Revisiting GPD evolution

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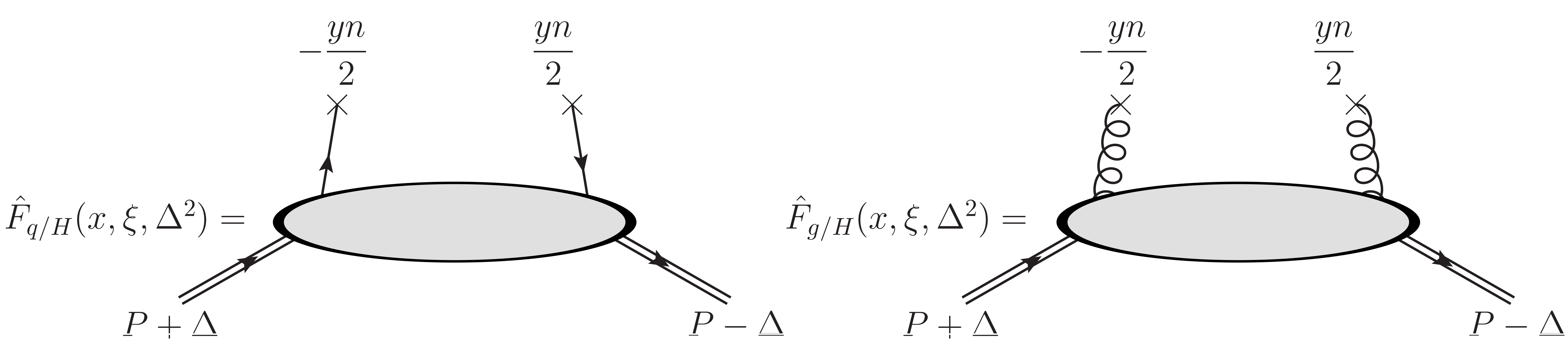
GPD definition

- Generalised parton distributions (GPDs) are a "byproduct" of factorisation of *amplitudes* for **exclusive** processes such as deeply-virtual Compton scattering.
  [Collins, Freund, *Phys.Rev.D* 59 (1999) 074009] [Ji, *Phys.Rev.D* 55 (1997) 7114-7125]
- An operator definition of the GPDs in the **light-cone gauge** ($n \cdot A = 0$) reads:

$$\hat{F}_{q/H}(x,\xi,\Delta^2) = \frac{1}{\sqrt{1-\xi^2}} \int \frac{dy}{2\pi} e^{-ix(n\cdot P)y} \left\langle P-\Delta \left| \overline{\psi}_q\left(\frac{yn}{2}\right) \frac{\not{n}}{2} \psi_q\left(-\frac{yn}{2}\right) \right| P+\Delta \right\rangle$$

$$\hat{F}_{g/H}(x,\xi,\Delta^2) = -x(n\cdot P) \int \frac{dy}{2\pi} e^{-ix(n\cdot P)y} \left\langle P-\Delta \left| A_a^{\alpha}\left(\frac{yn}{2}\right) A_{a\alpha}\left(-\frac{yn}{2}\right) \right| P+\Delta \right\rangle$$

$$\xi = \frac{\Delta^+}{P^+}$$



- No Wilson line, simpler gluon GPD, more complicated gluon propagator:

$$\mathcal{D}_{\mu\nu}(k) = \frac{1}{k^2 + i\varepsilon}\left(-g_{\mu\nu} + \frac{k_\mu n_\nu + k_\nu n_\mu}{k\cdot n}\right)$$

- These definitions are affected by UV divergences that need to be renormalised.

GPD evolution

- Using dim. reg., the renormalisation of GPDs can be implemented as follows:

$$F_{i/H}(x,\xi,\mu) = \sum_{j=q,g}\int_{-1}^{1}\frac{dy}{|y|}Z_{ij}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s(\mu),\varepsilon\right)\hat{F}_{j/H}(y,\xi,\varepsilon)$$

- In the $\overline{\mathrm{MS}}$ scheme renormalisation constants have the following structure:

$$Z_{ij}\left(z,\kappa,\alpha_s,\varepsilon\right) = \delta_{ij}\delta\left(1-z\right) + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\sum_{p=1}^{n}\frac{1}{\varepsilon^p}Z_{ij}^{[n,p]}\left(z,\kappa\right)$$

- Exploiting the independence of the bare GPDs on μ, one can derive a **RGE**:

$$\frac{dF_{i/H}(x,\xi,\mu)}{d\ln\mu^2} = \sum_{k=q,g}\int_{-1}^{1}\frac{dz}{|z|}\mathcal{P}_{i/k}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s(\mu)\right)F_{k/H}(z,\xi,\mu)$$

- The evolution kernel are finite quantities computable in perturbation theory:

$$\mathcal{P}_{i/k}\left(z,\kappa,\alpha_s\right) = \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^{n+1}\mathcal{P}_{i/k}^{[n]}\left(z,\kappa\right)$$

- They are related to the renormalisation constants Z_{ij}. At LO one finds:

$$\mathcal{P}_{i/k}^{[0]}\left(z,\kappa\right) = -Z_{ik}^{[1,1]}\left(z,\kappa\right)$$

- Bottomline: the computation of the **evolution kernels** boils down to computing the GPD **renormalisation constants**.

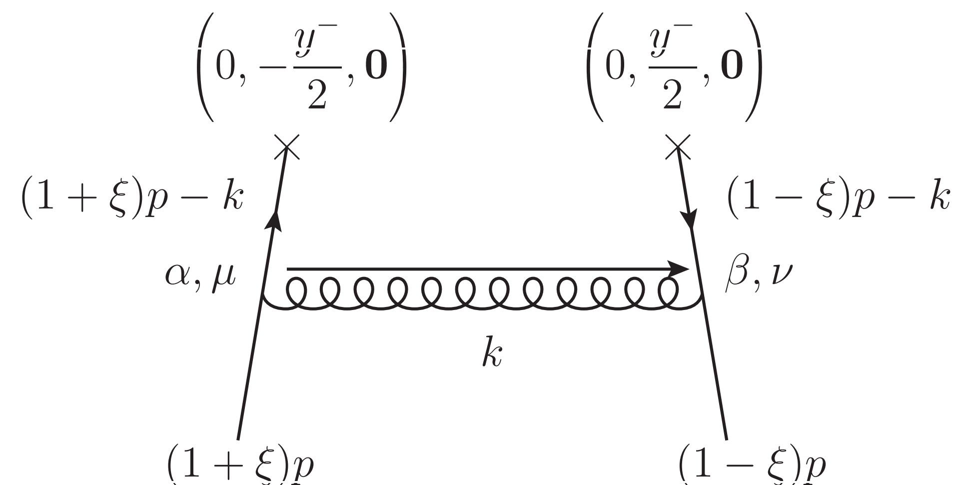
GPD evolution

- The Z_{ij} renormalisation constants can be computed using **parton-in-parton** GPDs, *i.e.* GPDs where the *hadronic* states are replaced by *partonic* states.
- Parton-in-parton GPDs can be perturbatively computed and their **pole part** gives the Z_{ij} that is related to the evolution kernels. At LO one finds:

$$\mathcal{P}^{[0]}_{i/k}\left(x, \frac{\xi}{x}\right) = \text{pole part of } \hat{F}^{[1]}_{i/k}(x,\xi,\varepsilon)$$

- For example:

$$\hat{F}^{[1]}_{q/q}(x,\xi,\varepsilon) \longrightarrow$$



- The LO RGE can finally be written in a **DGLAP-like** fashion:

$$\frac{dF^{\pm}(x,\xi,\mu)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\int_x^{\infty}\frac{dy}{y}\mathcal{P}^{\pm,[0]}\left(y, \frac{\xi}{x}\right)F^{\pm}\left(\frac{x}{y}, \xi, \mu\right)$$

$$F^- = F_{q/H} - F_{\overline{q}/H}$$

$$F^+ = \begin{pmatrix} \sum_{q=1}^{n_f} F_{q/H} + F_{\overline{q}/H} \\ F_{g/H} \end{pmatrix}$$

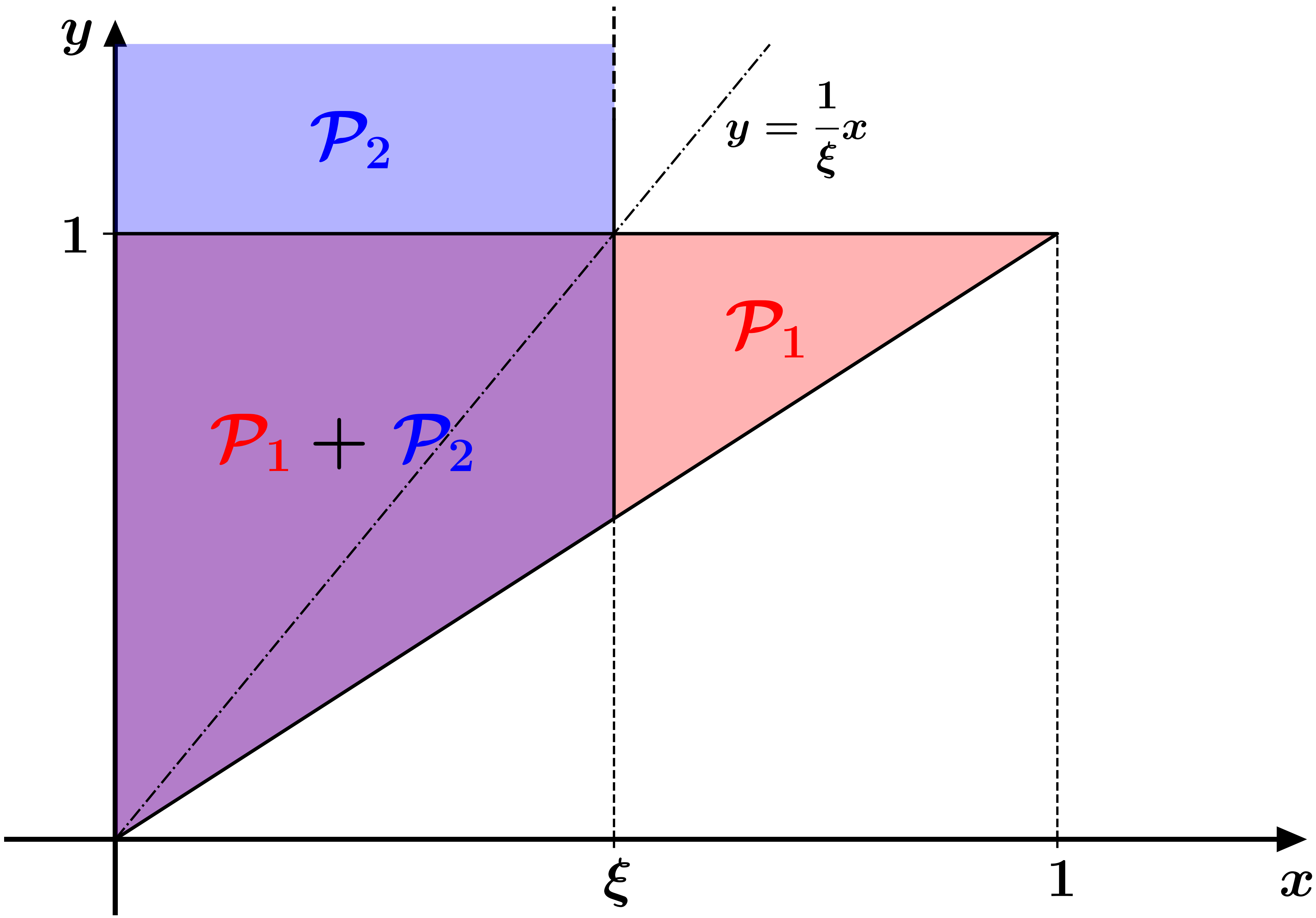
$$\mathcal{P}^{\pm,[0]}\left(y, \frac{\xi}{x}\right) = \theta(1-y)\mathcal{P}^{\pm,[0]}_1\left(y, \frac{\xi}{x}\right) + \theta(\xi - x)\mathcal{P}^{\pm,[0]}_2\left(y, \frac{\xi}{x}\right)$$

DGLAP region — ERBL contribution

GPD evolution

$$\frac{dF^{\pm}(x,\xi,\mu)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\int_{x}^{\infty}\frac{dy}{y}\mathcal{P}^{\pm,[0]}\left(y,\frac{\xi}{x}\right)F^{\pm}\left(\frac{x}{y},\xi,\mu\right)$$

$$\mathcal{P}^{\pm,[0]}\left(y,\frac{\xi}{x}\right) = \theta(1-y)\mathcal{P}_1^{\pm,[0]}\left(y,\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{\pm,[0]}\left(y,\frac{\xi}{x}\right)$$



Properties of the kernels

$$\mathcal{P}^{\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- In the limit $\kappa \to 0$ the **DGLAP** splitting functions are recovered:

$$\lim_{\kappa\to 0} \mathcal{P}^{\pm,[0]}(y,\kappa) = \theta(1-y)P^{\pm,[0]}(y)$$

- In the limit $\kappa \to 1/x$ the **ERBL** non-singlet kernel is recovered:

e.g. [Mikhailov, Radyushkin, *Nucl.Phys.B* 254 (1985) 89-126]
or [Blümlein, Geyer, Robaschik, *Phys.Lett.B* 406 (1997) 161-170]

$$\frac{1}{2u-1}\mathcal{P}^{-,[0]}\left(\frac{2t-1}{2u-1},\frac{1}{2t-1}\right) = C_F\left[\theta(u-t)\left(\frac{t-1}{u}+\frac{1}{u-t}\right) - \theta(t-u)\left(\frac{t}{1-u}+\frac{1}{u-t}\right)\right]_+$$

with $[f(t,u)]_+ \equiv f(t,u) - \delta(u-t)\int_0^1 du'\, f(t,u')$

- **Continuity** of GPDs at the crossover point $x = \xi$ ($\kappa = 1$) guaranteed:

$$\lim_{\kappa\to 1} \mathcal{P}_1^{\pm,[0]}(y,\kappa) = \text{finite} \qquad \mathcal{P}_2^{\pm,[0]}(y,\kappa) \propto (1-\kappa)$$

- Cancellation of **spurious divergencies** (stable numerical implementation)

$$\lim_{y\to\kappa^{-1}} (1-\kappa^2y^2)^\alpha \mathcal{P}_1^{\pm,[0]}(y,\kappa) = -\lim_{y\to\kappa^{-1}} (1-\kappa^2y^2)^\alpha \mathcal{P}_2^{\pm,[0]}(y,\kappa) = \text{finite}$$

Properties of the kernels

$$\mathcal{P}^{\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- Valence **sum rule** (polynomiality for the first moment of the non-singlet) conserved:

$$\int_0^1 dx F^-(x,\xi,\mu) = \text{FF} \quad \Rightarrow \quad \int_0^1 dz\, \mathcal{P}_1^{-,[0]}\left(z, \frac{\xi}{yz}\right) + \int_0^{\xi/y} dz\, \mathcal{P}_2^{-,[0]}\left(z, \frac{\xi}{yz}\right) = 0$$

- As consequence of the **Ji's sum rule** one also finds: [Ji, *Phys.Rev.Lett.* 78 (1997) 610-613]

$$\int_0^1 dx\, x\left[F_q^+(x,\xi,\mu) + F_g^+(x,\xi,\mu)\right] = \text{constant}$$

- that leads to:

$$\int_0^1 dz\, z\left[\mathcal{P}_{1,qq}^{+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{1,gq}^{+,[0]}\left(z,\frac{\xi}{yz}\right)\right] + \int_0^{\xi/y} dz\, z\left[\mathcal{P}_{2,qq}^{+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{2,gq}^{+,[0]}\left(z,\frac{\xi}{yz}\right)\right] = 0$$

$$\int_0^1 dz\, z\left[\mathcal{P}_{1,qg}^{+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{1,gg}^{+,[0]}\left(z,\frac{\xi}{yz}\right)\right] + \int_0^{\xi/y} dz\, z\left[\mathcal{P}_{2,qg}^{+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{2,gg}^{+,[0]}\left(z,\frac{\xi}{yz}\right)\right] = 0$$

- Explicit computation of conformal moments reveals that Gegenbauer polynomials of rank 3/2 diagonalise the LO non-singlet evolution kernel with **ξ-independent kernels**:

$$\int_{-1}^1 \frac{dx}{|\xi|} C_{2n}^{(3/2)}\left(\frac{x}{\xi}\right) \mathbb{V}^{-,[0]}\left(\frac{x}{\xi},\frac{y}{\xi}\right) = V_{2n}^{[0]} C_n^{(3/2)}\left(\frac{y}{\xi}\right)$$

Numerical setup

- The evolution kernels for *unpolarised* evolution that we have recomputed are implemented in **APFEL++** and available through **PARTONS** allowing for LO GPD evolution in momentum space.

- The properties of the evolution kernels allowed us to obtain a stable numerical implementation over the full range $\mathbf{0 \leq \xi \leq 1}$:

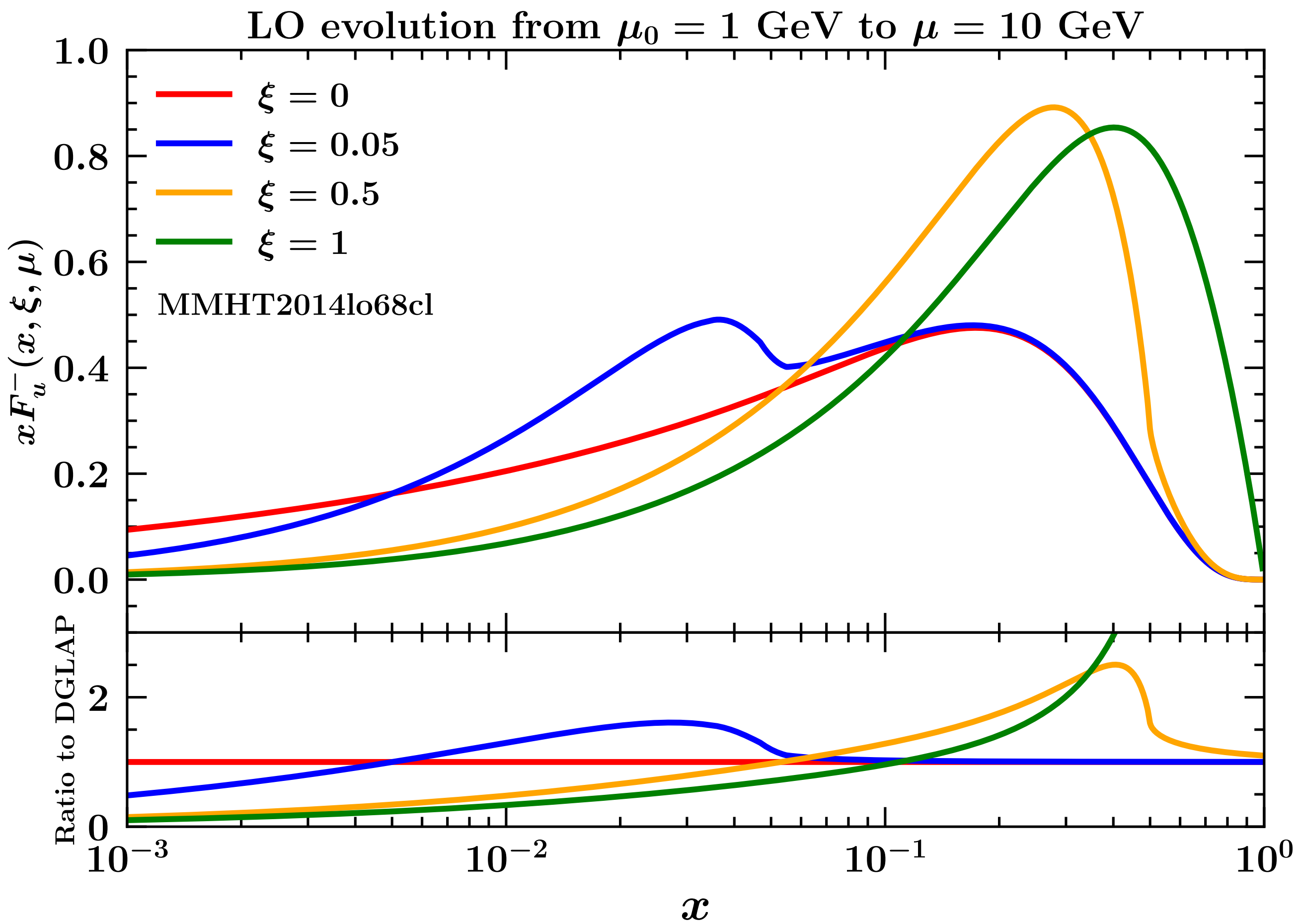
  - numerical check that both the **DGLAP** and **ERBL** limits are recovered,

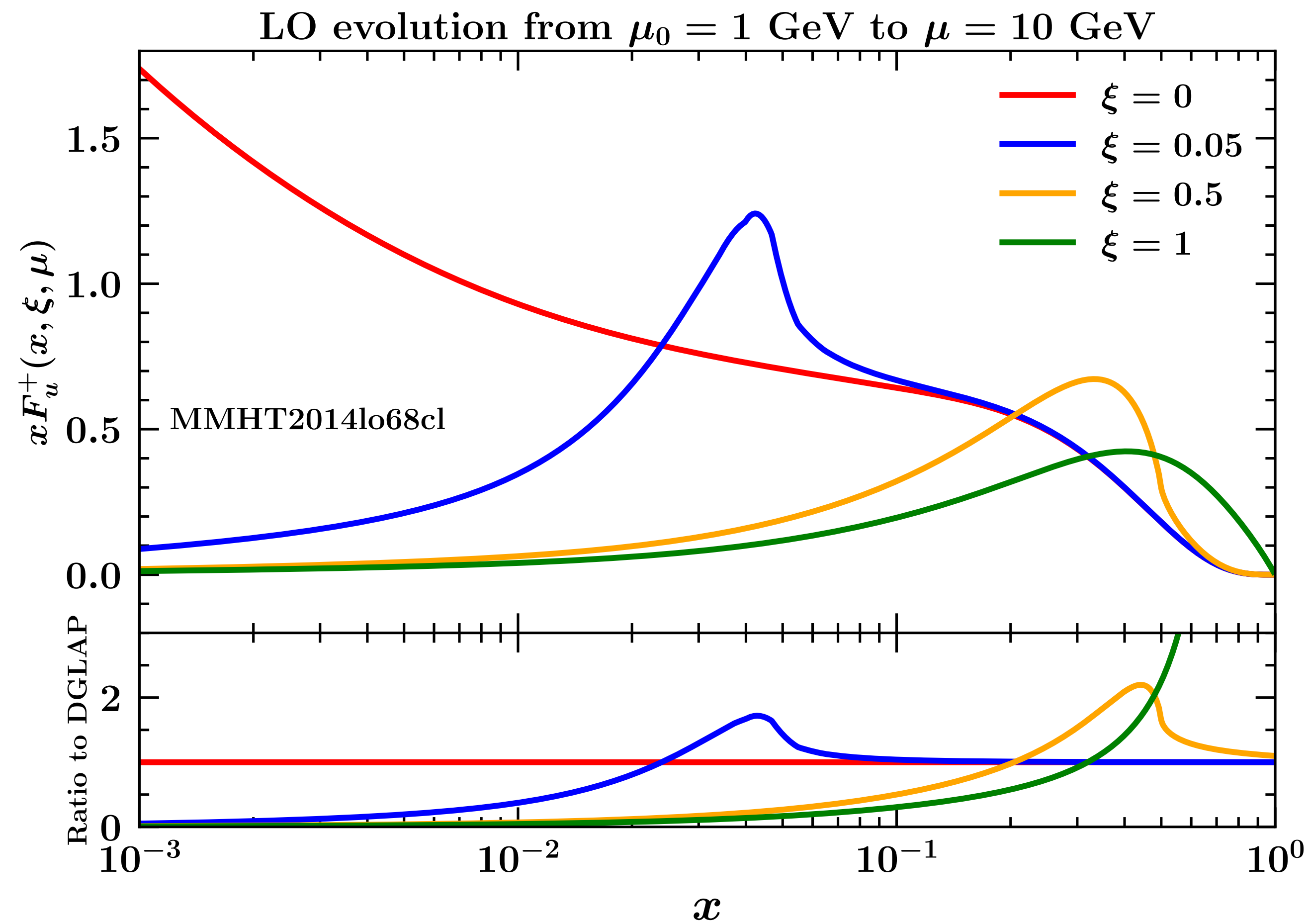
  - numerical check of **polynomiality** conservation.

- Numerical tests mostly use the MMHT14 PDF set at LO as an initial-scale set of distributions evolved from 1 to 10 GeV for the first time in the **variable-flavour-number scheme**, *i.e.* accounting for heavy-quark-threshold crossing.

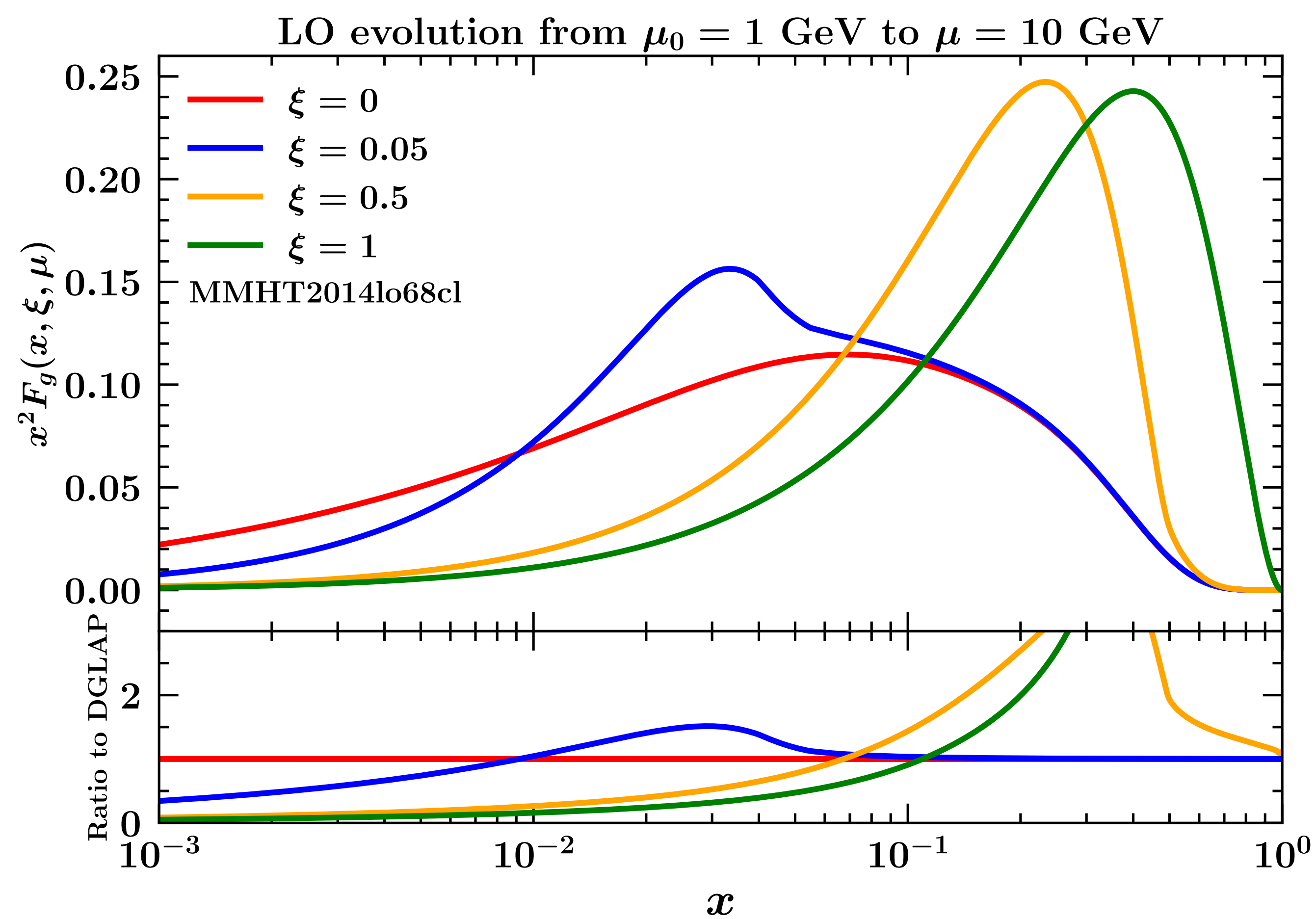
- Tests have also been performed using more realistic GPD models such as the Goloskokov-Kroll model [*Eur.Phys.J.C* 53 (2008) 367-384] based on the Radyushkin double-distribution ansatz [*Phys.Lett.B* 449 (1999) 81-88].

The DGLAP limit





- **DGLAP limit** reproduced within 10^{-5} relative accuracy.
- GPD evolution may significantly deviate from DGLAP evolution.
- The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.



The ERBL limit

- The limit $\xi \to \mathbf{1}$ ($\kappa \to 1/x$) should reproduce the **ERBL equation**.

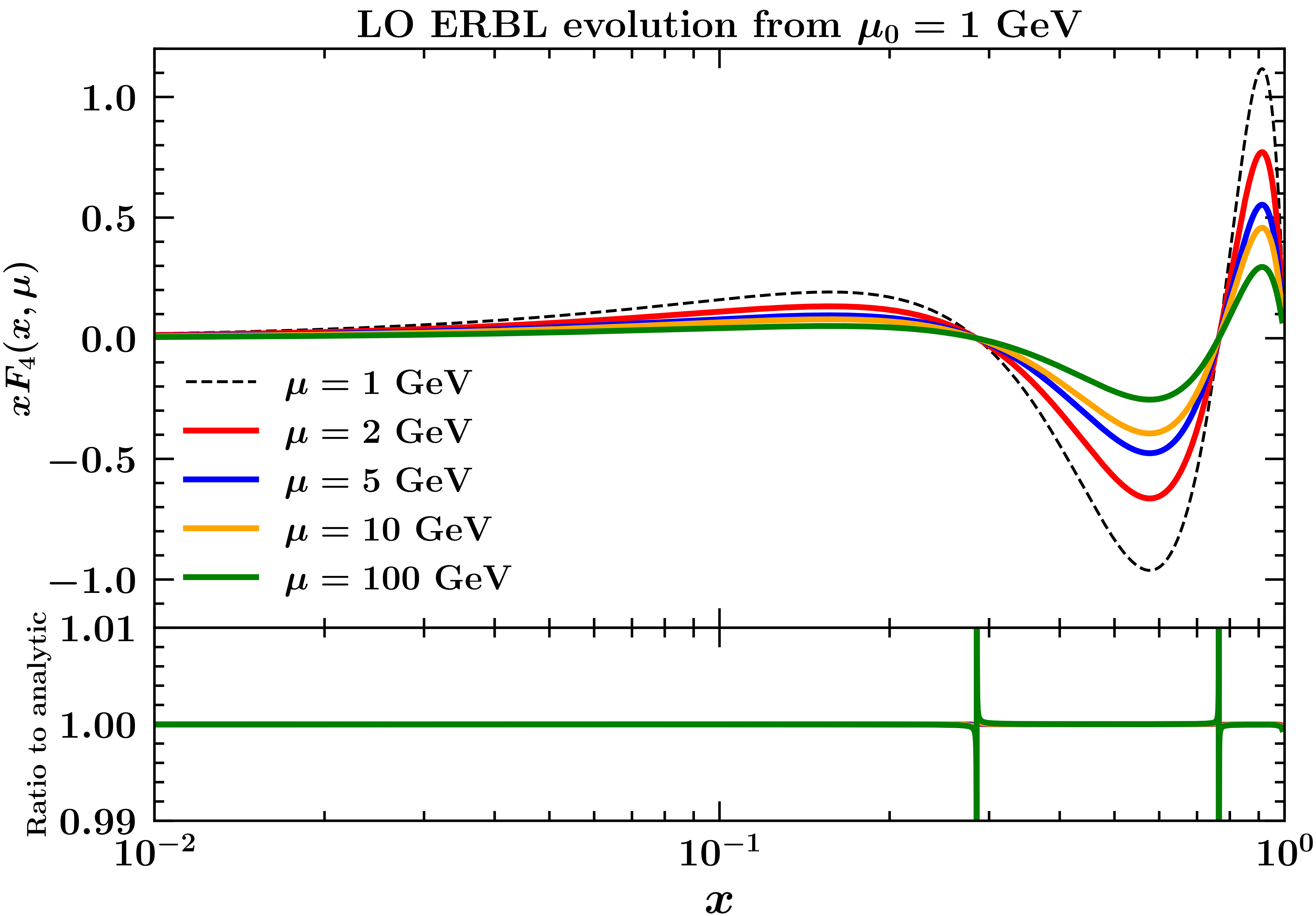
- It is well known that in this limit **Gegenbauer polynomials** decouple upon LO evolution, such that:

$$F_{2n}(x,\mu_0) = (1-x^2)C_{2n}^{(3/2)}(x) \quad \Rightarrow \quad F_{2n}(x,\mu) = \exp\left[\frac{V_{2n}^{[0]}}{4\pi}\int_{\mu_0}^{\mu} d\ln\mu^2\alpha_s(\mu)\right]F_{2n}(x,\mu_0)$$

- where the kernels $V_{2n}^{[0]}$ can be read off, for example, from [Brodsky, Lepage, *Phys.Rev.D* 22 (1980) 2157] or [Efremov, Radyushkin, *Phys.Lett.B* 94 (1980) 245-250].

- We have compared this expectation with the numerical results for GPD evolution by setting $\kappa = 1/x$ and using a Gegenbauer polynomial as an initial-scale GPD.

The ERBL limit



- **ERBL limit** reproduced within less than 10^{-5} relative accuracy,
- Same accuracy for **higher-degree** Gegenbauer polynomials.

Conformal-space evolution

- In order to check that LO GPD evolution ($\xi \neq 0$) in conformal space is diagonal in a **realistic** case, we have considered the RDDA:

$$H_q(x,\xi,\mu_0) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \delta\left(x-\beta-\xi\alpha\right) q(|\beta|)\pi(\beta,\alpha)$$

with:

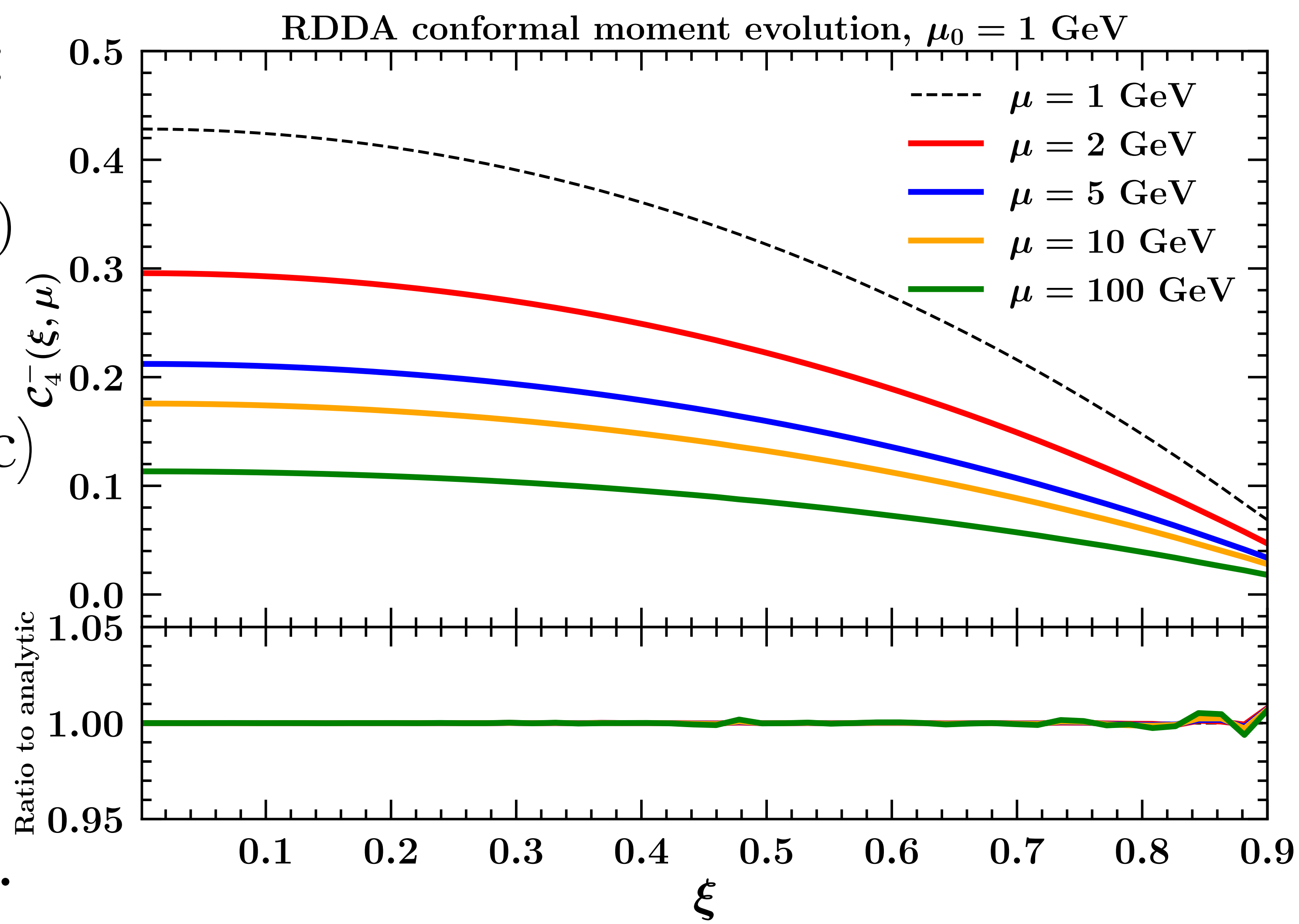
$$q(x) = \frac{35}{32} x^{-1/2}(1-x)^3, \quad \pi(\beta,\alpha) = \frac{3}{4}\frac{\left((1-|\beta|)^2-\alpha^2\right)}{(1-|\beta|)^3}$$

We have evolved the 4th moment:

$$\mathcal{C}_4^-(\xi,\mu) = \xi^4 \int_{-1}^{1} dx\, C_4^{(3/2)}\left(\frac{x}{\xi}\right) H_q(x,\xi,\mu)$$

from $\mu_0 = 1$ GeV using the (analytic) conformal-space evolution and the (numerical) momentum-space evolution.

Excellent agreement was found.



Polynomiality

- GPD evolution should preserve **polynomiality**.
[Xiang-Dong Ji, *J.Phys.G* 24 (1998) 1181-1205] [A.V. Radyushkin, *Phys.Lett.B* 449 (1999) 81-88]
- The following relations for the Mellin moments must hold at **all scales**:

$$\int_0^1 dx\, x^{2n} F_q^-(x,\xi,\mu) = \sum_{k=0}^{n} A_k(\mu)\xi^{2k}$$

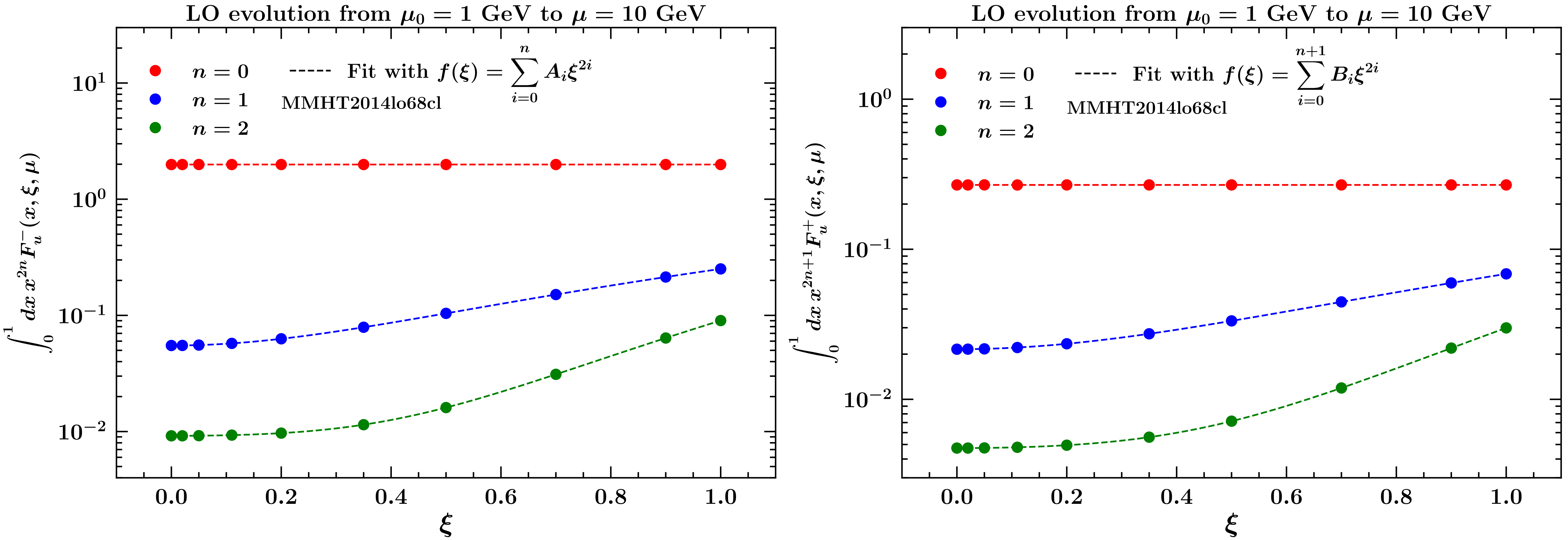
$$\int_0^1 dx\, x^{2n+1} F_q^+(x,\xi,\mu) = \sum_{k=0}^{n+1} B_k(\mu)\xi^{2k}$$

- Polynomiality predicts that the first moment ($n = 0$) of the *non-singlet* distribution is **constant** in ξ.

- The coefficient of the ξ^{2n+2} term of the *singlet* (D-term) is absent in our initial conditions and it is *not* generated by evolution, so that also the first moment of the singlet is expected to be **constant** in ξ.

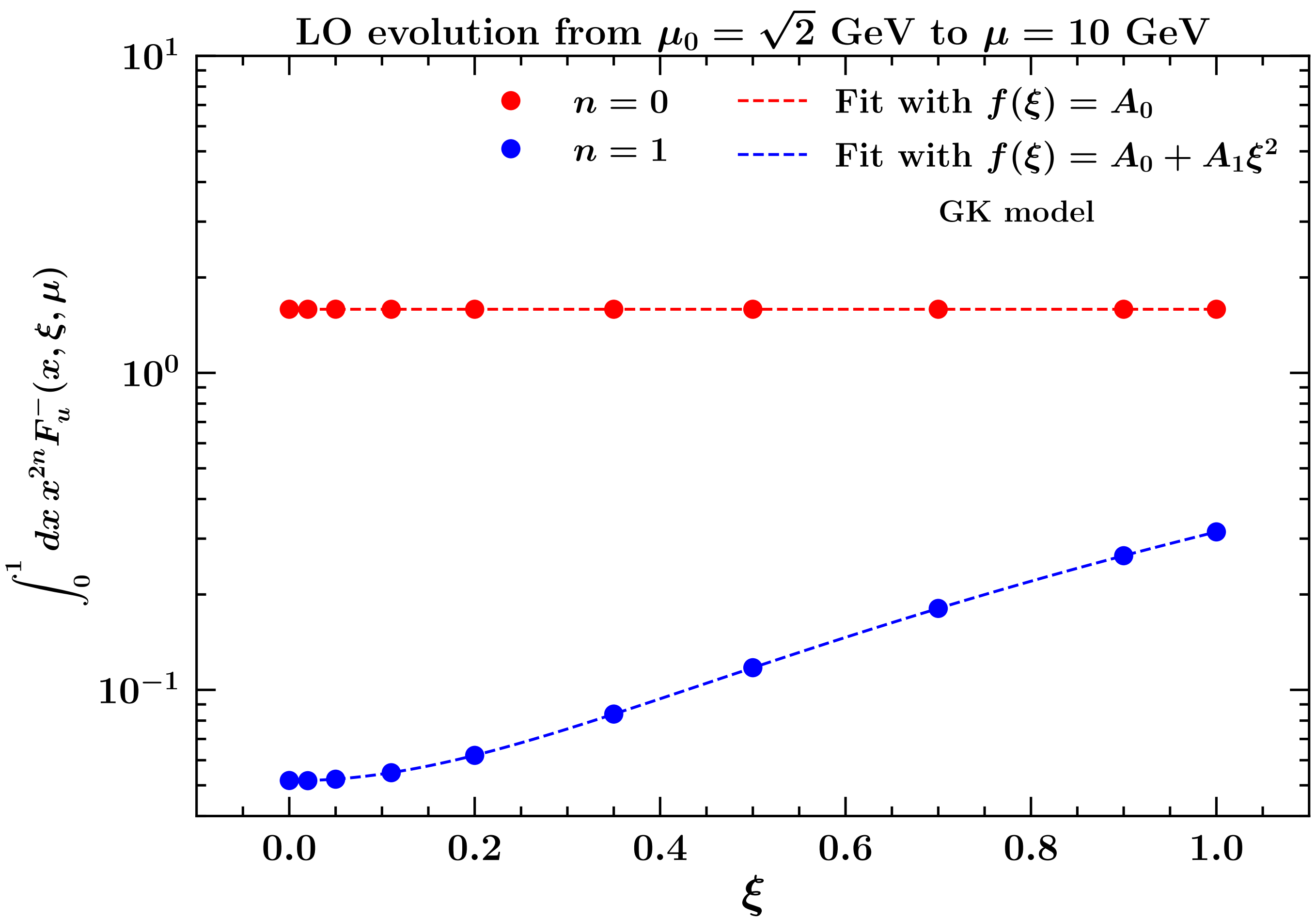
- For the other values of n one can just **fit** the behaviour in ξ and check that it follows the **expected power law**.

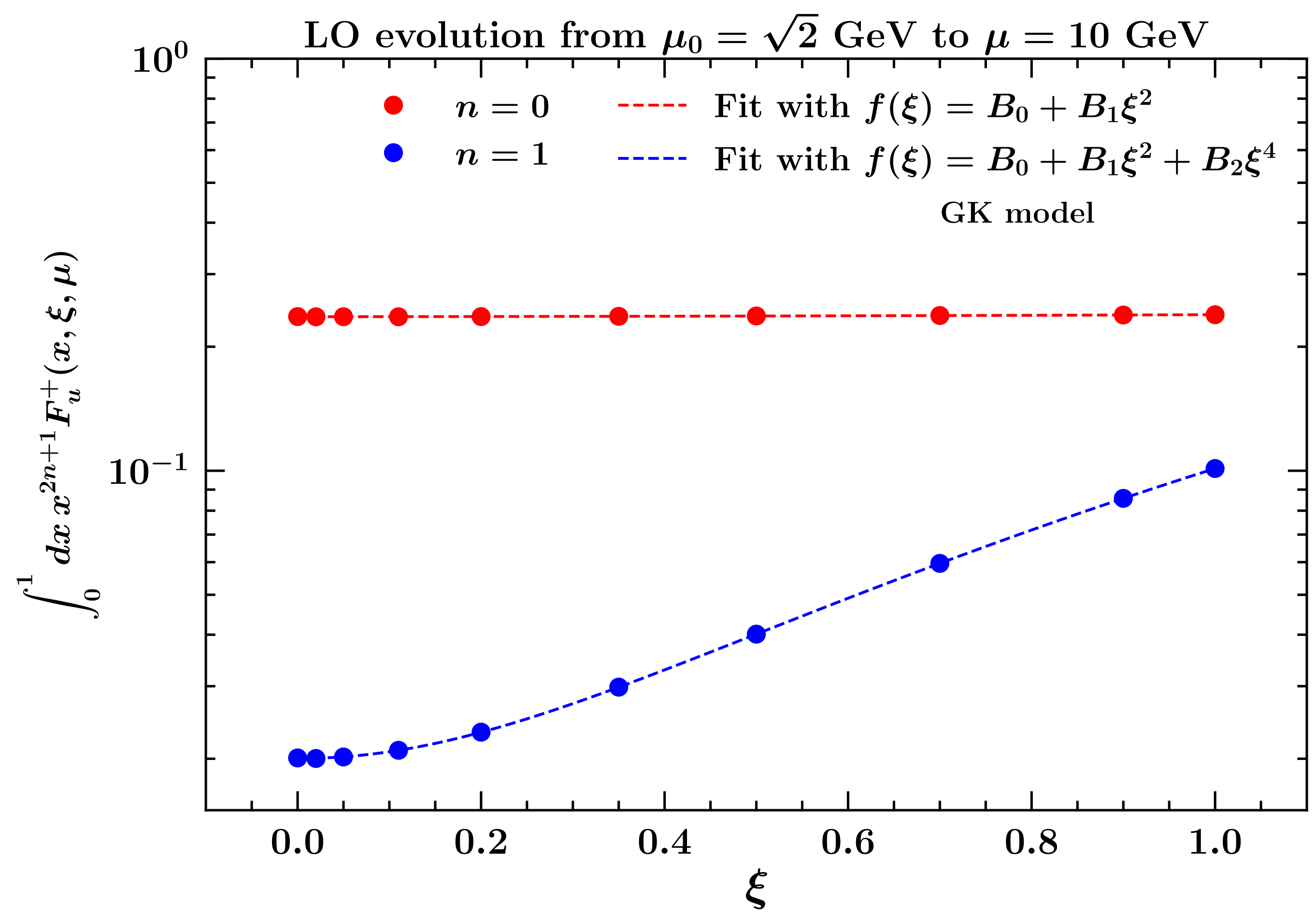
Polynomiality



- **First moment** for both singlet and non-singlet is indeed **constant** in ξ:
  - this was expected and the expectation is very nicely fulfilled.
- **Second and third moments** follow the expected law:
  - including odd-power terms in the fit gives coefficients very close to zero.
  - B_{n+1} in the singlet is consistently found to be compatible with zero (no D-term).

Polynomiality





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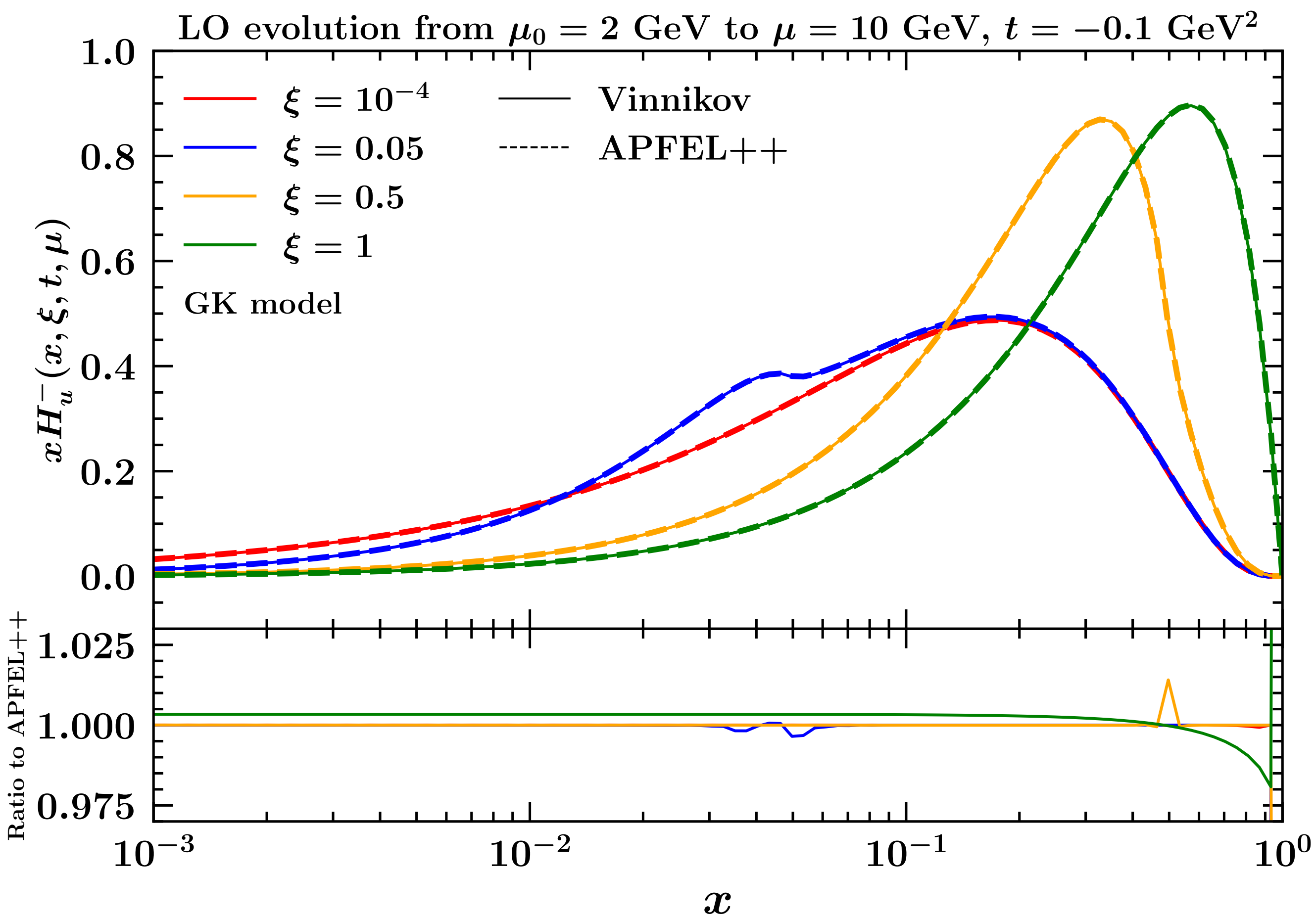
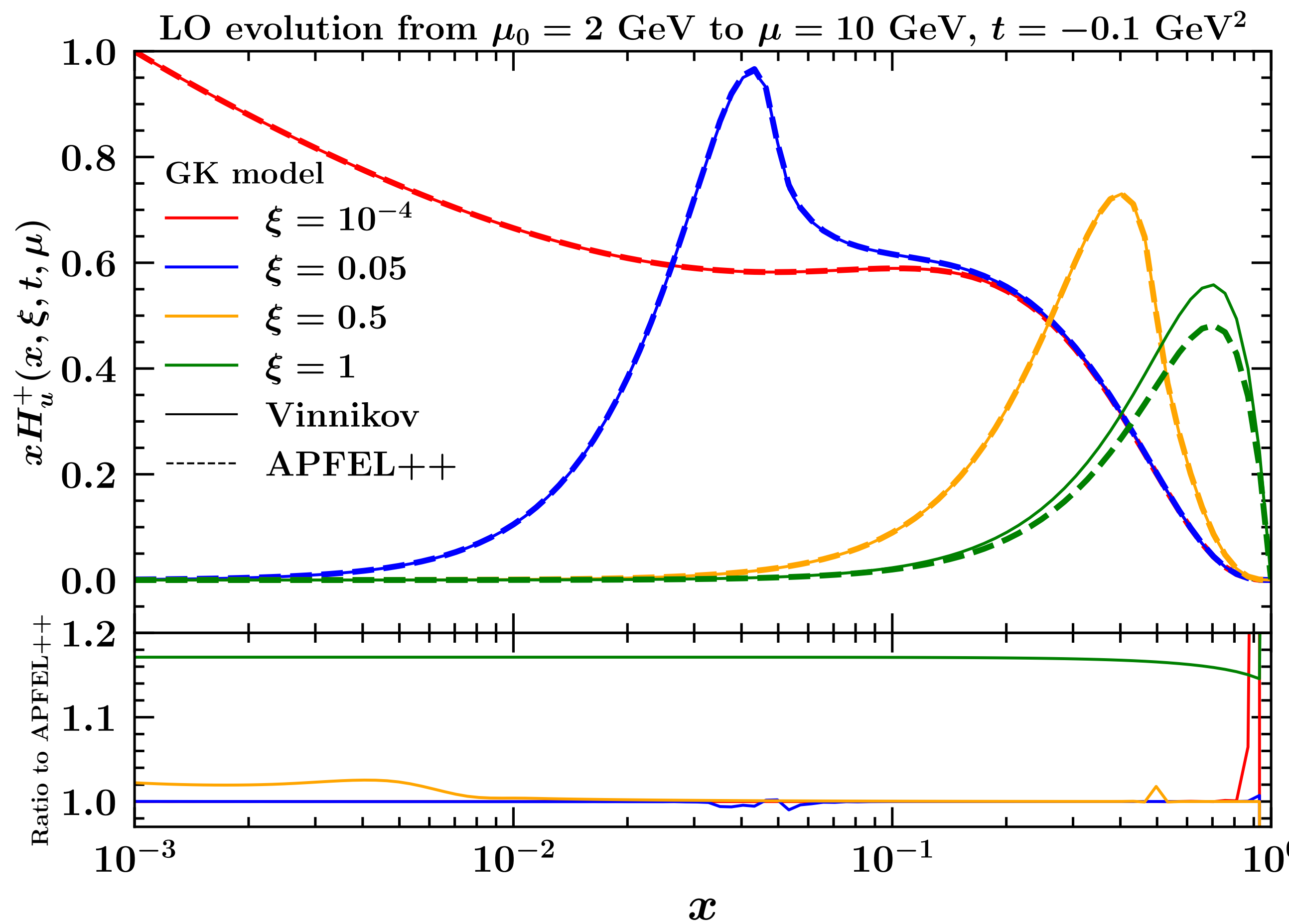
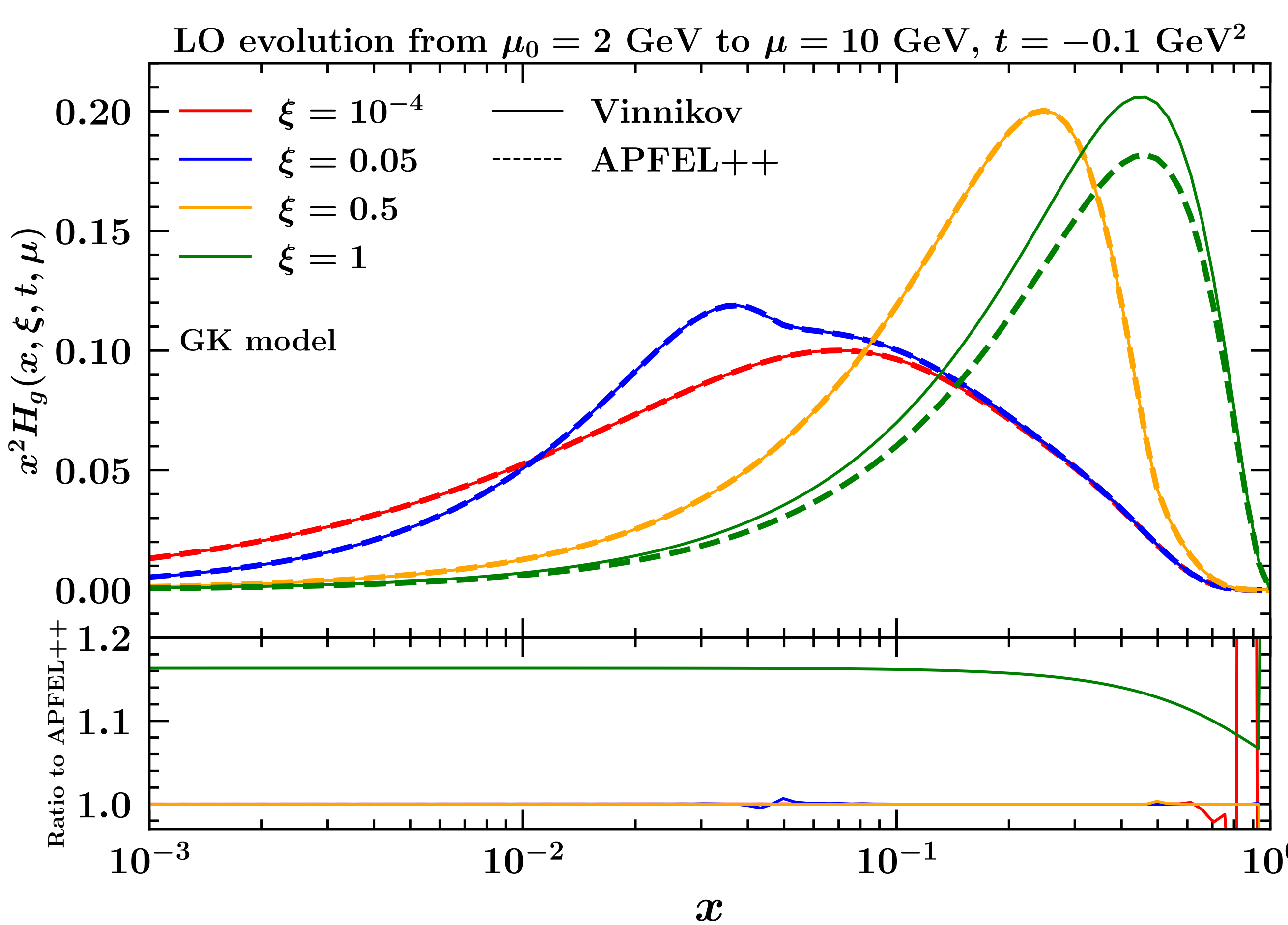
APFEL vs. Vinnikov's code





- **Excellent agreement** between the two code for $\xi \lesssim 0.6$.
- Agreement deteriorates for $\xi \gtrsim 0.6$:
  - discrepancy larger for the singlets (~20%) than for the non-singlet (~1%).
  - possible numerical instabilities of Vinnikov's code?
  - Inability to check the ERBL limit.



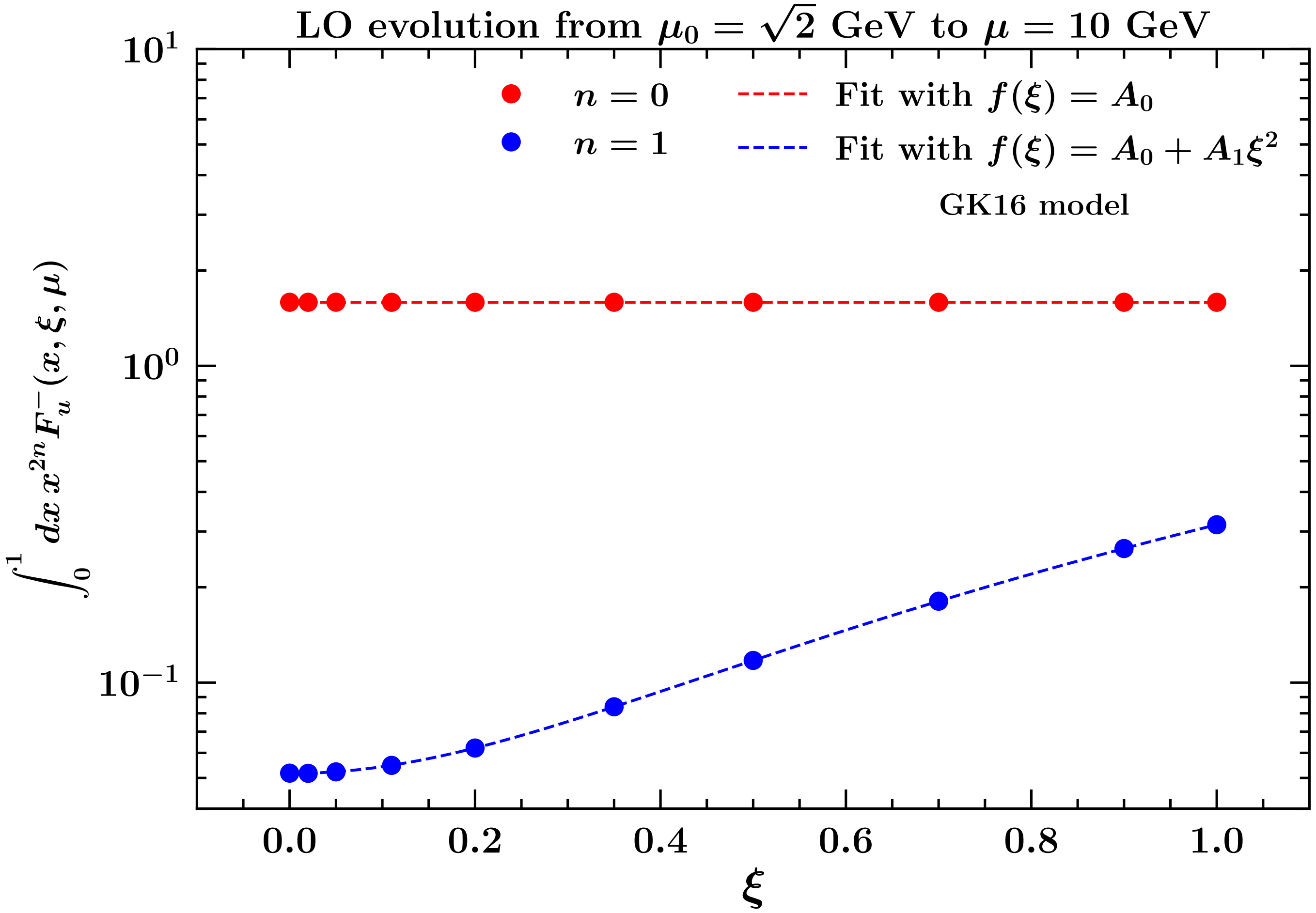
Conclusions and outlook

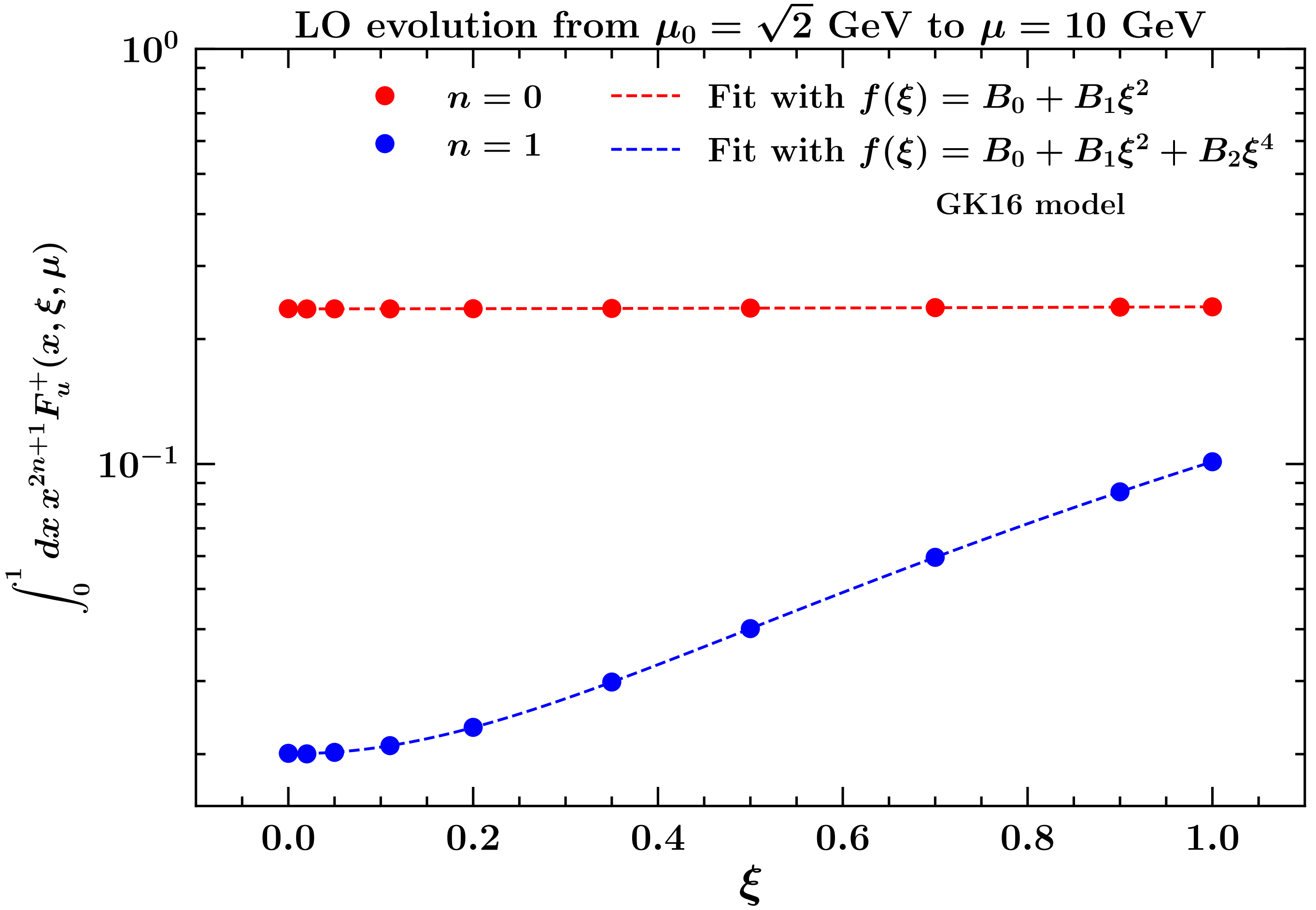
- We have **revisited LO GPD evolution** in momentum space:
  - *Ab-initio* calculation of the LO unpolarised splitting kernels based on Feynman diagrams in light-cone gauge.
  - GPD evolution equations recasted in a DGLAP-like form convenient for implementation.
  - Various analytical properties of the kernels highlighted and numerically checked.
  - DGLAP and ERBL limits correctly recovered within excellent accuracy.
  - Evolution conserves polynomiality and agrees with conformal-space evolution.
  - the code (APFEL++) is public and available within PARTONS
    https://github.com/vbertone/apfelxx

    http://partons.cea.fr/partons/doc/html/index.html
- **Next steps**:
  - **short term:** calculation/implementation of polarised (long. and trans. (?)) evolutions,
  - **middle term:** benchmark of the public evolution codes (discussion already started),
  - **longer term:** (re)calculation and implementation of the NLO corrections.

Back up

Numerics: polynomiality





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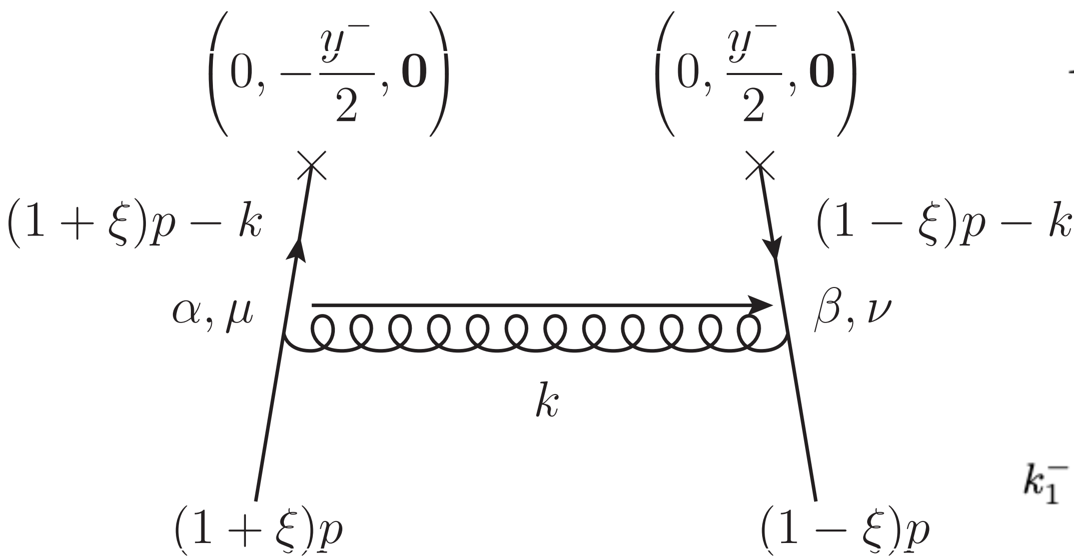
On the calculation of P_{qq} at LO

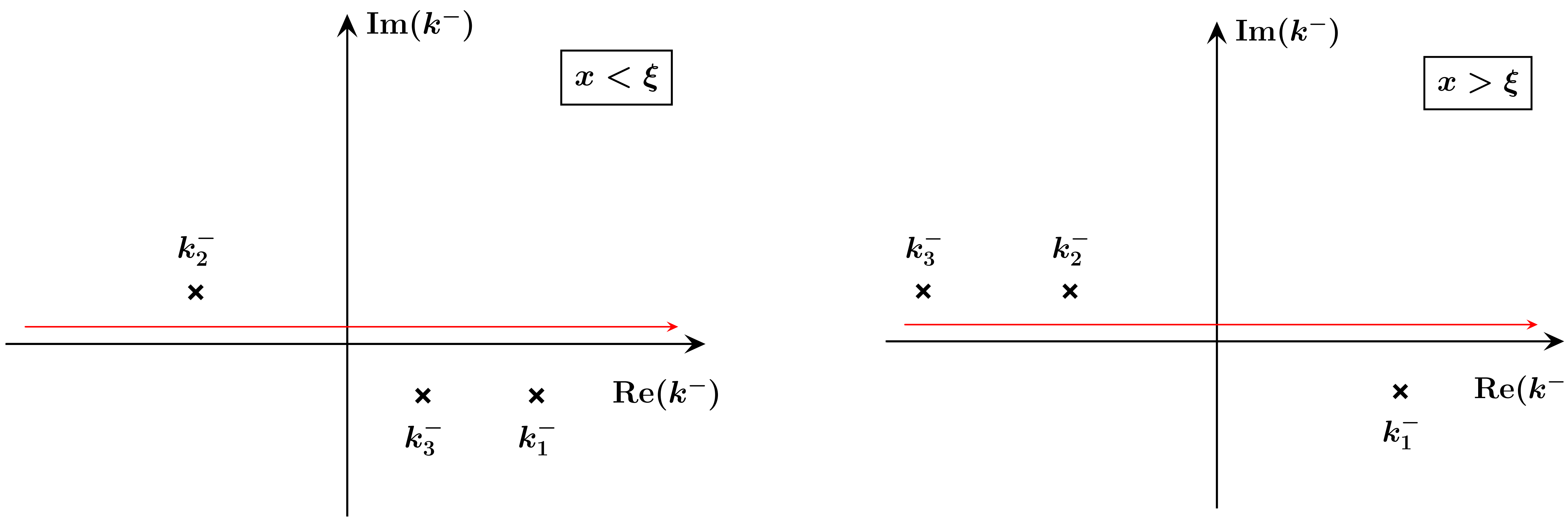
- In light-cone gauge, there is one single real diagram:



$$\hat{F}^{[1],(g^{\mu\nu})}_{(0),q/q}(x,\xi) = \sqrt{1-\xi^2}\,\frac{i}{2}C_F\frac{1}{(p^+)^2(1-x)(x^2-\xi^2)}\mu^{2\epsilon}\int\frac{d^{2-2\epsilon}\mathbf{k}_T}{(2\pi)^{2-2\epsilon}}\mathbf{k}_T^2 \times \int_{-\infty}^{+\infty}\frac{dk^-}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}\,,$$

$$k_1^- = \frac{\mathbf{k}_T^2}{2(1-x)p^+} - i\varepsilon\,,\quad k_2^- = -\frac{\mathbf{k}_T^2}{2(x+\xi)p^+} + i(x+\xi)\varepsilon\,,\quad k_3^- = -\frac{\mathbf{k}_T^2}{2(x-\xi)p^+} + i(x-\xi)\varepsilon\,,$$

- Pole structure:



On the calculation of P_{qq} at LO

- The real diagram gives:

$$
\begin{aligned}
\hat{F}^{[1]}_{(0),q/q}(x,\xi) &= \hat{F}^{[1],(n^\mu)}_{(0),q/q}(x,\xi) + \hat{F}^{[1],(g^{\mu\nu})}_{(0),q/q}(x,\xi) \\
&= C_F \frac{\sqrt{1-\xi^2}}{\xi(1-x)} \left[\frac{(x+\xi)(1-x+2\xi)}{1+\xi} - \theta(x-\xi) \frac{(x-\xi)(1-x-2\xi)}{1-\xi} \right] \mu^{2\epsilon} S_\epsilon \int \frac{dk_T^2}{k_T^{2+2\epsilon}},
\end{aligned}
$$

- The virtual contribution can be computed using the sum rule.

- Including the virtual diagram and isolating the UV divergence gives:

$$
\begin{aligned}
P^{[1]}_{qq}(x,\xi) &= 2C_F \Bigg\{ \frac{1}{2\xi(1-x)} \left[\frac{(x+\xi)(1-x+2\xi)}{1+\xi} - \theta(x-\xi) \frac{(x-\xi)(1-x-2\xi)}{1-\xi} \right] \\
&- \delta(1-x) \left[\int_0^1 dz \frac{1+z^2}{1-z} + \ln(|1-\xi^2|) \right] \Bigg\}.
\end{aligned}
$$