An efficient implicit numerical solver for relativistic hydrodynamics

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Motivations

- Relativistic hydrodynamics can well reproduce heavy-ion collisions results such as flow.
- Relativistic hydrodynamics in this context is currently only solved with explicit methods.
- Implicit time integrators have interesting stability properties that could contribute to viscous and fluctuating hydrodynamics.
- Implicit numerical solver are expected to be slow in general, however we show that they can actually be more computationally efficient.

Ideal hydrodynamic equations

We have implemented a new method to implicitly solve the 1+1 and 1+2 dimensional ideal hydrodynamic equations. The latter correspond to the conservation of the energy–momentum tensor $T^{\mu\nu}$,

$$\partial_t T^{t\nu} = -\partial_x T^{x\nu} - \partial_y T^{y\nu}, \qquad (\nu = t, x, y, z).$$

with

$$T^{\mu\nu} = \epsilon u^\mu u^\nu + P\Delta^{\mu\nu},$$

Space and time discretization

We use the Kurganov-Tadmor method to discretize the hydrodynamic equations as

$$\partial_t \vec{y} = \vec{h}_{\mathrm{KT}}(t, \vec{y})$$

- $\vec{y}$: vector of cells of size Δx containing the values of $T^{\mu\nu}$
- $\vec{h}_{\mathrm{KT}}$: discretized fluxes in all cells

The time is discretize in time interval Δt with the multi-stages Runge-Kutta method,

$$\vec{y}_{(n+1)} = \vec{y}_{(n)} + \Delta t \sum_i^S b_i \vec{k}_i,$$

$$\vec{k}_i = \vec{f}_i(\vec{k}) = \vec{h}(t_{(n)} + c_i \Delta t, \vec{y}_{(n)} + \sum_j^S a_{ij}\vec{k}_j).$$

We use the following coefficients to compare explicit and explicit:

	Explicit	Implicit
$\begin{array}{c\|c} c_i & a_{ij} \\ \hline & b_j \end{array}$	$\begin{array}{c\|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 0.5 & 0.5 \end{array}$	$\begin{array}{c\|c} 0 & 0.5 \\ \hline & 1 \end{array}$
Stages S	2	1

Fixed-point method

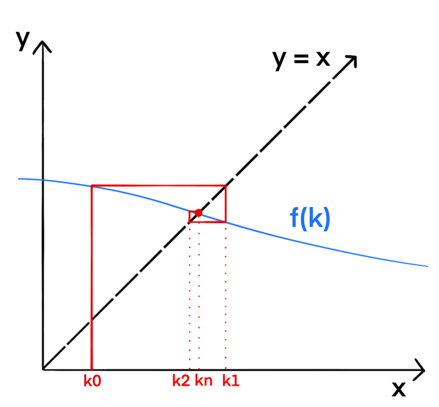
We use the simple fixed-point method

$$\vec{k}_i^{\{s+1\}} = \vec{f}_i(\vec{k}^{\{s\}})$$

until

$$\|\vec{k}_i^{\{s+1\}} - \vec{f}_i(\vec{k}^{\{s\}})\| \ll 1,$$

where the initial guess $\vec{k}_i^{\{0\}}$ is chosen to be the converged solution from the last Runge-Kutta time step. For the initial time $t = 0$ only, $\vec{k}_i^{\{0\}} = \vec{0}$.



Local optimization

Due to the locality of the space discretization, cells that have already converged are not computed anymore unless some of its surrounding cells are not converged.

Computational-cost

$$n_{\mathrm{KT}} = \frac{N_{\mathrm{KT}}}{N_{\mathrm{cell}}}$$

- N_{KT} : number of computation of $\vec{h}_{\mathrm{KT}}$
- N_{cell} : total number of cells

Quantities

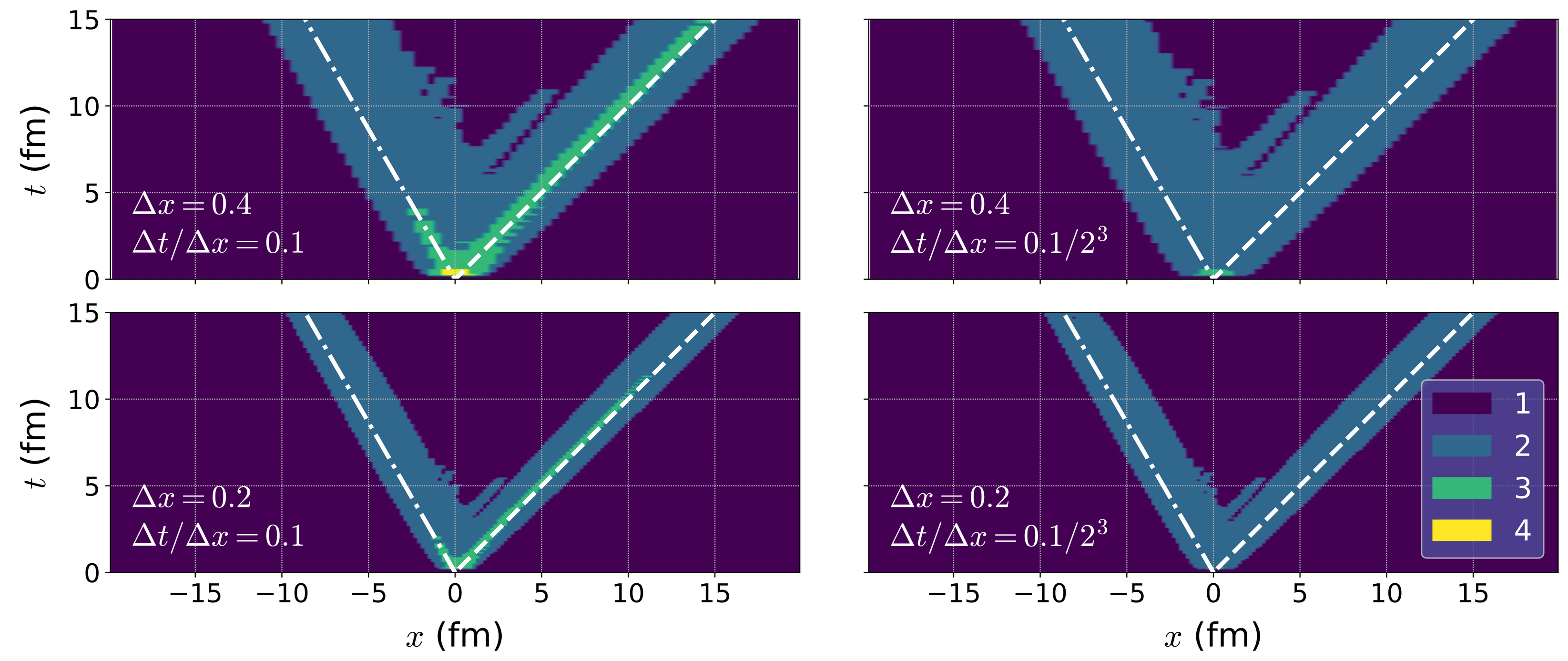
- ϵ_{num} : numerical result
- $\epsilon_{\mathrm{exact}}$: exact solution
- ϵ_{ref} : vanishing Δt numerical reference
- $\Delta_{\mathrm{exact}} = D(\epsilon_{\mathrm{num}}, \epsilon_{\mathrm{exact}})$
- $\Delta_{\mathrm{ref}} = D(\epsilon_{\mathrm{num}}, \epsilon_{\mathrm{ref}})$

$$D(e_1, e_2) = \frac{e_1 - e_2}{\max(|e_1|, |e_2|)}.$$

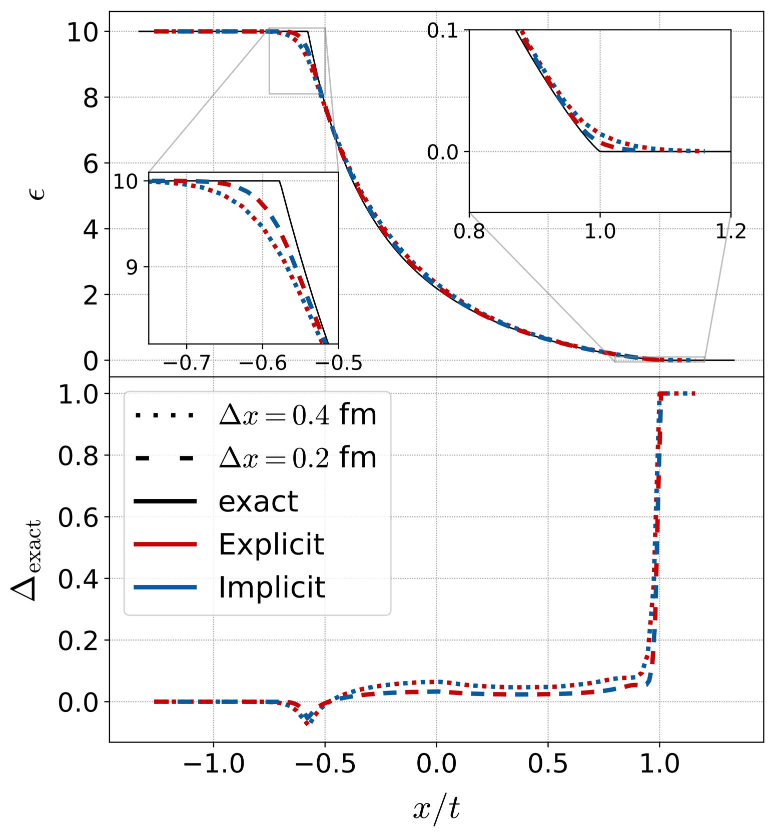
Summary

- We implemented an implicit method to numerically solve ideal relativistic hydrodynamics
- We use the Kurganov-Tadmor for space scheme and fixed-point solver with a local optimization.
- **Implicit method is more efficient than explicit**

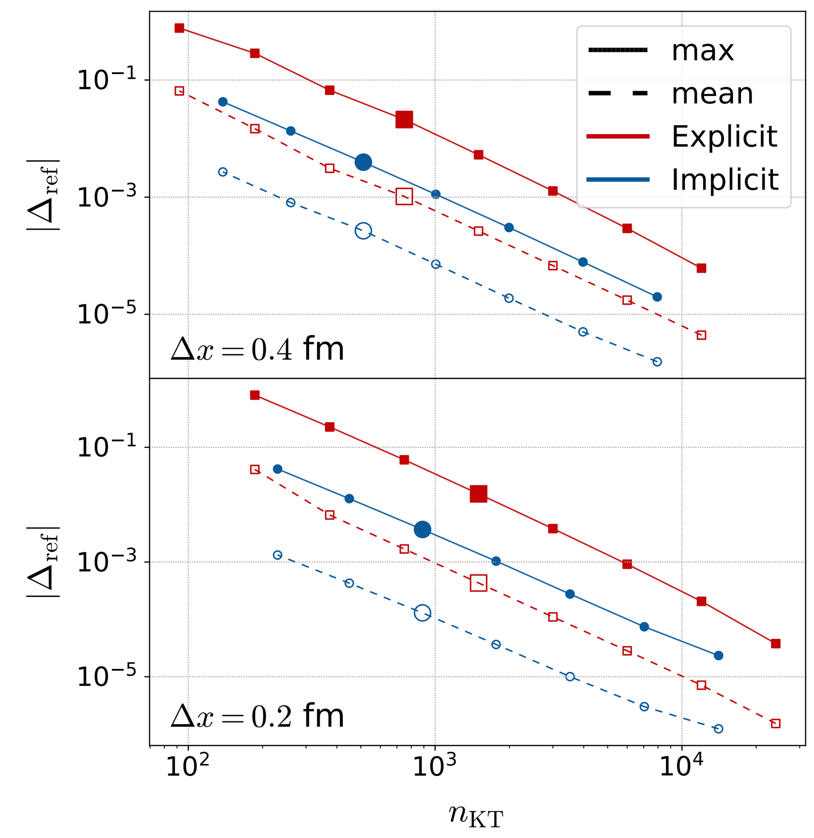
1+1d results for the Riemann problem



Number of iterations required for the fixed-point solver at each space-time point for the Riemann problem. The dash-dotted and dashed lines show the positions of rarefaction ($x = -c_s t$) and shock ($x = t$), respectively.

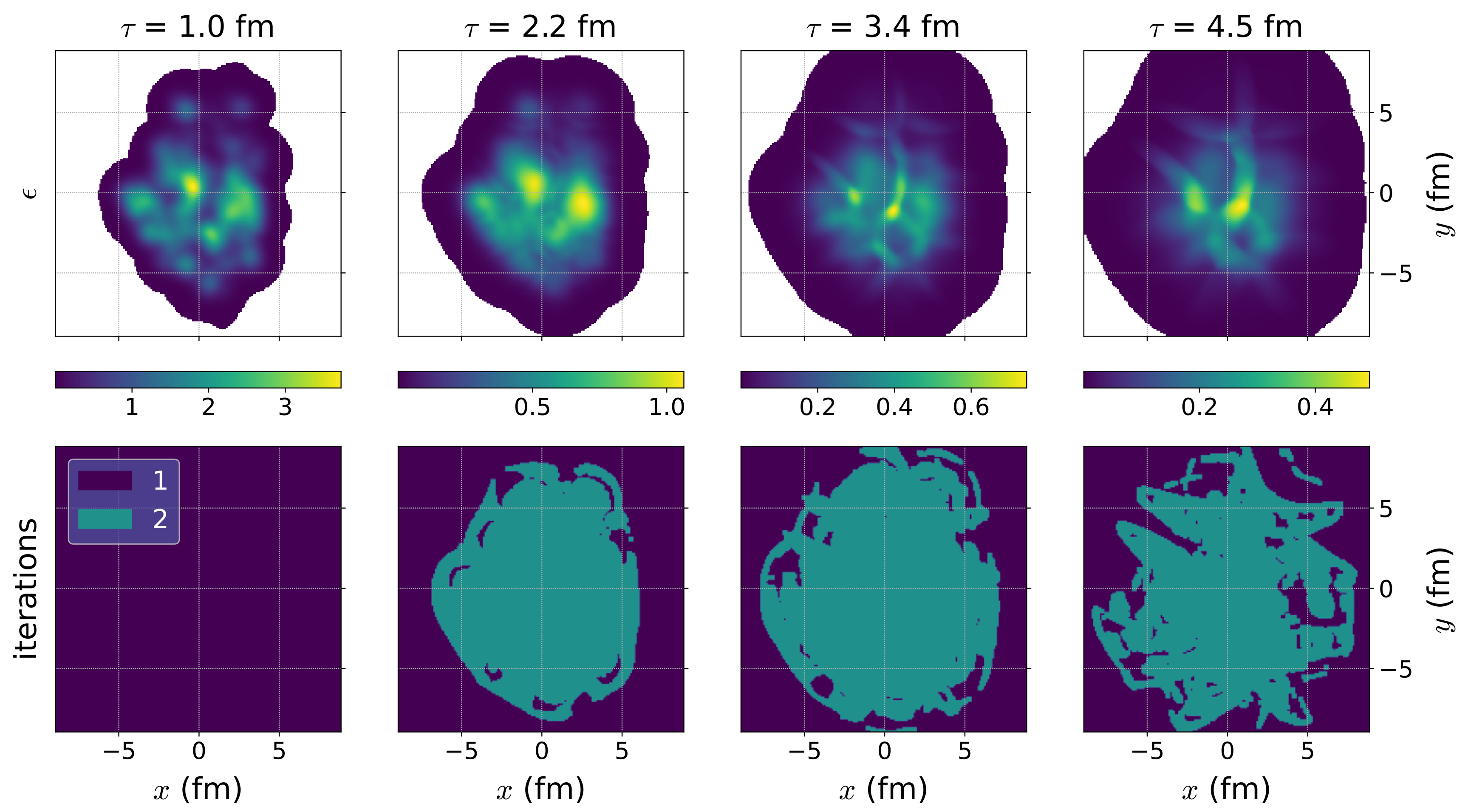


Energy density ϵ (upper panel) and its error compared with the exact solution Δ_{exact} (lower panel) for the Riemann problem at $t = 7.5$ fm with $\Delta t/\Delta x = 0.1 \times 2^{-6}$ fm.

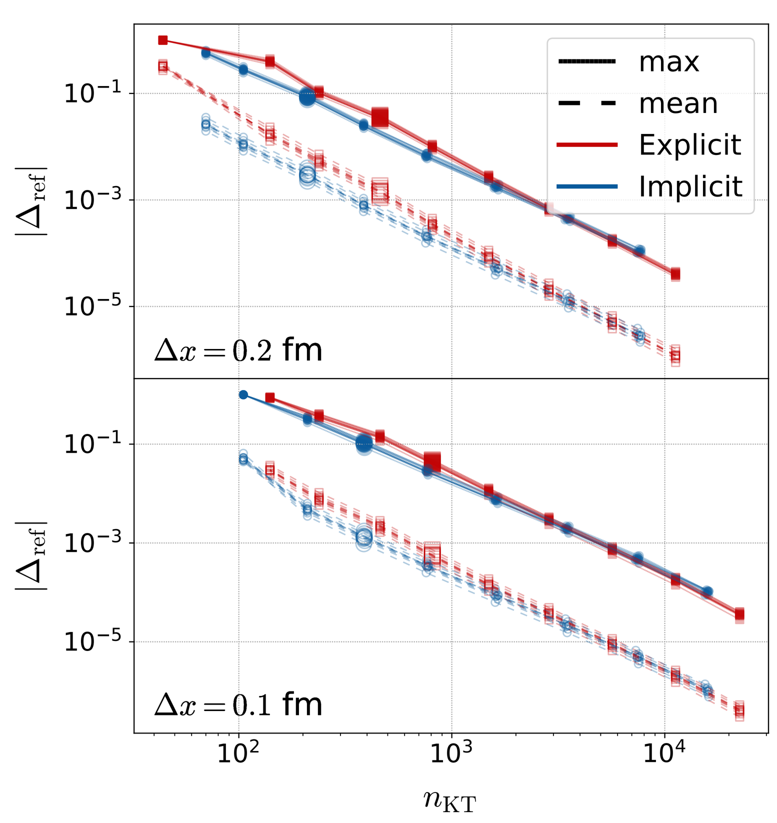


Maximum and average of the numerical error $|\Delta_{\mathrm{ref}}|$ as a functions of the computational cost n_{KT} for the Riemann problem. The large points show the results at $\Delta t = 0.1\Delta x$.

2+1d results for TRENTo heavy-ions initial condition



Numerical results of energy density ϵ (upper panel) and number of iterations of the fixed-point numerical solver in each cells (bottom panel) for $\Delta t = 0.1\Delta x \times 2^{-6}$ fm and $\Delta x = 0.2$ fm. The collision considered is a lead-lead collisions with impact parameter $b = 7$ fm, collision energy $\sqrt{s_{\mathrm{NN}}} = 2.76$ TeV.



Maximum and average of the numerical error $|\Delta_{\mathrm{ref}}|$ as a functions of the computational cost n_{KT} for 10 TRENTo events. The large points show the results at $\Delta t = 0.1\Delta x$.