Constraining the equation of state with heavy quarks in the quasi-particle model of QCD matter

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1. Background

2. The quasi-particle linear Boltzmann transport (QLBT) model

3. Application of QLBT: Bayesian extraction for QCD EoS and transport coefficients

4. Summary
Background

The quasi-particle linear Boltzmann transport (QLBT) model

Application of QLBT: Bayesian extraction for QCD EoS and transport coefficients

Summary
Heavy Quark: Hard Probe in QGP

- Ideal probe of QGP.
  - $m_Q \gg T_{QGP}$
  - $m_Q \gg \Lambda_{QCD}$

- Explore the transport properties of QGP and EoS for the QGP through the energy loss of heavy quarks.
  - Extract $\dot{q}$ and diffusion coefficient $D_s$
  - Extract EoS
Jet(Bulk) transport coefficients and EoS are constrained separately by jet quenching (bulk observables).

In this work, a direct Bayesian extraction of the QGP EOS using heavy flavor observables based on QLBT model.
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The quasi-particle linear Boltzmann transport (QLBT) model


\[
m^2_g(T) = \frac{1}{6} \left( N_c + \frac{1}{2} N_f \right) g^2(T) T^2,
\]

\[
m^2_{u,d,s}(T) = \frac{N_c^2 - 1}{8N_c} g^2(T) T^2,
\]

The pressure can be calculated by summing over the contributions from different constituents:

\[
P(T) = \sum_i d_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E_i(p, T)} f_i(p, T) - B(T),
\]

Quasi Particle model: describe the equation of state

The energy density of the system can be obtained as follows:

$$\epsilon(T) = \sum_i d_i \int \frac{d^3p}{(2\pi)^3} E_i(p, T)f_i(p, T) + B(T).$$

As for the entropy density, the bag constant $B(T)$ cancels:

$$s(T) = \frac{\epsilon(T) + P(T)}{T}.$$

Motivated by the perturbative QCD calculation, we use the following parametric form to model the temperature dependence of the coupling $g(T)$:

$$g^2(T) = \frac{48\pi^2}{(11N_c - 2N_f) \ln \left( \frac{(aT/T_c+b)^2}{1+ce^{-d(T/T_c)^2}} \right)}$$

where $a$, $b$, $c$ and $d$ are parameters.

Standard process: Lattice QCD EOS $\rightarrow g^2(T) \rightarrow$ observables

Inverse question: observables $\rightarrow g^2(T) \rightarrow$ Lattice QCD EOS?
Linear Boltzmann transport model

The evolution of the phase space distribution $f_1$ of a given parton (denoted as “1” below) is described using the Boltzmann equation as follows:

$$p_1 \cdot \partial f_1(x_1, p_1) = E_1 \left( C_{el}[f_1] + C_{inel}[f_1] \right),$$

where $C_{el}$ and $C_{inel}$ are collision integrals arising from elastic and inelastic processes.
Elastic Process

The elastic scattering rate for parton “1” through a given channel can be obtained by integrating the transition rate over the exchange momentum $\vec{k}$:

$$\Gamma_{12\rightarrow34}(\vec{p}_1) = \int d^3 k w_{12\rightarrow34}(\vec{p}_1, \vec{k}) = \frac{\gamma_2}{2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} \int \frac{d^3 p_4}{(2\pi)^3 2E_4}$$

$$\times f_2(\vec{p}_2)[1 \pm f_3(\vec{p}_3)][1 \pm f_4(\vec{p}_4)]S_2(s, t, u)$$

$$\times (2\pi)^4 \delta^{(4)}(p_1 + p_2 - p_3 - p_4)|M_{12\rightarrow34}|^2,$$

- The leading order pQCD matrix elements are taken for $|M_{12\rightarrow34}|^2$ ($Qq \rightarrow Qq, Qg \rightarrow Qg$).
- Kinematic cut: $S_2(s, t, u) = \theta(s \geq 2\mu_D^2)\theta(t \leq -\mu_D^2)\theta(u \leq -\mu_D^2)$
- Note that the quasi-particle masses of the thermal partons have been included in evaluating the Mandelstam variables $s$, $t$, $u$ and density distribution of quasi particles.
Elastic Process

Elastic Scattering ($1 + 2 \rightarrow 3 + 4$ process), after introducing thermal mass, elastic scattering rate can be reduced to

$$\Gamma_{12\rightarrow34} = \frac{\gamma_2}{16E_1(2\pi)^4} \int dE_2 d\theta_2 d\theta_4 d\phi_4$$
$$\times f_2(E_2, T)(1 \mp f_4(E_4, T))S_2(s, t, u)|\mathcal{M}_{12\rightarrow34}|^2$$
$$\times \frac{p_2 p_4 \sin \theta_2 \sin \theta_4}{E_1 + E_2 - p_1 \cos \phi_4 \frac{E_4}{p_4} - p_2 \cos \theta_2 \cos \theta_4}$$

$$\cos \theta_2 = \sin \theta_2 \sin \theta_4 \cos \phi_4 + \cos \theta_2 \cos \theta_4$$

$$E_4 = \frac{(E_1 + E_2)B \pm \sqrt{A^2(\frac{m_4^2 A^2 + B^2}{(E_1 + E_2)^2} - m_4^2(E_1 + E_2)^2)}}{(E_1 + E_2)^2 - A^2}$$

with $A = |\vec{p}_1| \cos \theta_4 + |\vec{p}_2| \cos \theta_2$, and

$$B = p_1 \cdot p_2 + m_4^2$$
Inlastic Process: The distribution function of radiated gluon (hight twist) \(^2\)

\[
\frac{dN_g}{dx dk^2_\perp dt} = \frac{2\alpha_s C_A P(x) k^4_\perp}{\pi (k^2 + x^2 M^2)^4} \hat{q} \sin^2 \left( \frac{t - t_i}{2\tau_f} \right),
\]

where \(k_\perp\) is the gluon transverse momentum with respect to the parent parton, \(\alpha_s\) is the strong coupling. \(P(x)\) is the vacuum splitting function, \(\tau_f\) is the formation time of the radiated gluon in the form of \(\tau_f = 2Ex(1 - x)/(k^2_\perp + x^2 M^2)\) with \(M\) being the mass of the parent parton, and \(t_i\) denotes the initial time or the production time of the parent parton.

Two types of coupling vertices

- For vertices that connect to the jet partons (heavy quarks):
  \[ g^2(E) = \frac{48\pi}{(11N_c - 2N_f) \log \left( (AE/T_c + B)^2 \right)} , \]

- For vertices that connect to the medium partons
  \[ g^2(T) = \frac{48\pi^2}{(11N_c - 2N_f) \ln \left[ \frac{(aT/T_c + b)^2}{1 + ce^{-d(T/T_c)^2}} \right]} , \]

where A, B, a, b, c, d are parameters.
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Bayesian analysis

Parameters
\( \{ \theta = a, b, c, d, A, B \} \) → Latin-Hypercube

QLBT

Gaussian process emulator

LHC & RHIC data
\( R_{AA}, v_2 \) and \( T_c \)

\[ P( \text{data} | \theta) = \prod_i \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(y_i - \hat{y}_i)^2}{2\sigma_i^2}} \]

\( \sigma_i \): Experimental error and the interpolation error from GP

MCMC

Equilibrium

Posterior \( P(\theta | \text{data}) \)
Evaluate the EoS and transport coefficient
Calibration of the QLBT calculation (using $T_c = 150$ MeV)

One observes that after the Bayesian calibration, our QLBT model is able to provide a reasonable description of the D meson observables in heavy-ion collisions.
Calibration of the QLBT calculation (using $T_c = 154$ MeV)

One observes that after the Bayesian calibration, our QLBT model is able to provide a reasonable description of the D meson observables in heavy-ion collisions.

- Similar results can be obtained for $T_c = 154$ MeV compared to $T_c = 150$ MeV.
Over 800 sets of $\theta$ for prior calculation.

Posterior distributions of the model parameters, together with their correlations, upper triangle (green) for using $T_c = 154$ MeV, lower triangle (blue!50!dark) for $T_c = 150$ MeV.
Posterior distributions of the model parameters

<table>
<thead>
<tr>
<th>mean</th>
<th>WB</th>
<th>HotQCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1.1060</td>
<td>1.0936</td>
</tr>
<tr>
<td>B</td>
<td>0.0867</td>
<td>0.1131</td>
</tr>
<tr>
<td>a</td>
<td>2.0634</td>
<td>3.0117</td>
</tr>
<tr>
<td>b</td>
<td>0.7148</td>
<td>0.3362</td>
</tr>
<tr>
<td>c</td>
<td>4.7047</td>
<td>8.2149</td>
</tr>
<tr>
<td>d</td>
<td>0.5105</td>
<td>0.4524</td>
</tr>
</tbody>
</table>

- Reasonable constraints on these model parameters have been obtained for same $T_c$.
- Extracted parameters is sensitive to $T_c$. 
The extract EoS ( $T^3$ rescaled entropy density )

One observes that the heavy flavor observables do provide constraints on the QGP EoS.

The extracted EoS with $T_c = 150$ MeV agrees well with the WB lattice data that shares the same $T_c$.

Some deviation can be observed from HQ data: a larger $T_c$, terminates early, higher entropy density.
Our constraints are consistent with other model results as well as direct lattice QCD results.

One may also calculate the temperature dependent shear viscosity of the quasi-particle system using the relaxation time approximation.
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Summary

- Within the QLBT model for jet and heavy quarks transport in QGP which evolves according to the CLVisc hydrodynamics and a hybrid fragmentation-coalescence model for hadronization, we have carried out a Bayesian analysis of the experimental data on D meson spectra and anisotropy $v_2$ at both RHIC and LHC.

- We realize a simultaneous constraint on the properties of the QGP and hard probes.
  - The EoS we extract for the QGP is consistent with the lattice QCD data.
  - The heavy quark diffusion coefficient we obtain agrees with results from other model and lattice calculations.
Thanks
Backup: Temperature dependent bag constant

The energy density $\epsilon$ of the system is obtained from the pressure through the thermodynamic relationship $\epsilon(T) = TdP(T)/dT - P(T)$, where pressure $P$ of system only depend on $T$.

In order to have above thermodynamic consistency, the following relationship has to be satisfied

$$\left( \frac{\partial P_{qp}}{\partial m_i} \right)_{T,\mu} = 0, \quad i = u, d \ldots$$

which gives rise to a set of equations of the form

$$\frac{\partial B}{\partial m_i} + d_i \int \frac{d^3 p}{(2\pi)^3} \frac{m_i}{E_i} f_i(E_i) = 0$$

Bag constant $B$ is Temperature dependent due to Temperature dependent effective thermal mass of quasi particles.
Backup: Spatial diffusion coefficient

\[
\mathbf{q} = \sum_{bcd} \frac{\gamma b}{2E_a} \int \prod_{i=b,c,d} \frac{d^3 p_i}{2E_i(2\pi)^3} f_b |\mathcal{M}_{ab\rightarrow cd}|^2 S_2(\hat{s}, \hat{t}, \hat{u}) \times (2\pi)^4 \delta^4 (p_a + p_b - p_c - p_d) [\hat{p}_c - (\hat{p}_c \cdot \hat{p}_a) \hat{p}_a]^2
\]

\[
D_s(2\pi T) = 8\pi T^3 / \mathbf{q}
\]
## Backup: Prior range of parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Prior Range  $(T_c = 150 \text{ MeV})$</th>
<th>Prior Range  $(T_c = 154 \text{ MeV})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>[0.18, 4.5]</td>
<td>[0.26, 15.0]</td>
</tr>
<tr>
<td>$b$</td>
<td>[0.5, 1.2]</td>
<td>[0.1, 0.8]</td>
</tr>
<tr>
<td>$c$</td>
<td>[-2.0, 5.0]</td>
<td>[-4.0, 20.0]</td>
</tr>
<tr>
<td>$d$</td>
<td>[0.35, 0.7]</td>
<td>[0.25, 0.65]</td>
</tr>
<tr>
<td>$A$</td>
<td>[0.05, 0.16]</td>
<td>[0.05, 0.18]</td>
</tr>
<tr>
<td>$B$</td>
<td>[0.6, 3.0]</td>
<td>[0.6, 3.2]</td>
</tr>
</tbody>
</table>

**Table:** The ranges of model parameters used in the prior distributions, for two different values of $T_c$. 
Backup: Bayesian analysis

Parameters 
\( (\theta = a, b, c, d, A, B) \)

QLBT

Latin-Hypercube

Gaussian process emulator

LHC & RHIC data 
\( R_{AA}, \nu_2 \) and \( T_c \)

\[ P(\text{data} \mid \theta) = \prod_i \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{[x_i - m_i]^2}{2\sigma_i^2}} \]

\( \sigma_i \): Experimental error and the interpolation error from GP

MCMC

\( p(\theta \mid \text{data}) \propto P(\text{data} \mid \theta) p(\theta) \)

Equilibrium

Posterior \( P(\theta \mid \text{data}) \)

Evaluate the EoS and transport coefficient

| Parameters | \( \tau_f \)  
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( T_c = 150 \text{ MeV} )</td>
<td>( T_c = 154 \text{ MeV} )</td>
</tr>
<tr>
<td>( a )</td>
<td>178.242</td>
</tr>
<tr>
<td>( b )</td>
<td>16.175</td>
</tr>
<tr>
<td>( c )</td>
<td>310.174</td>
</tr>
<tr>
<td>( d )</td>
<td>1.329</td>
</tr>
<tr>
<td>( A )</td>
<td>3.764</td>
</tr>
<tr>
<td>( B )</td>
<td>14.365</td>
</tr>
</tbody>
</table>
Backup: Shear viscosity

The formulas for the viscosities $\eta$ are derived for a quasi-particle description with bosonic and fermionic constituents S. Plumari, W. M. Alberico, V. Greco, C. Ratti, Phys. Rev. D84 (2011)

$$\eta = \frac{1}{15 T} \sum_i d_i \int \frac{d^3 p}{(2\pi)^3} \tau_i \frac{p^4}{E_i^2} f_i (1 \mp f_i)$$

$$d^3 p = p^2 dp \sin \theta d\theta d\psi.$$

In relaxtime approximation, shear viscosity depends on collision relax time $\tau_i$ given by (HTL):

$$\tau_q^{-1} = 2N_C \left( 2 - \frac{1}{2N_C} \right) g^2 \frac{T}{8\pi} \ln \frac{2k}{g^2}, \quad \tau_g^{-1} = 2N_C \frac{g^2 T}{8\pi} \ln \frac{2k}{g^2},$$

where $g$ is the coupling obtained and $k$ is a parameter which is fixed by requiring that $\tau_i$ yields a minimum of one for the quantity $4\pi \eta / s$. 
Backup: Guarantee E and p conservation for \(2 \rightarrow 2+n\) process

First do \(2 \rightarrow 2\) process, record \(q_{\perp}\) and \(p_2\)

\[
p_3 = p_1 - q - k,
\]
\[
p_4 = p_2 + q,
\]

Using on shell condition and E and p conservation:

\[
(p_1 - q - k)^2_0 = (p_1 - q - k)^2_x + (p_1 - q - k)^2_y + (p_1 - q - k)^2_z + M^2,
\]
\[
(p_2 + q)^2_0 = (p_2 + q)^2_x + (p_2 + q)^2_y + (p_2 + q)^2_z
\]

One can get \(q_0\) and \(q_z\) from above equation which used E and p conservation.