: use the Non-Abelian Stokes theorem

[RB, Mehtar-Tani]



$$\mathcal{P}\exp\left[\oint_{\mathcal{C}} dx_\mu A^\mu(x)\right] = \mathcal{P}\exp\left[\int_{\mathcal{S}} d\sigma_{\mu\nu}\, W F^{\mu\nu} W^\dagger\right]$$

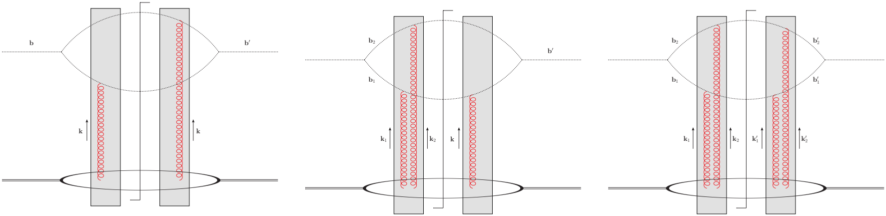
$$U_{x_{1\perp}} U^\dagger_{x_{2\perp}} = [\hat{x}_{1\perp}, \hat{x}_{2\perp}]$$

Inclusive low x cross section

Inclusive low x cross section = TMD cross section

[Altinoluk, RB, Kotko], [Altinoluk, RB]

Generalizes [Dominguez, Marquet, Xiao, Yuan]



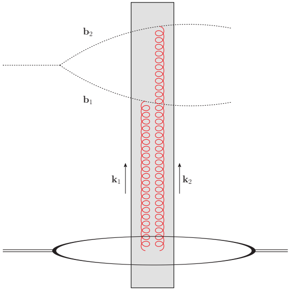
$$\sigma = \mathcal{H}_2^{ij}(\boldsymbol{k}) \otimes f_2^{ij}(x = 0, \boldsymbol{k})$$
$$+ \mathcal{H}_3^{ijk}(\boldsymbol{k}, \boldsymbol{k}_1) \otimes f_3^{ijk}(x = 0, x_1 = 0, \boldsymbol{k}, \boldsymbol{k}_1)$$
$$+ \mathcal{H}_4^{ijkl}(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1') \otimes f_4^{ijkl}(x = 0, x_1 = 0, x_1' = 0, \boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_1')$$

All distributions are evaluated in the strict $x = 0$ limit

Exclusive low x cross section

Exclusive low x amplitude = GTMD amplitude

[Altinoluk, RB]

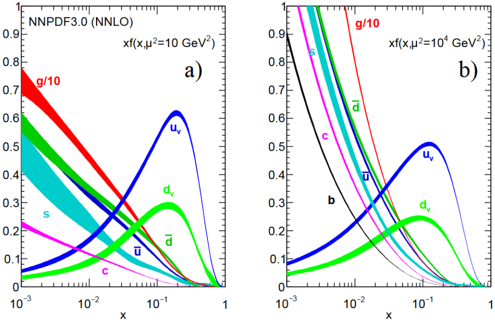


$$\mathcal{H}^{ij}(\boldsymbol{k}_1, \boldsymbol{k}_2) \otimes f^{ij}(x = 0, \xi = 0; \boldsymbol{k}, \Delta)$$

Every exclusive low x process probes a Wigner distribution!

All distributions are evaluated in the strict $x = 0$ limit

All distributions are evaluated in the strict $x = 0$ limit



[NNLO NNPDF3.0 global analysis, taken from PDG2018]

Instabilities in the collinear corner of the phase space

All distributions are evaluated in the strict $x = 0$ limit

Hard part $\mathcal{H}$ and gluon distribution f for an inclusive observable:

Bjorken limit	Leading twist of the CGC
$s \sim Q^2$	$s \gg Q^2, Q^2 \to \infty$
$\int \mathrm{d}x f(x)\mathcal{H}(x)$	$f(0) \int \mathrm{d}x \mathcal{H}(x)$

Strong mismatch beyond LL: the PDF is not a constant in $x \simeq 0$.

Too late to restore a dependence on x via evolution: x is already integrated over

Summary so far

Distributions involved in pQCD observables

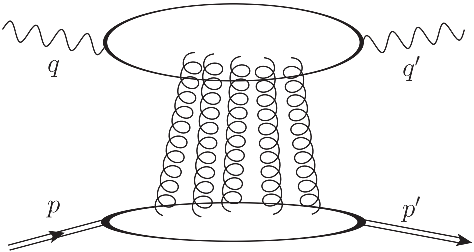
Overarching scheme?

$$f(x_1 \dots x_n; k_{\perp 1} \dots k_{\perp n})$$

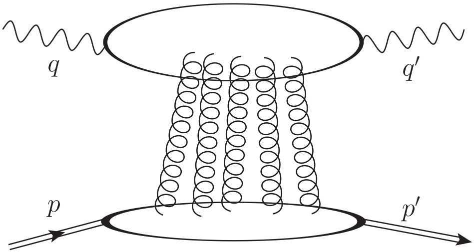
Bjorken limit	Regge limit
$s \sim Q^2$	$s \gg Q^2$
$f(x; 0_\perp) + O(Q^{-2})$	$f(0 \dots 0, k_{\perp 1} \dots k_{\perp n}) + O(x_{\mathrm{Bj}})$

Look for an interpolating scheme for simple observables

An interpolating scheme for the $\gamma^{(*)}(q)P(p) \to \gamma^{(*)}(q')P(p')$ amplitude



Double, Spacelike, and Timelike exclusive Compton Scattering



Longitudinal momentum variables:

$$x, \quad \xi \sim \frac{-q^2 + q'^2}{2q \cdot (p + p')}, \quad x_{\mathrm{Bj}} = \frac{-q^2 - q'^2}{2q \cdot (p + p')}$$

Can we restore the dependence on all 3 variables in a CGC-like scheme?

Bjorken limit

$$s \sim Q^2$$

$$f(x, k_\perp = 0) + O(Q^{-2})$$

Regge limit

$$s \gg Q^2$$

$$f(x = 0, k_\perp) + O(x_{\mathrm{Bj}})$$

Interpolation?

$$s \gtrsim Q^2$$

$$f(x, k_\perp) + O(x_{\mathrm{Bj}} Q^{-2})$$

Basic observation: in both limits, $k^+ \simeq 0$ for t-channel gluons

Factorization in k^+ space is consistent

[Balitsky, Tarasov]

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, x^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, x^-, \boldsymbol{x})$$

Building a semi-classical picture

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Dependence on x^-: sub-sub-leading in twist counting

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

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$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x}) n_2^\mu$$

Necessary gluon fields in the Bjorken limit?

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu + A_\perp^\mu(x^+, 0^-, \boldsymbol{x})$$

Non-zero $A_\perp$: only two A^i contribute to DDVCS

They can be computed using Ward-Takahashi: only necessary for consistency checks, can be dropped.

Building a semi-classical picture

Still factorizing gluons depending on k^+ in $A^+ = 0$ gauge

Necessary gluon fields in the Regge limit:

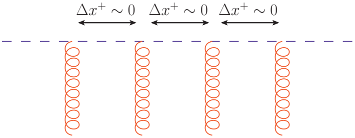
$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})n_2^\mu$$

Necessary gluon fields in the Bjorken limit:

$$A^\mu(x) = A^-(x^+, 0^-, \boldsymbol{x})\, n_2^\mu$$

Effective Feynman rules in the slow background field

Effective fermion propagator in the external classical field



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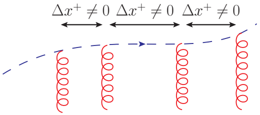

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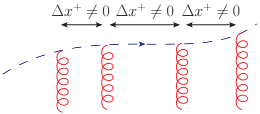
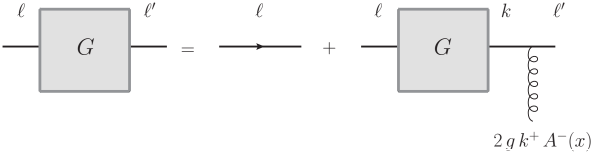


- $A^i_{\mathrm{cl}} = 0, \; A^+_{\mathrm{cl}} = 0$: the Dirac structure factorizes
- A_{cl} does not depend on x^-: conservation of + momentum

$$D_F(\ell', \ell) = i\frac{\gamma^+}{2\ell^+}(2\pi)^D \delta^D(\ell' - \ell) + i\frac{\not{\ell}'\gamma^+\not{\ell}}{2\ell^+} G_{\mathrm{scal}}(\ell', \ell)$$

Effective Feynman rules in the slow background field

Effective scalar propagator in the external classical field



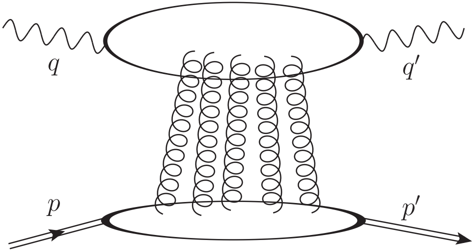
$$G_{\rm scal}(\ell', \ell) - G_0(\ell')(2\pi)^D \delta^D(\ell' - \ell)$$
$$= 2g \int \mathrm{d}^D z \int \frac{\mathrm{d}^D k}{(2\pi)^D} \mathrm{e}^{i(\ell' - k)\cdot z} G_0(\ell') \, (k \cdot A)(z) \, G_{\rm scal}(k, \ell).$$

In coordinate space, it satisfies the Klein-Gordon equation in a potential

$$[-\Box_z + 2igA(z) \cdot \partial_z] \, G_{\rm scal}(z, z_0) = \delta^D(z - z_0)$$

Computation

Application: $\gamma^{(*)} P \to \gamma^{(*)} P$



$$\mathcal{A} = \frac{e^2}{\mu^{d-2}} \varepsilon_q^{\mu} \varepsilon_{q'}^{\nu *} \sum_f q_f^2 \int \frac{\mathrm{d}^D \ell}{(2\pi)^D} \int \frac{\mathrm{d}^D k}{(2\pi)^D}$$
$$\times \langle p' | \mathrm{tr} \left[\gamma_\nu D_F(\ell + k, \ell) \gamma_\mu D_F(-q + \ell, -q' + \ell + k) \right] | p \rangle$$

(First) final result

Fully general result

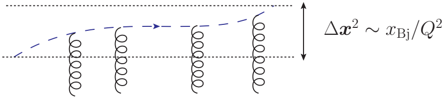
$$\mathcal{A} \propto \mathcal{U}^{ij}(z, q, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2) \otimes_{z,\boldsymbol{\ell}_1,\boldsymbol{\ell}_2} (\partial^i \Phi)(z, \boldsymbol{\ell}_1)(\partial^j \Phi^*)(z, \boldsymbol{\ell}_2)$$

- Φ: standard wave functions
- $\mathcal{U}^{ij}$: generalization of the dipole operator

Contains unnecessary subleading powers of x_{Bj}, ξ and Q, Q'

Further simplifcations

Partial twist expansion



Typical transverse recoil of a fast parton:
$\Delta x^2 \sim 1/(2q^+P^-) \sim \xi/(Q^2 + Q'^2)$

ξ-suppressed in the Regge limit

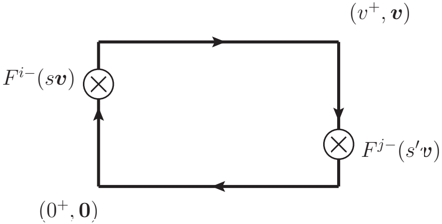
$1/(Q^2 + Q'^2)$-suppressed in in the Bjorken limit.

We can get rid of all corrections from transverse recoils without loss of accuracy

Further simplifications

Partial twist expansion

$$\frac{\langle p'|\mathcal{U}^{ij}(z, q, \boldsymbol{\ell}_1, \boldsymbol{\ell}_2)|p\rangle}{\langle p|p\rangle} \simeq -i\frac{(2\pi)^d}{8z\bar{z}(q^+)^2}\int \mathrm{d}x \frac{\mathcal{G}^{ij}(x, \boldsymbol{\ell}_2 - \boldsymbol{\ell}_1)}{x - \frac{\left(\frac{\boldsymbol{q}+\boldsymbol{q}'}{2}\right)^2}{2q^+P^-} - \frac{\left(\frac{\boldsymbol{\ell}_1+\boldsymbol{\ell}_2}{2}\right)^2}{2z\bar{z}q^+P^-} + i0},$$



(x, ξ)-dependent unintegrated GPD

$$\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta) \equiv \frac{1}{P^-}\int \frac{\mathrm{d}v^+}{2\pi}\mathrm{e}^{ixP^-v^+}\int \frac{\mathrm{d}^d\boldsymbol{v}}{(2\pi)^d}\mathrm{e}^{-i(\boldsymbol{k}\cdot\boldsymbol{v})}\int_0^1 \mathrm{d}s\mathrm{d}s'$$
$$\times \left\langle p'\middle|\mathrm{tr}_c\left\{\left[v^+, 0^+\right]_{\boldsymbol{0}} F^{i-}\left(0^+, s\boldsymbol{v}\right)\left[0^+, v^+\right]_{\boldsymbol{v}} F^{j-}\left(v^+, s'\boldsymbol{v}\right)\right\}\middle|p\right\rangle$$

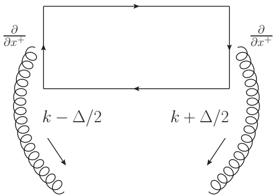
The unintegrated GPD

uGPD as a finite Wilson loop

$$\int \mathrm{d}^2\boldsymbol{k}\,\mathrm{e}^{i(\boldsymbol{k}\cdot\boldsymbol{r})}\boldsymbol{r}^i\boldsymbol{r}^j\mathcal{G}^{ij}(x,\xi,\boldsymbol{k},\Delta)$$

$$= \frac{1}{\alpha_s}\int\frac{\mathrm{d}^4v_1\mathrm{d}^4v_2}{(2\pi)^4}\delta(v_1^-)\delta(v_2^-)\mathrm{e}^{-i(k-\frac{\Delta}{2})\cdot v_1+i(k+\frac{\Delta}{2})\cdot v_2}$$

$$\times\frac{\partial}{\partial v_1^+}\frac{\partial}{\partial v_2^+}\frac{\langle p'|\mathrm{tr}[v_1^+,v_2^+]_{v_1}[\boldsymbol{v}_1,\boldsymbol{v}_2]_{v_2^+}[v_2^+,v_1^+]_{v_2}[\boldsymbol{v}_2,\boldsymbol{v}_1]_{v_1^+}|p\rangle}{\langle p|p\rangle}$$



(x,ξ)-dependent unintegrated GPD $\Leftrightarrow$ FT of a finite Wilson loop

(Actual) final result

Final expression for the amplitude

$$\mathcal{A} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \int \mathrm{d}^d \boldsymbol{k}$$
$$\times (\partial^i \Phi)(z, \boldsymbol{\ell} - \boldsymbol{k}/2)(\partial^j \Phi^*)(z, \boldsymbol{\ell} + \boldsymbol{k}/2)$$
$$\times \int \mathrm{d}x \, \frac{\mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta)}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Standard wave functions Φ

(x, ξ)-dependent unintegrated GPD $G^{ij}(x, \xi, \boldsymbol{k}, \Delta)$

Bjorken limit, Regge limit and their non-commutativity

The Bjorken limit

Recovering the Bjorken limit

The Bjorken limit is reached by neglecting transverse momentum transfert from the target:

$$|\boldsymbol{\ell}| \sim Q, Q' \gg |\boldsymbol{k}|$$

Key observation: $\mathcal{G}^{ij}$ integrates into GPDs

$$\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta) = G^{ij}(x, \xi, \Delta)$$

We fully recover the well-known one-loop exclusive Compton scattering amplitudes

The Regge limit

Recovering the Regge limit? What is x?

Naive argument

- In the Regge limit, the amplitude is dominated by its imaginary part
- Leading order amplitude:

$$\mathrm{Im}\mathcal{A}_{LO} \propto \mathrm{Im} \int \mathrm{d}x H^q(x,\xi,t) \frac{1}{x - x_{\mathrm{Bj}} + i\epsilon} = -\pi H^q(x_{\mathrm{Bj}},\xi,t)$$

- Hence take $x = x_{\mathrm{Bj}}$

Problems

- At NLO, the x cut is way more complicated
- For DDVCS and for TCS, s-channel cuts also contribute to the imaginary part

The Bjorken limit

Recovering the Regge limit

The Regge limit is reached by neglecting $x_{\rm Bj}$ and setting $\frac{\ell^2}{z\bar{z}} \ll q \cdot P$, then taking the x cut:

$$\frac{1}{x - x_{\rm Bj} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0} \to \frac{1}{x + i0} \to -i\pi\delta(x),$$

then taking $x_{\rm Bj}, \xi \ll 1$.
Rq: the $x = 0$ limit of the uGPD matches the dipole operator.

We recover the small x description of exclusive Compton scattering see e.g. [Hatta, Xiao, Yuan]

Rq: x is strictly 0 in the uGPD

Summary

Interpolating scheme for exclusive Compton scattering

Overarching scheme

$$\int \mathrm{d}x \int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta) H^{ij}(x, \xi, \boldsymbol{k}, \Delta)$$

Bjorken limit

$$\int \mathrm{d}x H^{ij}(x, \xi, 0, \Delta) \times \left[\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(x, \xi, \boldsymbol{k}, \Delta)\right]$$

Regge limit

$$\int \mathrm{d}^d \boldsymbol{k} \mathcal{G}^{ij}(0, \xi, \boldsymbol{k}, \Delta) \times \left[\int \mathrm{d}x H^{ij}(x, \xi, \boldsymbol{k}, \Delta)\right]$$

We found an interpolating scheme

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\to\infty} \mathcal{A}_{\mathrm{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i \Phi)(z,\boldsymbol{\ell})(\partial^j \Phi^*)(z,\boldsymbol{\ell})$$

Eikonal limit of the Bjorken limit

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \mathcal{A}_{\mathrm{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d \boldsymbol{\ell}}{(2\pi)^d} \times \lim_{x_{\mathrm{Bj}},\xi\to 0} \int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i \Phi)(z,\boldsymbol{\ell})(\partial^j \Phi^*)(z,\boldsymbol{\ell})}{x - x_{\mathrm{Bj}} - \frac{\boldsymbol{\ell}^2}{2z\bar{z}q^+P^-} + i0}$$

Double limit

Do the two limits commute?

If $G^{ij}(x,\xi,t)$ is a constant at $x = 0$:

$$\int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$\simeq G^{ij}(0,\xi,t) \int \mathrm{d}x \, \frac{(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$= G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)$$

$$\times \ln\left(\frac{1 - x_{\mathrm{Bj}} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0}{-1 - x_{\mathrm{Bj}} - \frac{\ell^2}{z\bar{z}\frac{Q^2+Q'^2}{2}}\xi + i0}\right)$$

and thus

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \int \mathrm{d}x \, \frac{G^{ij}(x,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)}{x - x_{\mathrm{Bj}} - \frac{\ell^2}{2z\bar{z}q^+P^-} + i0}$$

$$\simeq -i\pi G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\ell)(\partial^j\Phi^*)(z,\ell)$$

Double limit

Do the two limits commute?

Leading twist limit of the Regge limit

$$\lim_{Q^2+Q'^2\to\infty} \mathcal{A}_{\mathrm{Regge}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\boldsymbol{\ell})(\partial^j\Phi^*)(z,\boldsymbol{\ell})$$

Eikonal limit of the Bjorken limit **provided the GPDs are constant at $x = 0$**

$$\lim_{x_{\mathrm{Bj}},\xi\to 0} \mathcal{A}_{\mathrm{Bjorken}} = g^2 \sum_f q_f^2 \int_0^1 \frac{\mathrm{d}z}{2\pi} \int \frac{\mathrm{d}^d\boldsymbol{\ell}}{(2\pi)^d} \times (-i\pi) G^{ij}(0,\xi,t)(\partial^i\Phi)(z,\boldsymbol{\ell})(\partial^j\Phi^*)(z,\boldsymbol{\ell})$$

Checked with explicit final expressions for both double limits

Conclusion

Where do we stand?

Bad news

- Semi-classical small x physics has, at its core, issues with collinear logarithms
- The problem can be traced down to the very starting point

Good news

- We now have a minimal correction of semi-classical small x which solves the problem from first principles
- Wave functions, and thus hard parts, are not modified by the scheme
- All we need is the right evolution equation...