# Inverting the Radon Transform using Artificial Neural Networks

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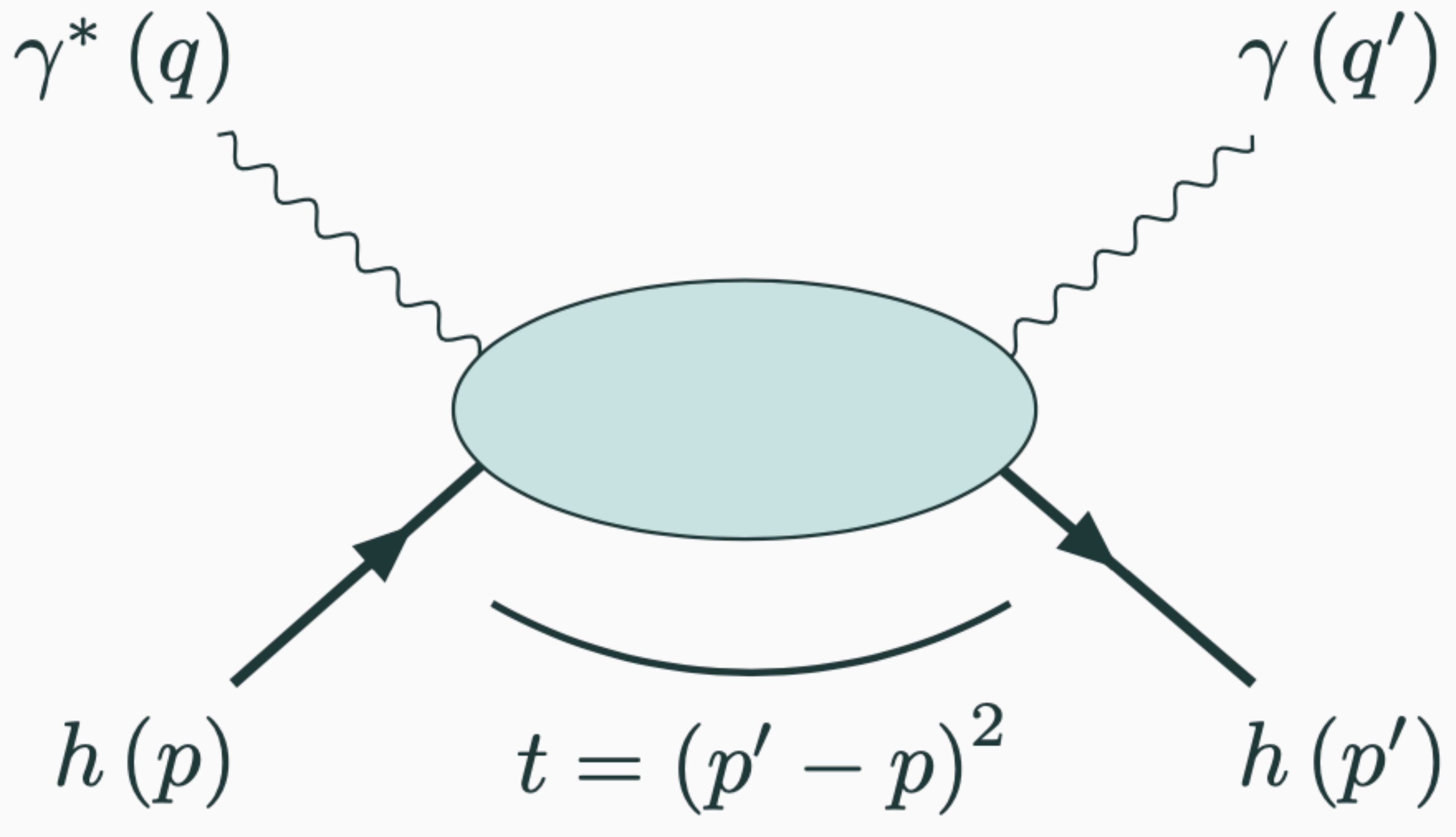
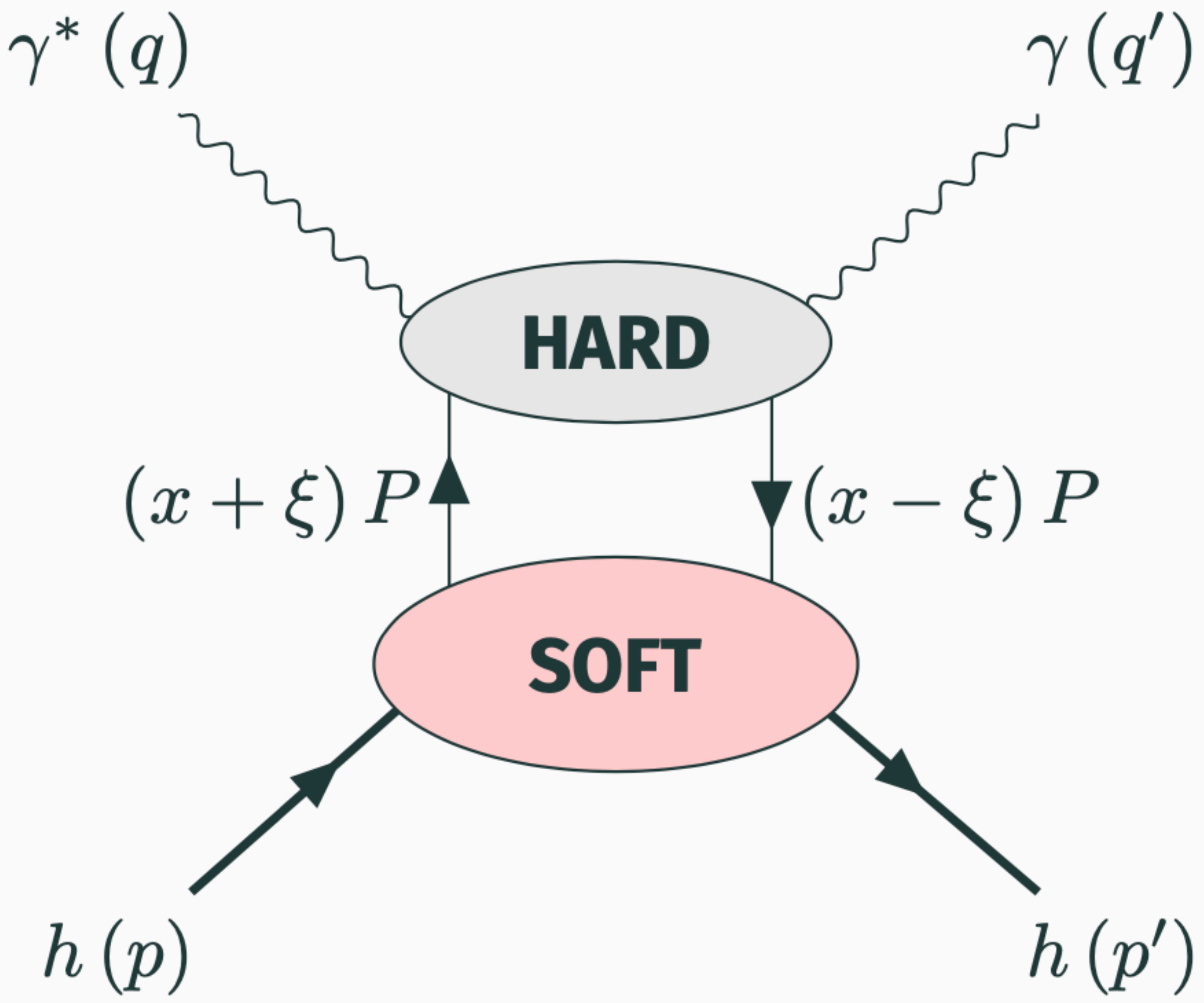
3D Partons Workshop 26-28 Oct 2022

# Generalized Parton Distribution




$Q^2 \to \infty$

$$Q = \frac{q + q'}{2}$$
$$P = \frac{p + p'}{2}$$
$$\xi = \frac{(p' - p)^+}{2P^+} = -\frac{\Delta^+}{2P^+}$$




$$\mathcal{M}(\xi, t; Q^2) = \sum_{p=q,g} \int_{-1}^{1} \frac{dx}{\xi} K^p\left(\frac{x}{\xi}, \frac{Q^2}{\mu_F^2}, \alpha_s(\mu_F)\right) H^p\left(x, \xi, t; \mu_F\right)$$

hard / perturbative

soft / non perturbative GPD

# Pion GPD

Twist 2 chiral even quark GPD of spineless hadron

$$H^q(x,\xi,t) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \left\langle \pi\left(P+\frac{\Delta}{2}\right)\middle| \bar{\psi}^q\left(-\frac{z}{2}\right)\gamma^+\psi^q\left(\frac{z}{2}\right)\middle|\pi\left(P-\frac{\Delta}{2}\right)\right\rangle\Bigg|_{z^+=0,\ z^\perp=0}$$

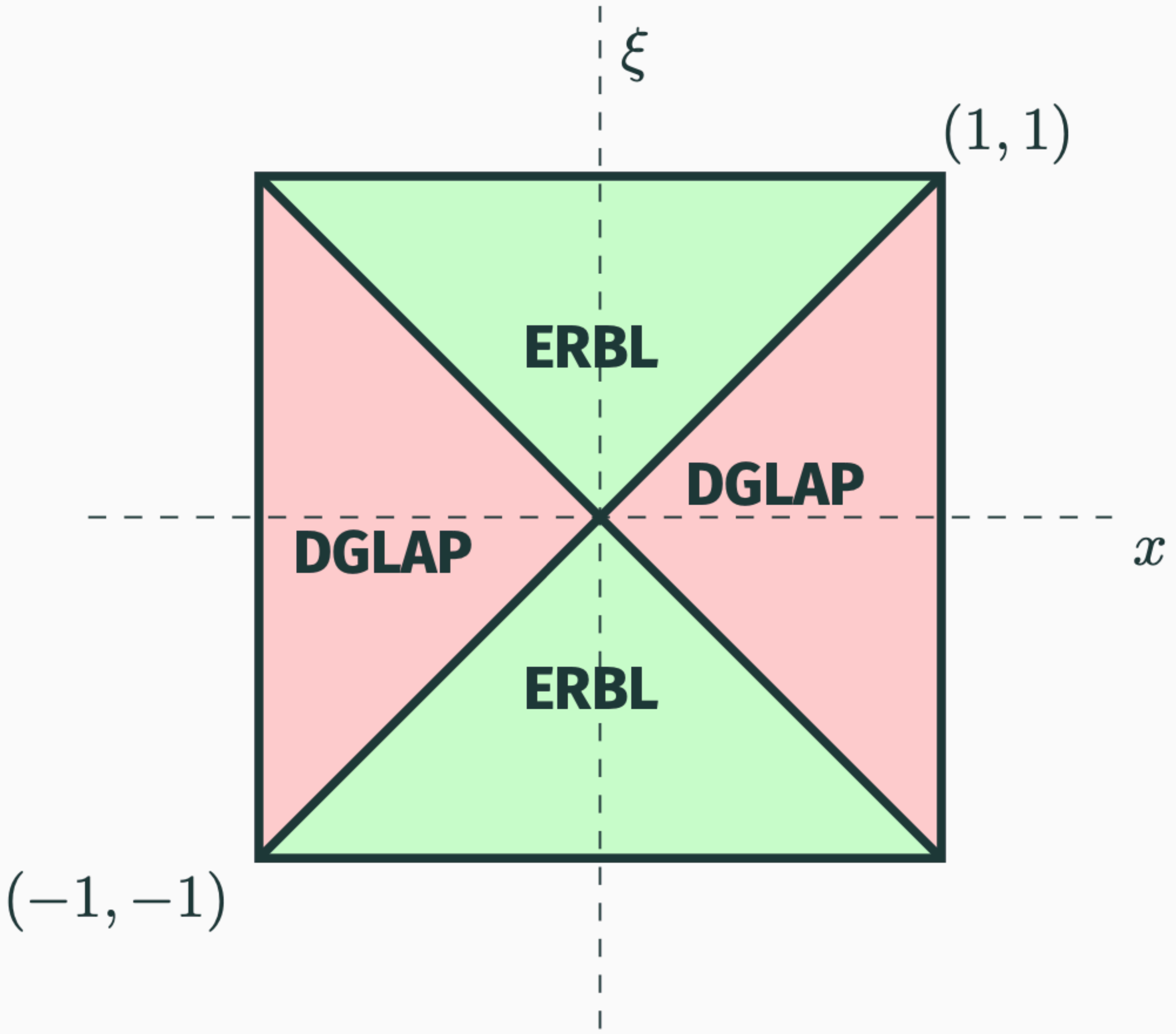
Support: $(x,\xi) \in [-1,1] \otimes [-1,1]$

DGLAP $|x| > |\xi|$

Emission/absorption quark $(x > 0)$ or antiquark $(x < 0)$

ERBL $|x| < |\xi|$

Emission of quark/antiquark pair



# GPD properties

- $\xi-$parity

$$H^q(x,-\xi,t)=H^q(x,\xi,t)$$

Time inversion symmetry

- Polynomiality

$$\mathcal{A}_m(\xi,t)=\int_{-1}^{1}dx\,x^m\,H^q(x,\xi,t)=\sum_{\substack{k=0\\ k\,\text{even}}}^{m+1}C_{k,m}(t)\,\xi^k$$

Lorentz symmetry

- Positivity

$$\left|H^q(x,\xi,t=0)\right|\le\sqrt{q\left(\frac{x+\xi}{1+\xi}\right)q\left(\frac{x-\xi}{1-\xi}\right)},\quad |x|>|\xi|$$

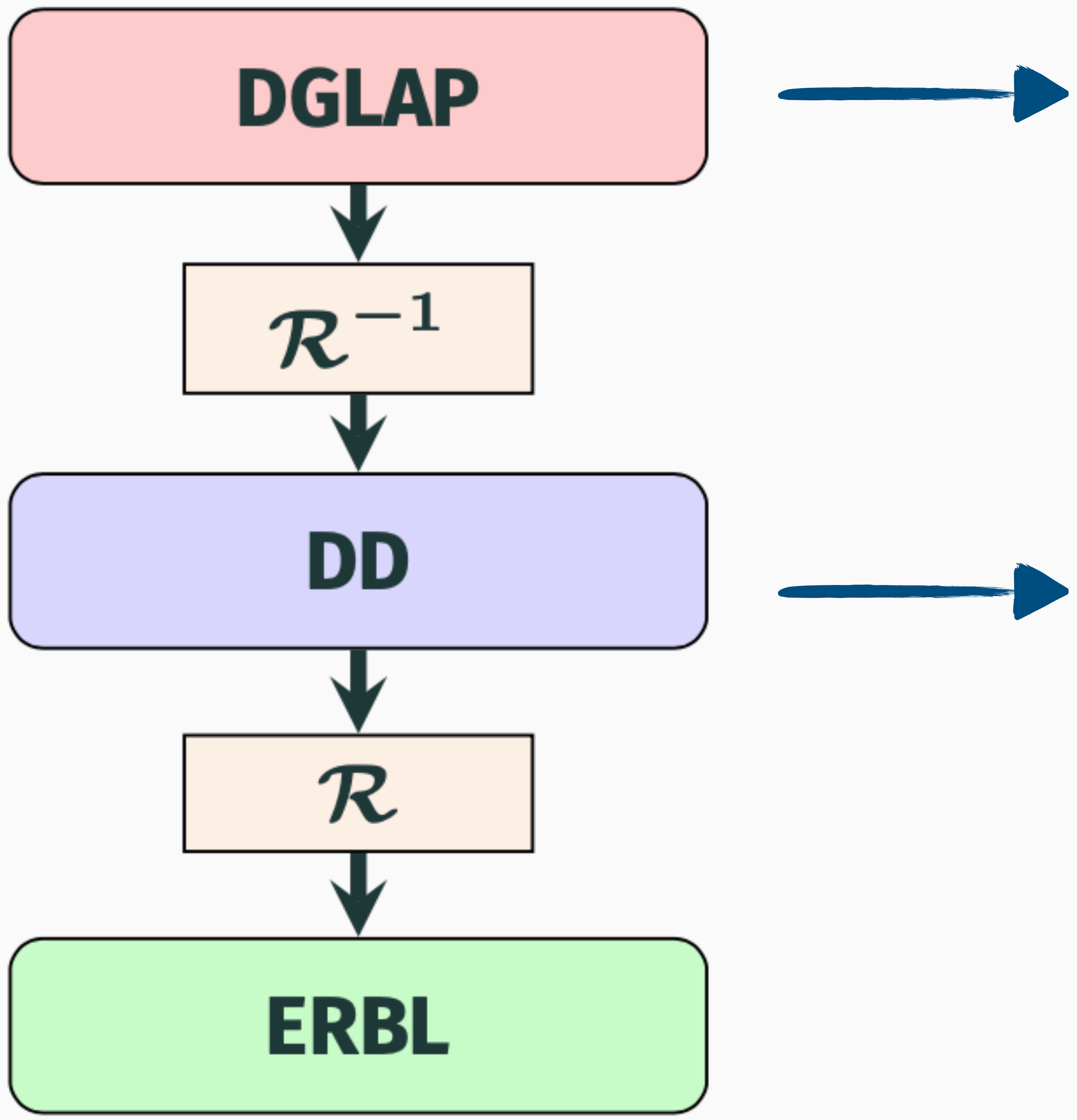
Hilbert space norm

PDF $\quad H^q(x,\xi=0,t=0)=\theta(x)q(x)-\theta(-x)\bar{q}(-x)$

# GPD modeling via Covariant Extension

N.Chouika et al.-Eur.Phys.J.C:77(2017)

Strategy




Overlap of LCWF

$$H^q(x,\xi,t)\Big|_{|x|>|\xi|} = \sum_{N,\beta} \sqrt{1-\xi^2}^{\,2-N} \int [d\bar{x}]_N [d^2\bar{\mathbf{k}}_\perp]_N \, \delta(x-\bar{x}_i)\, \psi^*_{N,\beta}(x_i^{out}, \mathbf{k}_{i\perp}^{out})\, \psi_{N,\beta}(x_i^{in}, \mathbf{k}_{i\perp}^{in})$$

Positivity ← truncation — states of same N

$$H^q(x,\xi) = \mathcal{R}[h] = \int_\Omega d\beta d\alpha\, \delta\left(x-\beta-\alpha\xi\right) h(\beta,\alpha) \quad + \quad \text{D-terms}$$

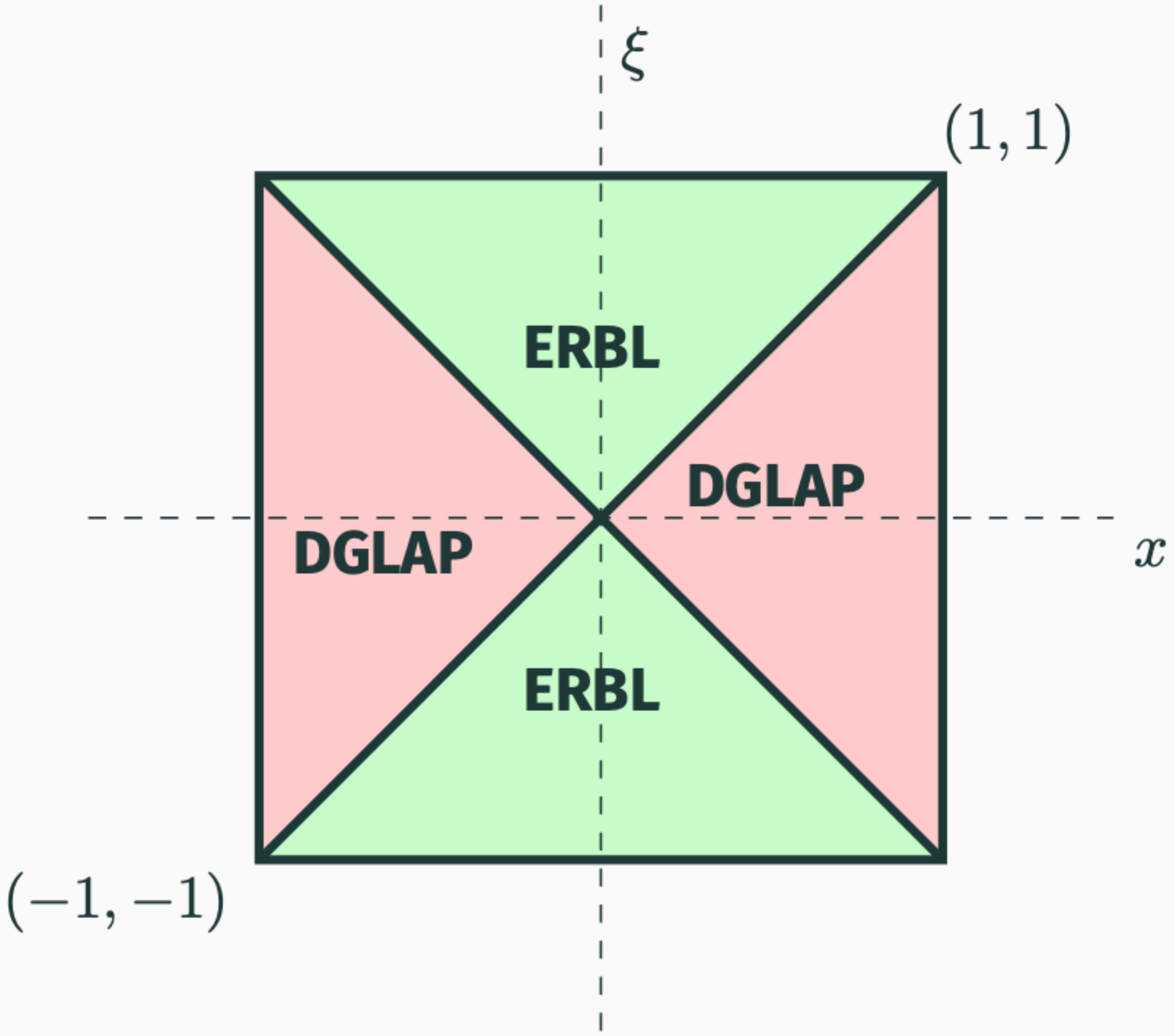
$$\mathcal{A}_m(\xi,t) = \sum_{k=0}^{m} \int_\Omega d\beta d\alpha \binom{m}{k} \alpha^k \beta^{m-k}\, h(\beta,\alpha)\, \xi^k$$

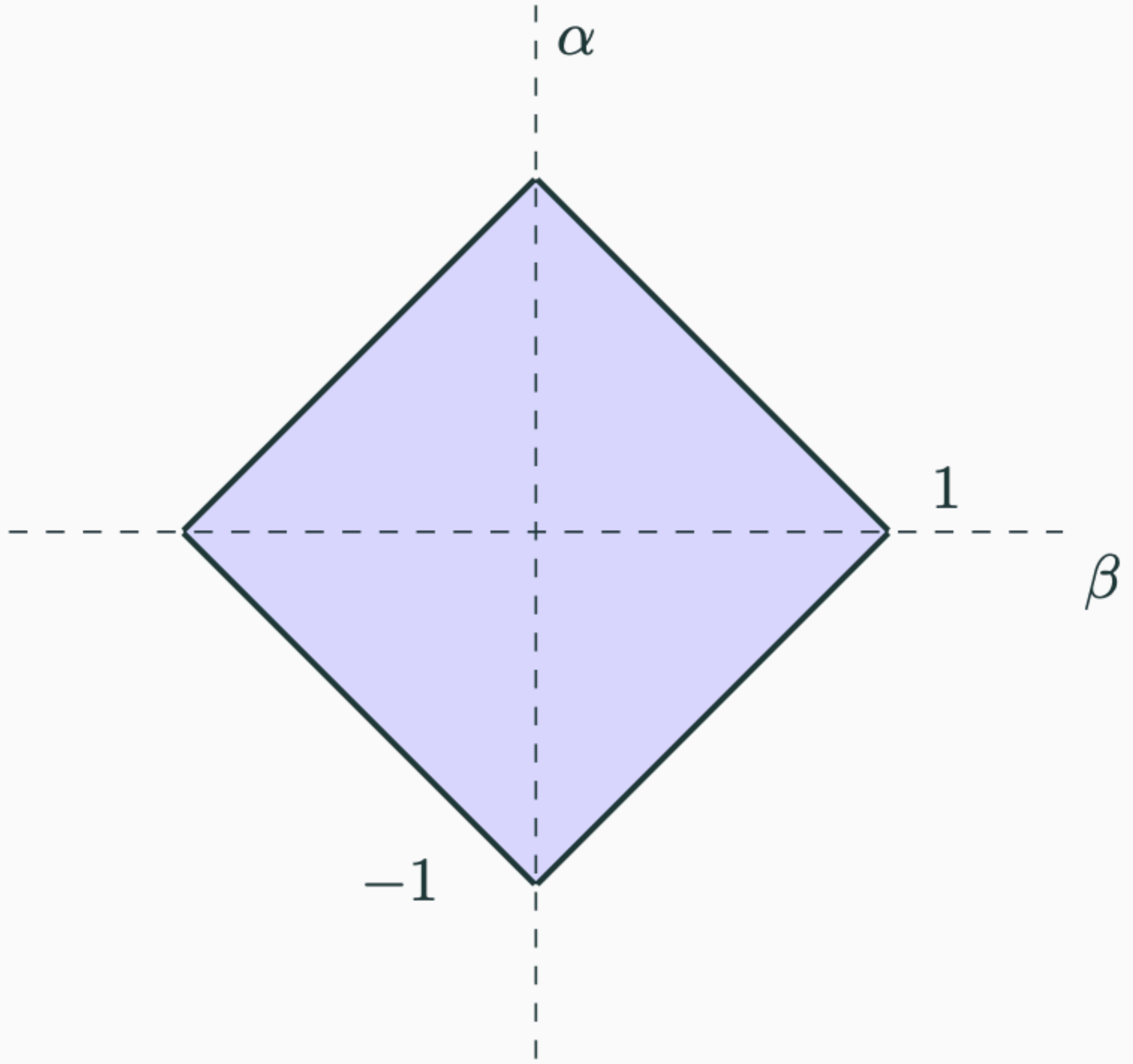
Polynomiality

# Inverse Radon Transform

$$H^q(x,\xi) = \mathscr{R}[h] = \int_\Omega d\beta d\alpha\, \delta\left(x - \beta - \alpha\xi\right) h(\beta,\alpha)$$

Support: $\Omega = \{(\beta,\alpha)\,|\,|\beta| + |\alpha| \leq 1\}$




line $\alpha = \frac{1}{\xi}(-\beta + x)$

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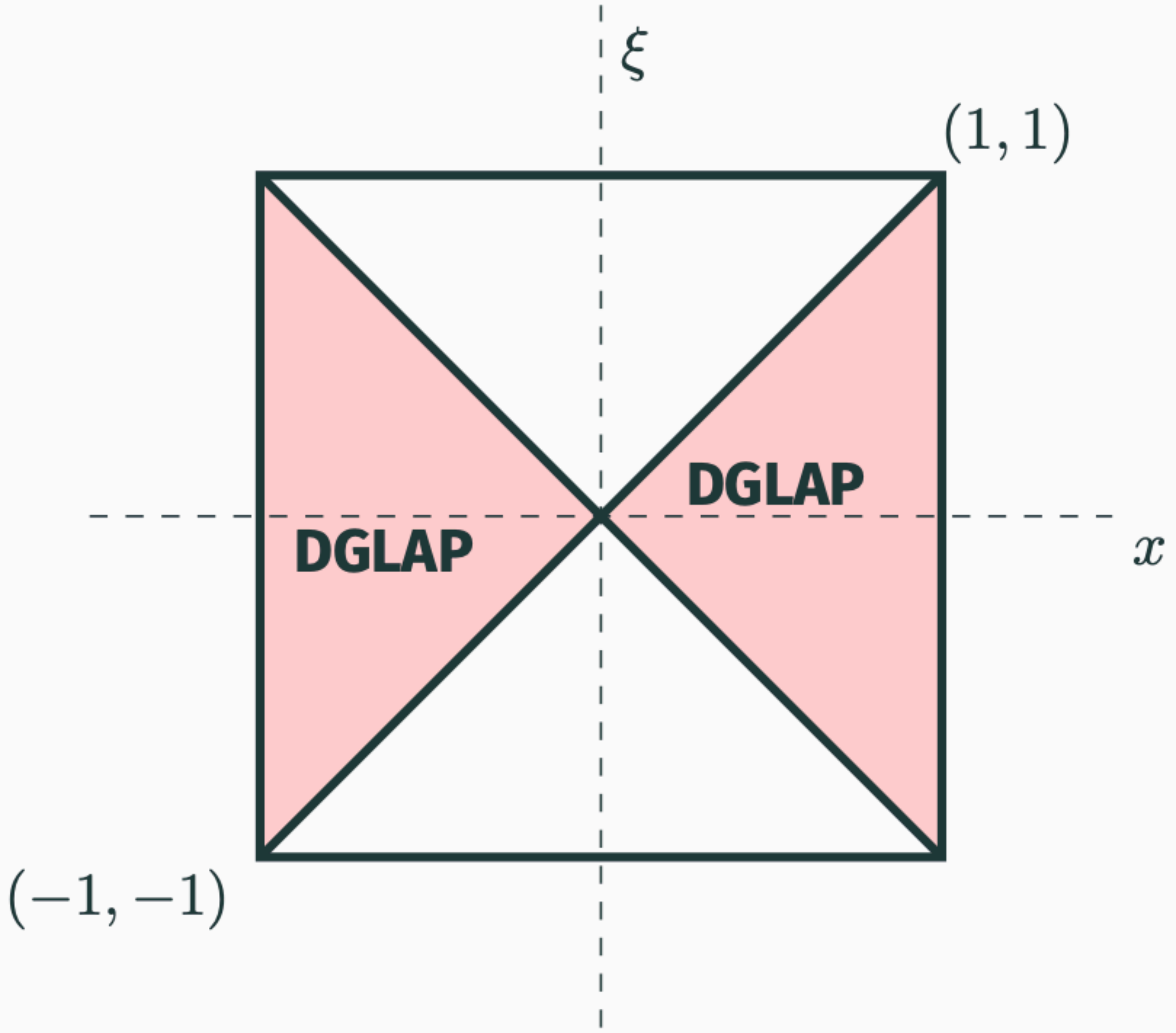


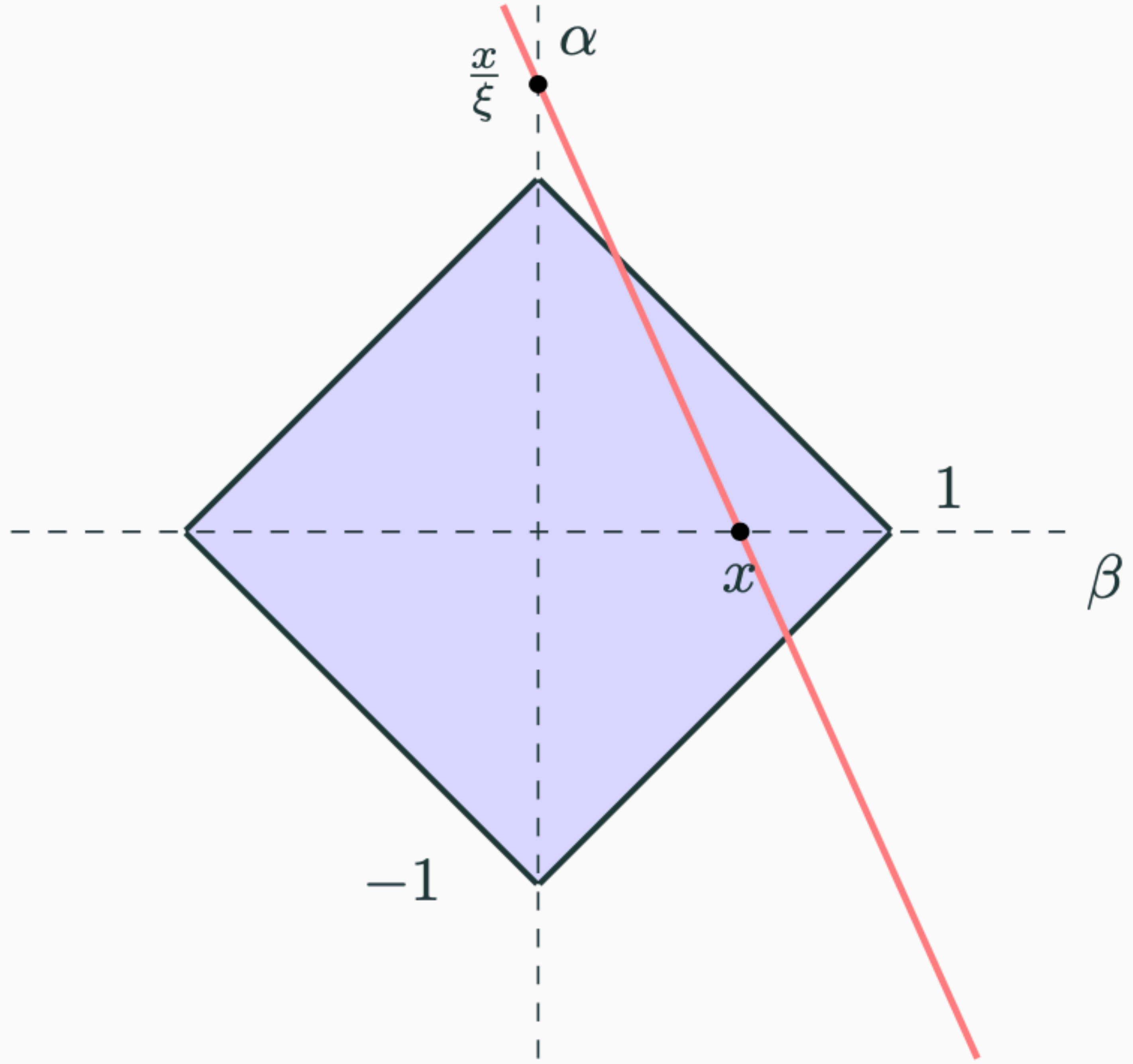


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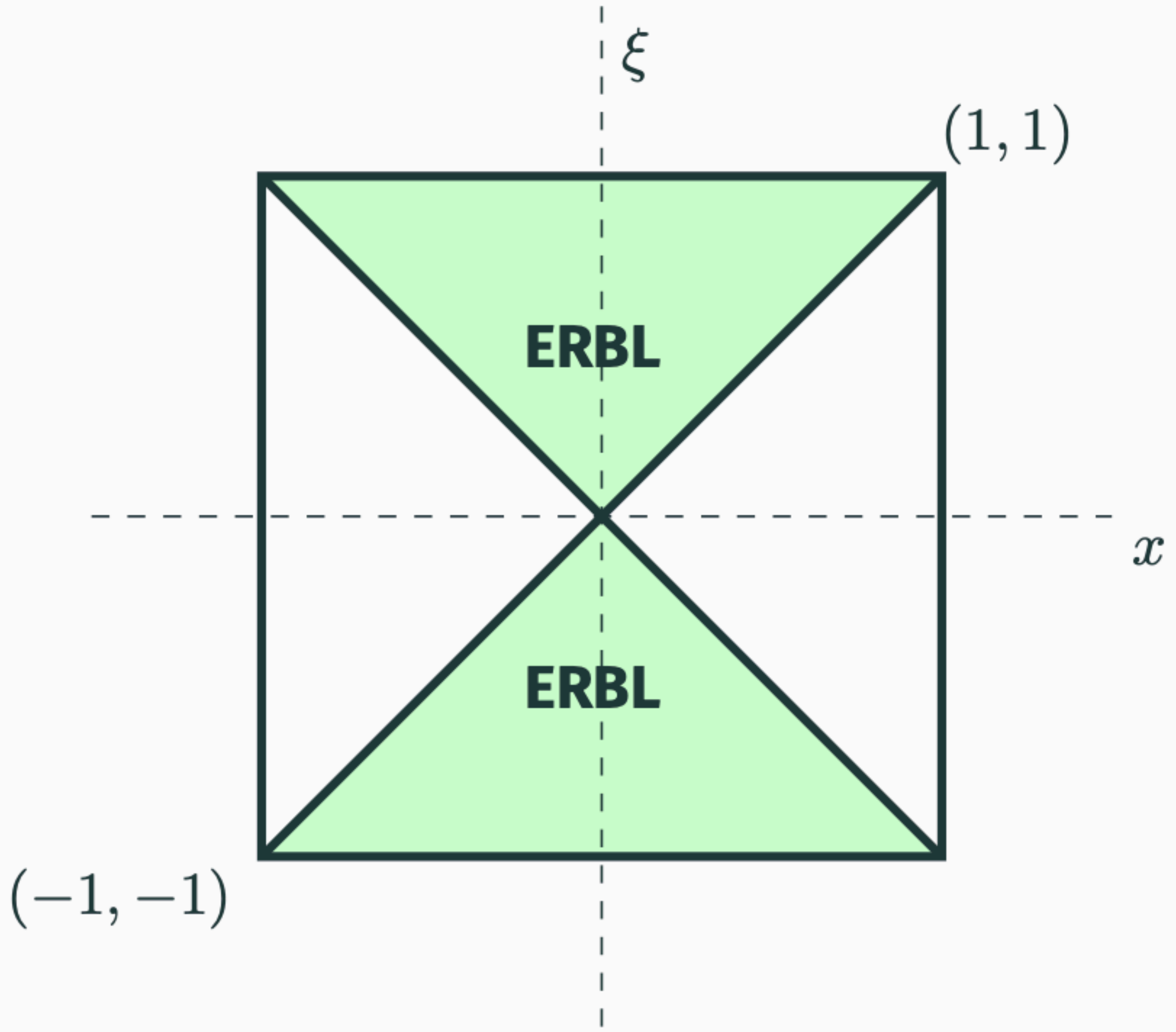


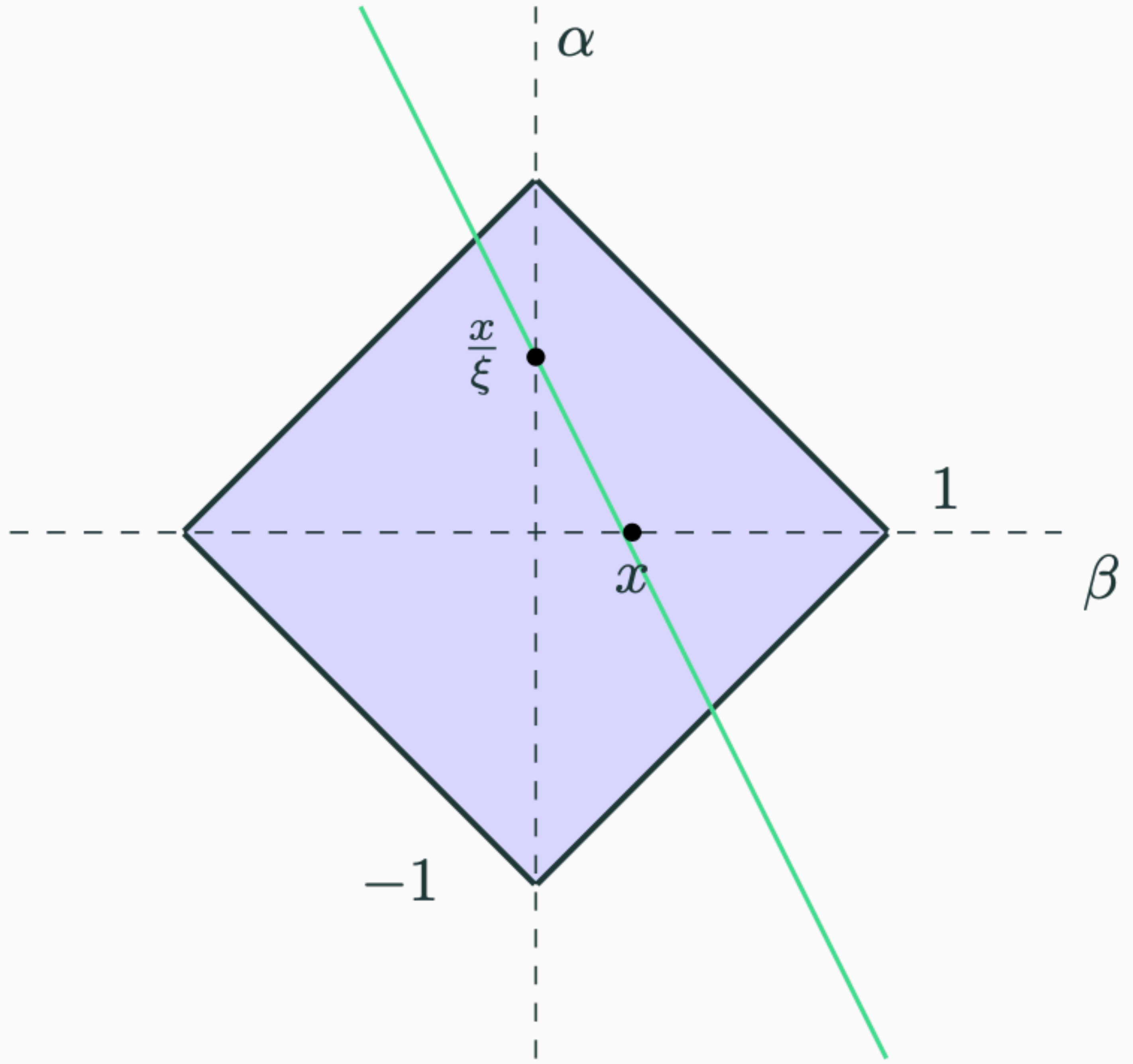


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# Finite Elements Methods

J.M. Morgado et al.-Phys.Rev.D 105(2022)

Reduce the continuum Radon transform in a discrete Matrix equation to be inverted

$$H(x,\xi) = \mathscr{R}[h] \longrightarrow \quad H(x_i,\xi_i) \equiv H_i = \mathrm{R}_{ij}\, h_j$$

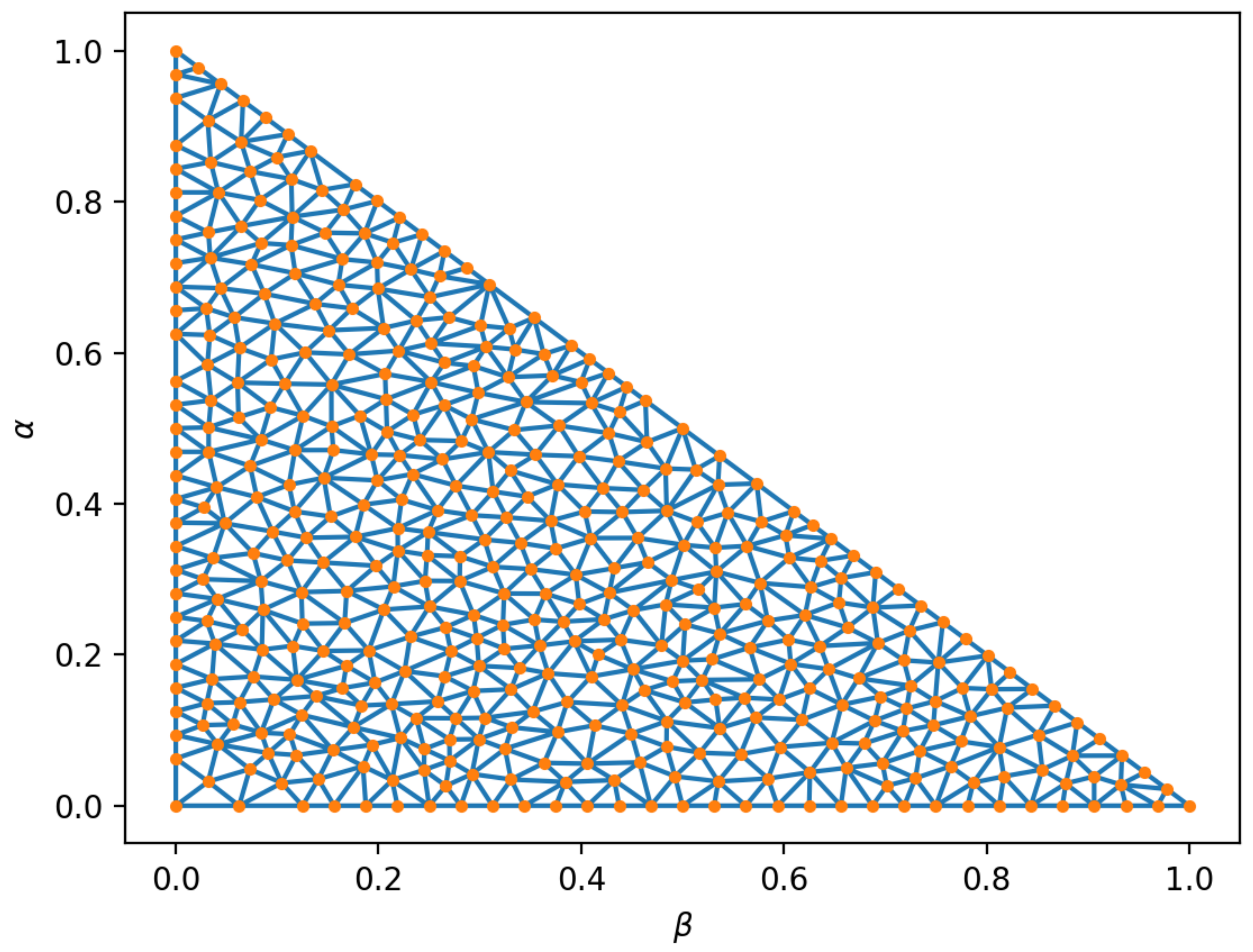
- Triangulation (Delaunay) of reduced domain $\Omega^+ = \Omega \cap \{\alpha, \beta \geq 0\}$
  - $\beta \geq 0$ quark sector (decouples from antiquark in DGLAP). $H(x,\xi)\Big|_{|x|\geq|\xi|} = H^{>}(x,\xi)\Big|_{x\geq\xi} + H^{<}(x,\xi)\Big|_{x\leq-\xi}$
  - $\alpha \geq 0$ because $H(x,-\xi) = H(x,\xi) \Rightarrow \quad h(\beta,-\alpha) = h(\beta,\alpha)$

- Interpolation $h(\beta,\alpha) \simeq \sum_j h_j\, v_j(\beta,\alpha)$

DD values at nodes ($h_j$)

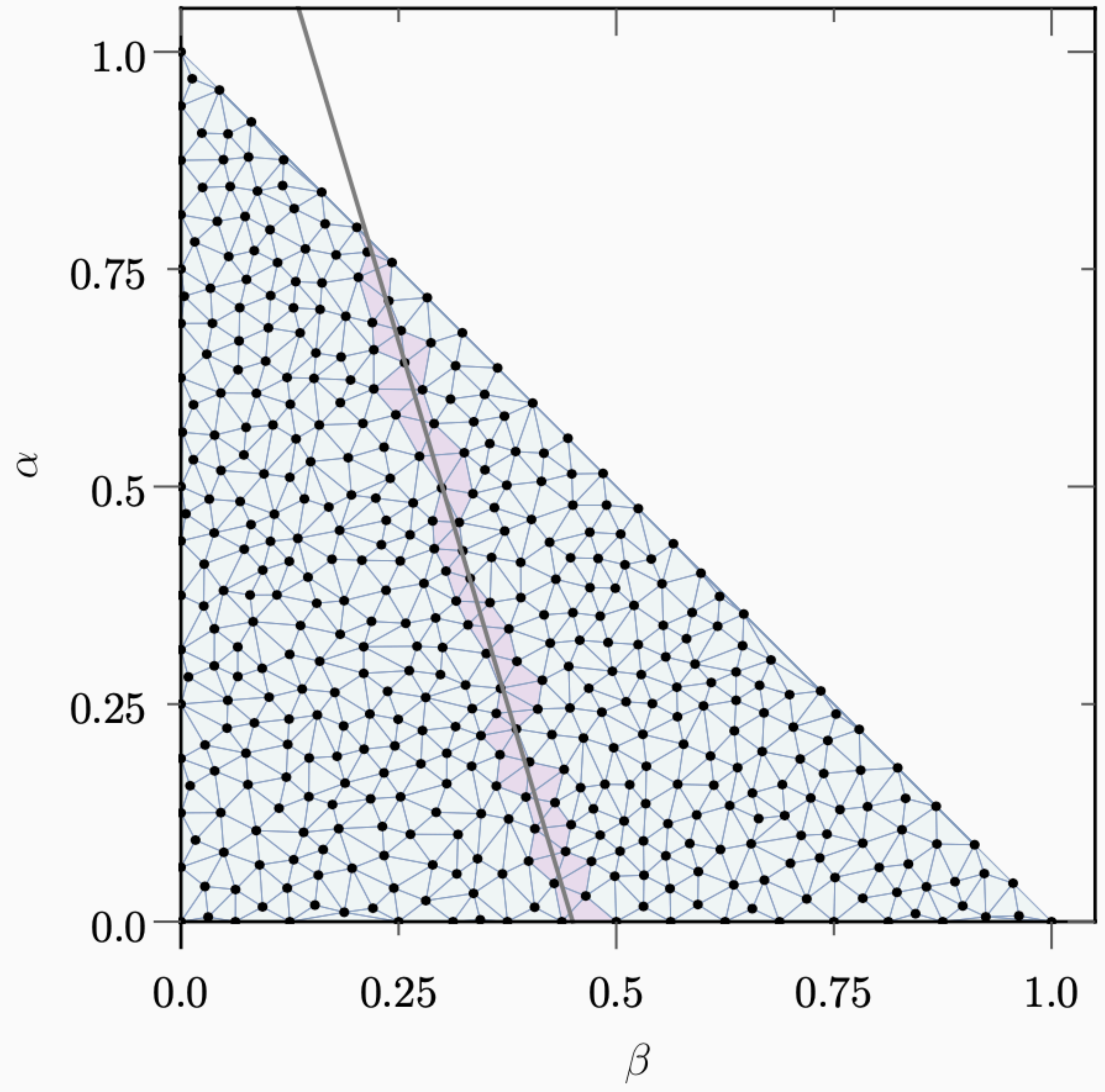
Lagrange P1 Polynomials ($v_j$)

- $v_j(\beta_j,\alpha_j) = 1$
- $v_j(\beta_{i\neq j},\alpha_{i\neq j}) = 0$
- Domain restricted to elements adjacent to node



- Matrix building by sampling in DGLAP region

$$H_i \equiv H(x_i, \xi_i) = \sum_j h_j \int_{\Omega^+} d\beta d\alpha \underbrace{\delta(x_i - \beta - \alpha\xi_i)\, v_j(\beta, \alpha)}_{\downarrow\ R_{ij}}$$




- Inverse Radon Transform

$$H = Rh \longrightarrow R^T H = R^T R\, h$$

Invertible if $R$ has max rank

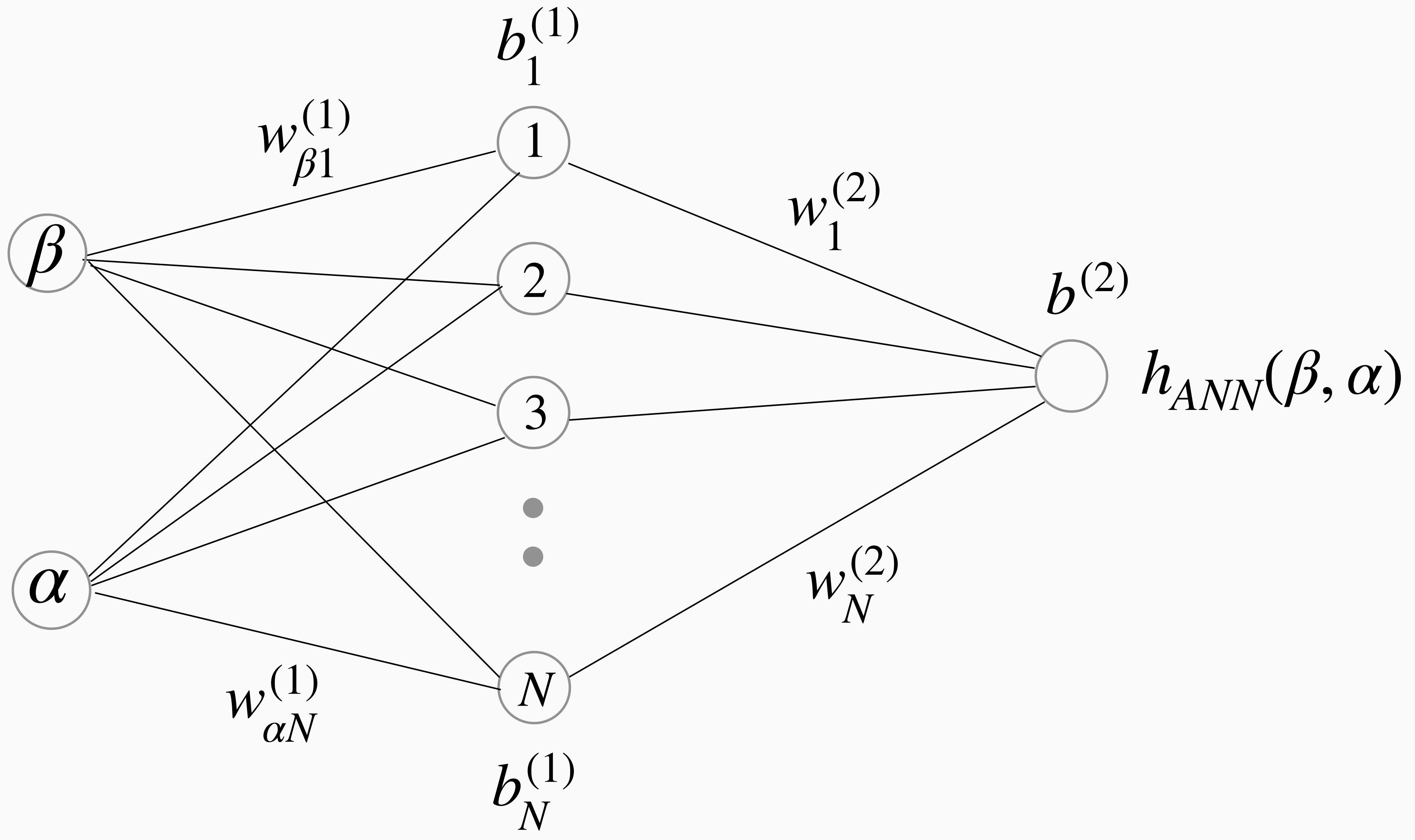
$$\rightarrow \boxed{h = \left(R^T R\right)^{-1} R^T H}$$

C++ algorithm by J.M. Morgado in NumA library of PARTONS

# Artificial Neural Networks

Universal approximation theorem: continuous function on compact approximated by ANN with one hidden layer

ANN used to approximate the DD $h(\beta, \alpha)$




Output of neuron $i$ at layer $j$ $\quad o_i^{(j)} = \varphi^{(j)}\left(\sum_k w_{ki}^{(j)} o_k^{(j-1)} + b_i^{(j)}\right)$

$$\varphi^{(1)}(x) = \frac{1}{1+e^{-x}}, \quad \varphi^{(2)}(x) = x$$

$$\longrightarrow \quad h_{ANN}(\beta, \alpha) = \sum_{i=1}^{N} \left\{ w_i^{(2)} \varphi\left(w_{\beta i}^{(1)}\beta + w_{\alpha i}^{(1)}\alpha + b_i^{(1)}\right) + b^{(2)} \right\}$$

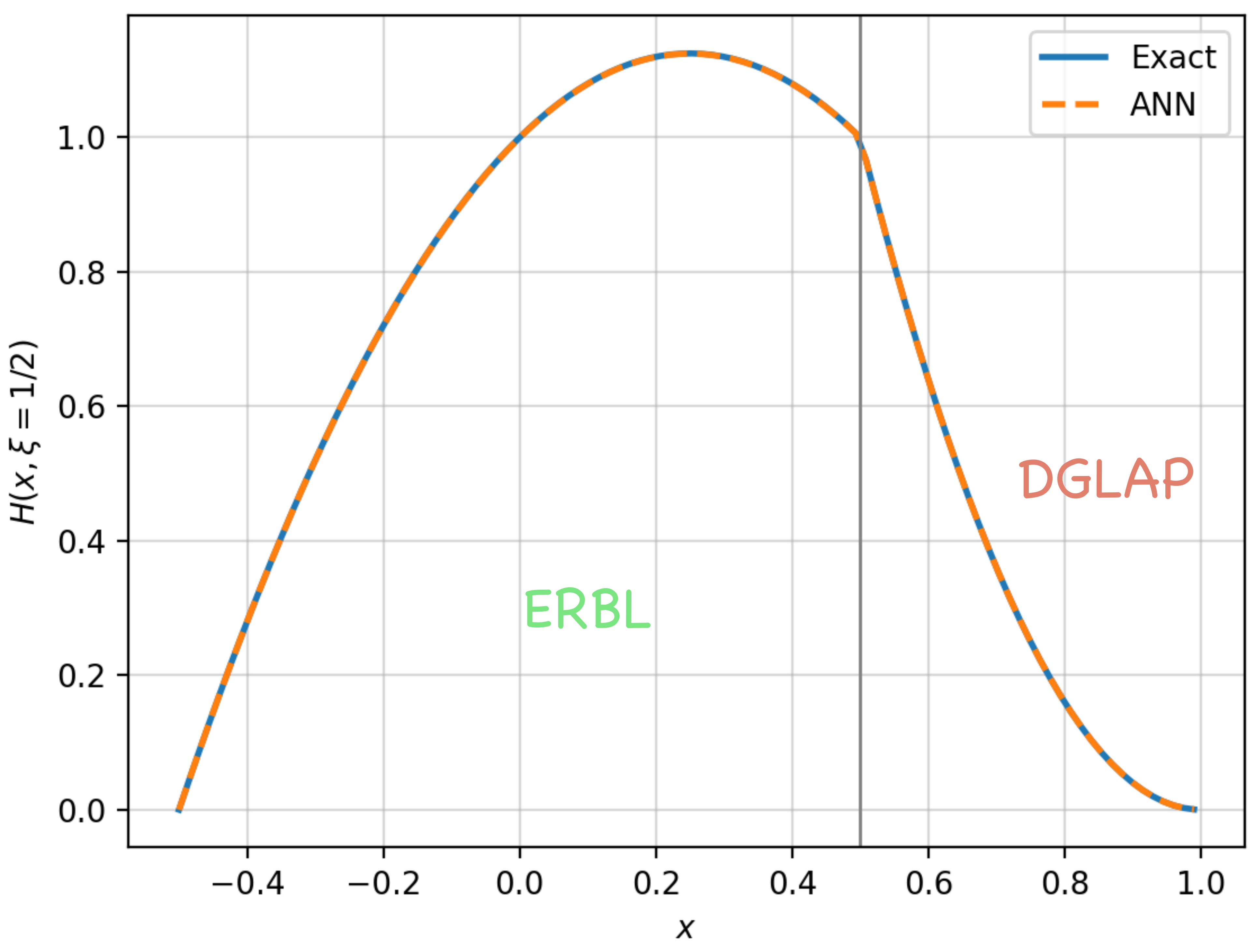
4N+1 parameters optimized by minimizing error of GPD data in DGLAP

RRMSE: $\sqrt{\dfrac{\sum_i^{N_{sample}} \left(H^{pred}(x_i, \xi_i) - H(x_i, \xi_i)\right)^2}{\sum_i^{N_{sample}} H^2(x_i, \xi_i)}}$

## Trivial test: $h(\beta, \alpha) = const = C$

$$H(x,\xi) = (1-x)\int_{\Omega^+} d\beta d\alpha \delta(x-\beta-\alpha\xi)\, h(\beta,\alpha) = \begin{cases} 2\,C\dfrac{(1-x)^2}{1-\xi^2}, & |x| > \xi \\ C\dfrac{(1-x)(x+\xi)}{\xi(1+\xi)}, & |x| < \xi \end{cases}$$

↓ *Pobylitsa* scheme




$N = 1\,, \quad N_{sample} = 10^4$

ANN implemented with Tensorflow

Optimization algorithm: Adam

RRMSE $\sim 10^{-4}$

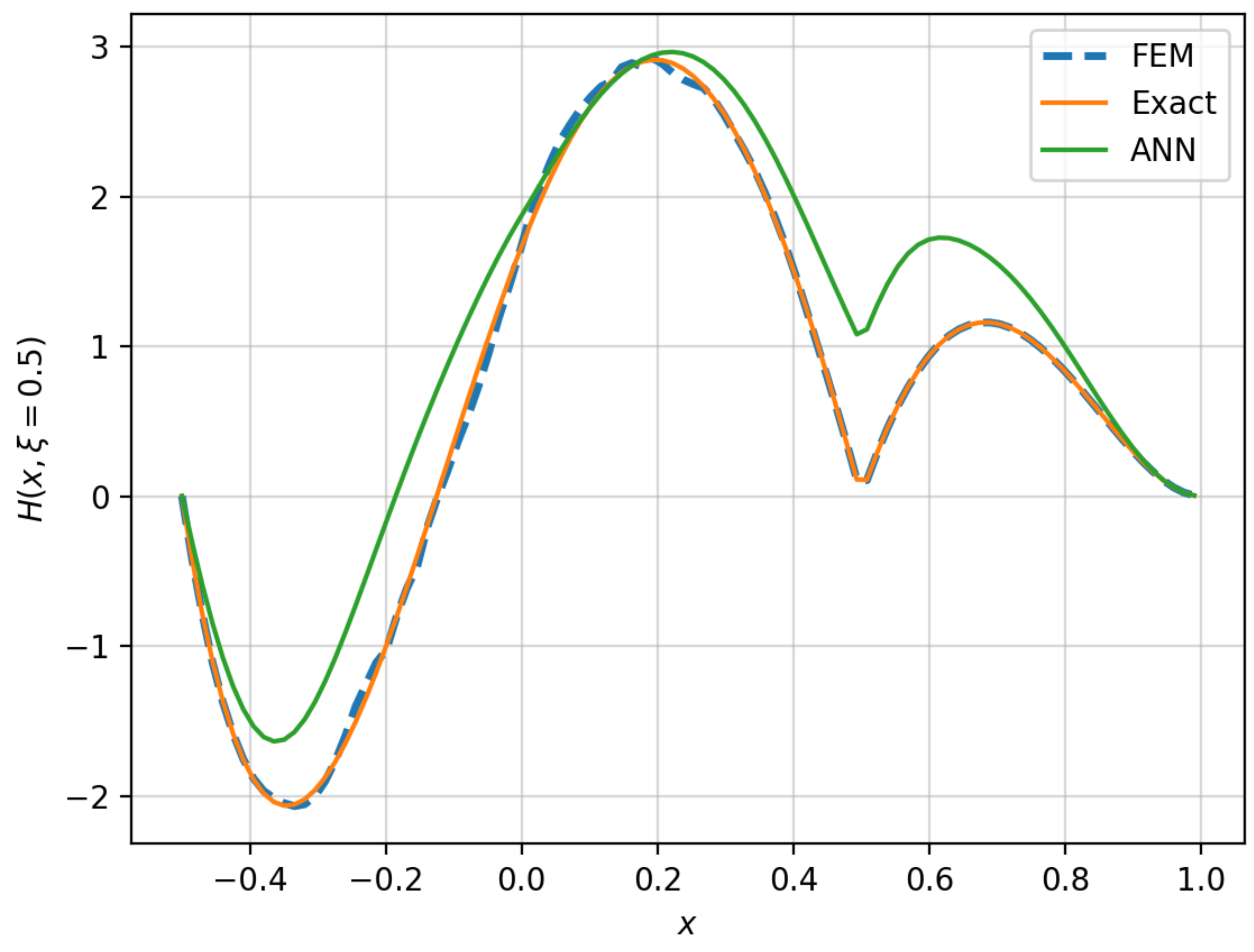
## Second test: Nakanishi based model for pion

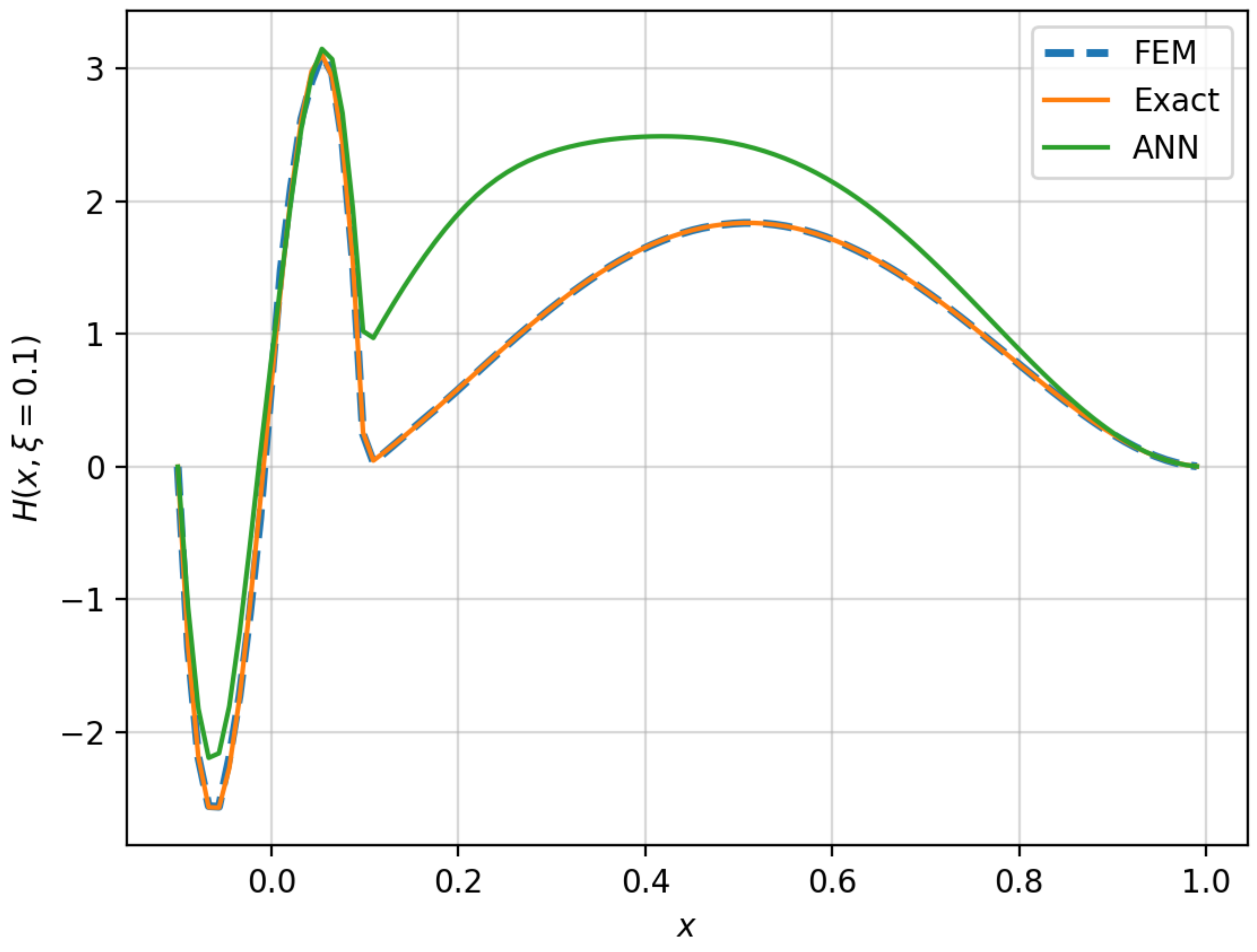
N.Chouika et al. -Phys.Lett.B:780(2018)

$$H(x,\xi,t=0) = \begin{cases} 30\,\dfrac{(1-x)^2(x^2-\xi^2)}{(1-\xi^2)^2}, & |x|>\xi \\ 15\,\dfrac{(1-x)(\xi^2-x^2)(x+2x\xi+\xi^2)}{2\xi^3(1+\xi)^2}, & |x|<\xi \end{cases}$$

$$H(x,\xi,t=0) = (1-x)\int_{\Omega^+} d\beta d\alpha \delta(x-\beta-\alpha\xi)\, h(\beta,\alpha)$$

$$h(\beta,\alpha) = \frac{15}{2}\left(1-3(\alpha^2-\beta^2)-2\beta\right)$$



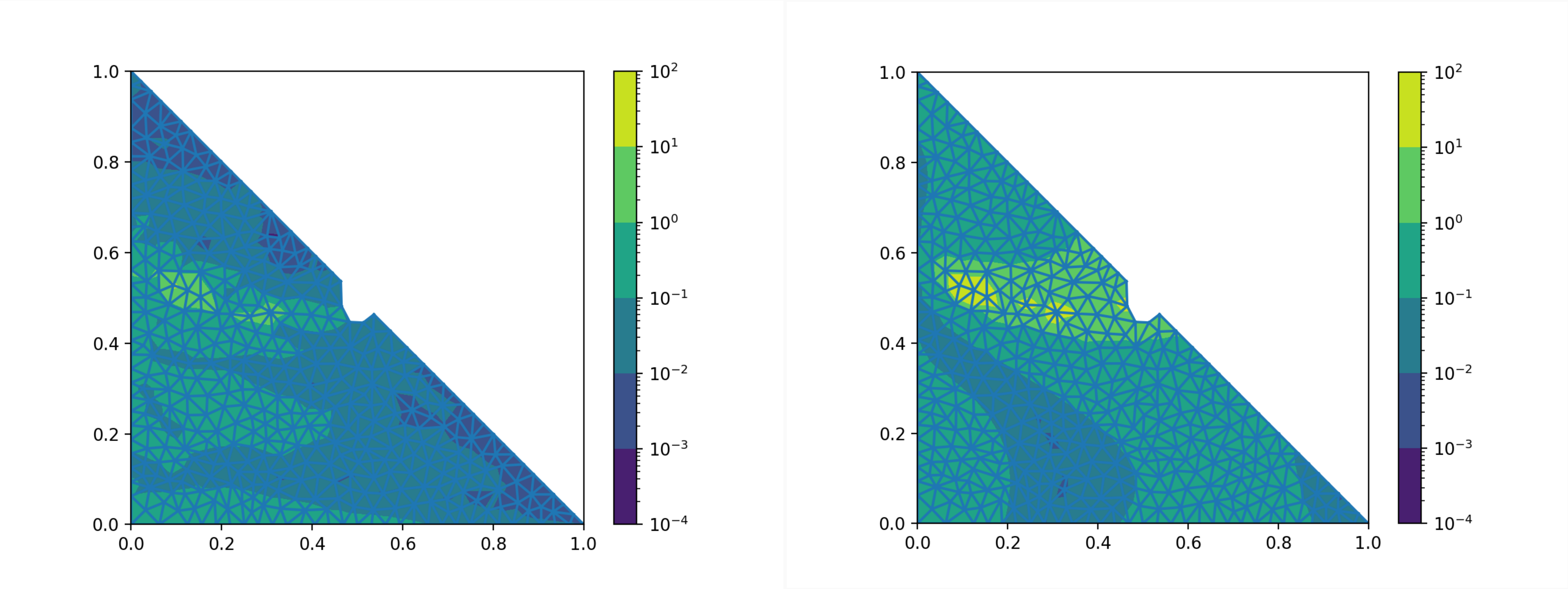

$N = 5\,, \quad N_{sample} = 10^4$

$\text{RRMSE}_{FEM} \sim 9\times 10^{-3}$

$\text{RRMSE}_{ANN} \sim 4\times 10^{-2}$

$$\left| \frac{h(\beta,\alpha) - h^{pred}(\beta,\alpha)}{h(\beta,\alpha)} \right|$$

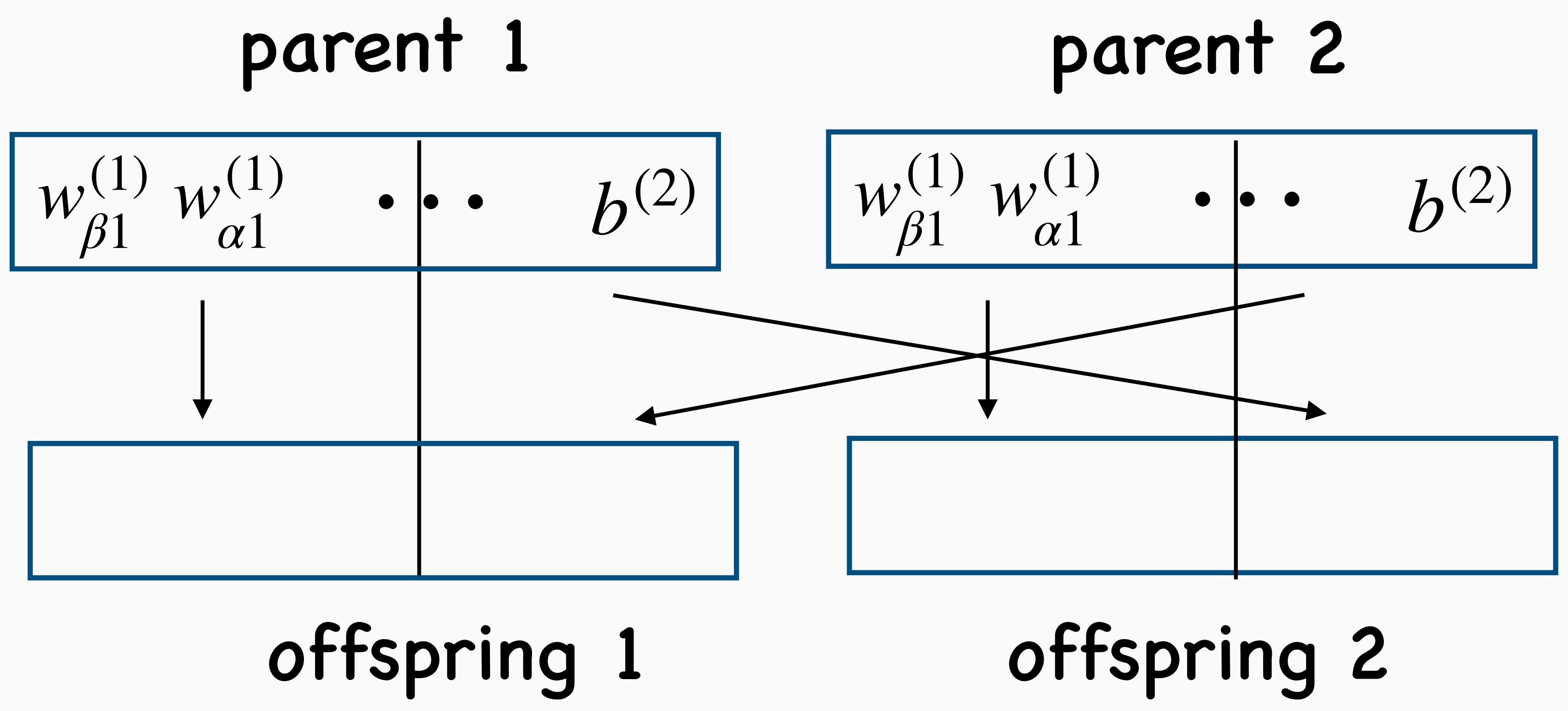
FEM ANN



# Genetic algorithm

Strategy: generate a population of potential solutions which "evolve" according to their fitness.

- Population initialization ($N_{pop}$ ANN)
- Mating pool selection
- Crossover




- Mutation

In progress

# Conclusions and outlook

- ANN need better optimization
- FEM seem more precise and more efficient
- More models as benchmarks
- Invert Radon Transform with partial DGLAP knowledge $\xi < \xi_{max}$