Effect of event-by-event fluctuations on light-nuclei yield ratio

+ Non-trivial background (geometry + flow) effect

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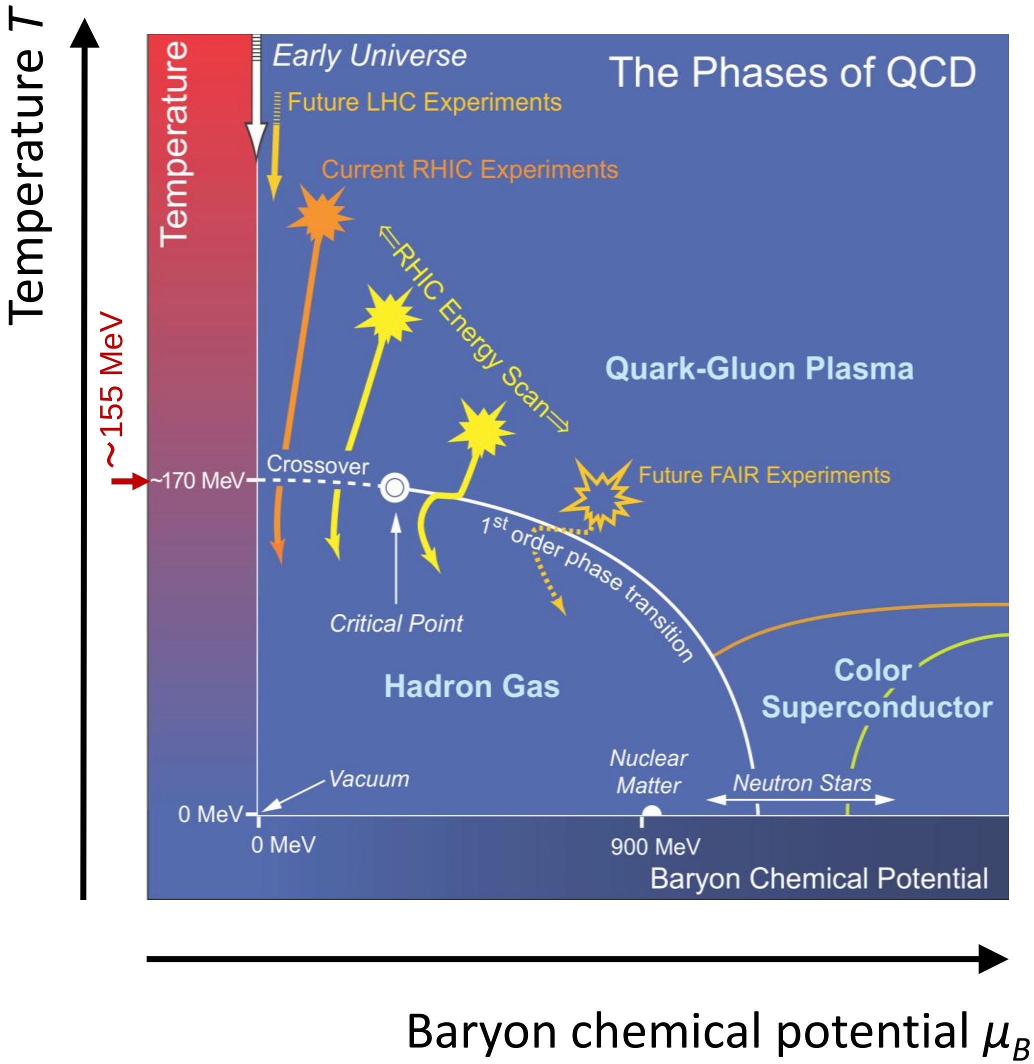
Critical-point search in high-energy heavy-ion collisions

Schematic QCD phase diagram

From the 2007 NSAC Long Range Plan



High density region ($\mu_B > 0$)

Sign problem in the lattice QCD

Effective models → Critical point (CP), 1st order phase transition?

Experiment: wide range of energies
RHIC Beam Energy Scan (BES),
FAIR, NICA, J-PARC-HI, HIAF, …

How to reliably identify the signals of CP in data?

Non-monotonic behaviors as functions of $\sqrt{s_{NN}}$

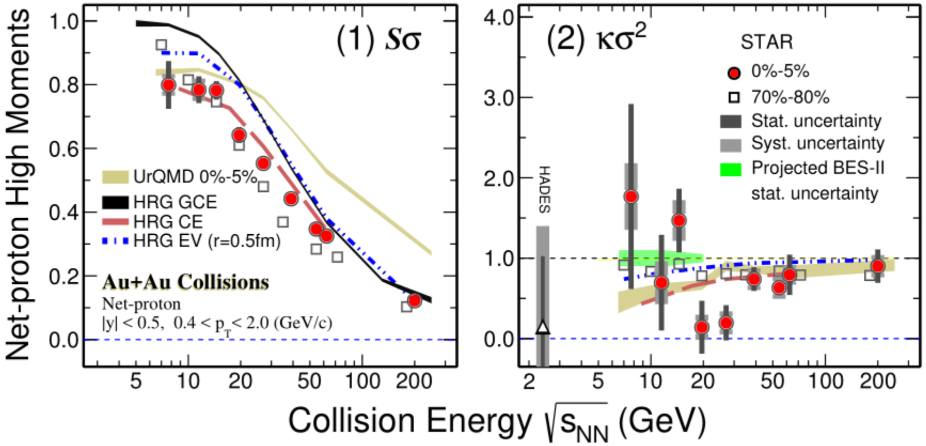
= Key to the search for critical point? ($\sqrt{s_{NN}} \leftrightarrow \mu_B$ … roughly corresponds)

Net-proton cumulant

∼ Net-baryon cumulant

∼ Susceptibilities $(\partial/\beta\partial\mu)^n \ln\Xi$

Stephanov, PRL **107**, 052301 (2011),
Kitazawa, Asakawa, PRC **86**, 024904 (2012),
STAR, PRL **126**, 092301 (2021)



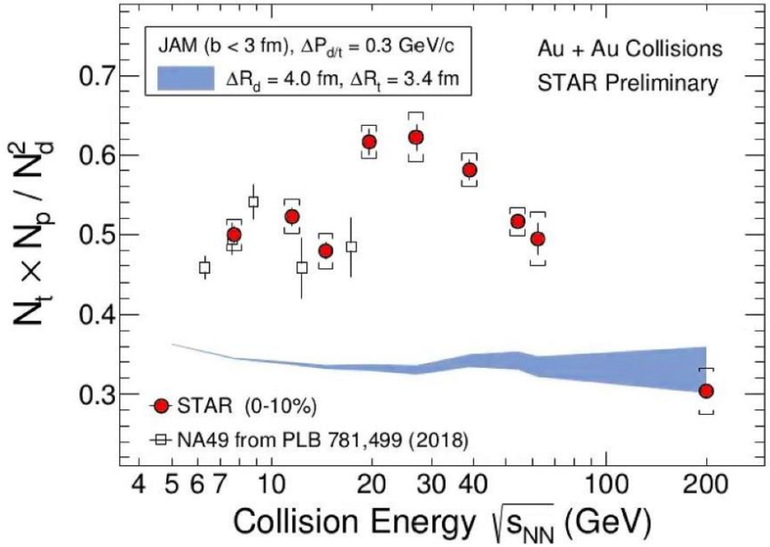
Light-nuclei yield ratio

Coalescence model

Cancels volume effect, etc. in ratio

$$N_t N_p / N_d^2 \sim \int \langle n(x) n(0) \rangle / \langle n \rangle^2$$

K-J Sun, L-W Chen, CM Ko, Z Xu, PLB **774**, 103 (2017),
H. Liu et al, PLB **805**, 135452 (2020)
[Recap. of NA49, STAR]



Model for yields of light nuclei in *this* study

Coalescence model (recombination model)

"Nucleus of mass number *A* is formed by *A*-nucleons close to one another"

$$N_A = g_A \int \left[\prod_i^A d^3\boldsymbol{r}_i d^3\boldsymbol{p}_i f_i(\boldsymbol{r}_i, \boldsymbol{p}_i) \right] W_A(\{\boldsymbol{r}_i, \boldsymbol{p}_i\}_{i=1}^A)$$

sudden frzout

Assumed to be

$W_A()$ … Wigner fn. corresponding to light-nucleus wave fn. (harmonic oscillator)

$f_i()$ … phase space distribution of *i*th nucleon g_A … statistical factor from spin

Relationship to the CP search

What is measured in experiments is event average
$\langle N_d \rangle \sim \langle \int_{xy} N_{p,x} N_{n,y} \rangle$, $\langle N_t \rangle \sim \langle \int_{xyz} N_{p,x} N_{n,y} N_{n,z} \rangle$

Note: Hereafter, we denote them as $\boldsymbol{N_A}$ omitting angle brackets **⟨…⟩**

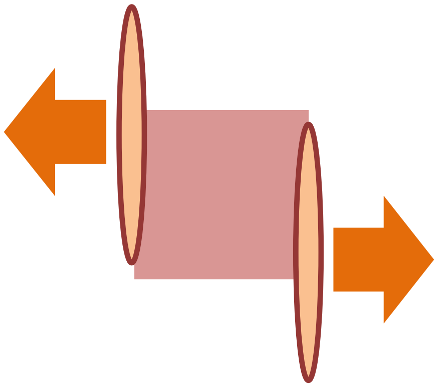
$\boldsymbol{N_d / N_p N_n}$ [aka $\boldsymbol{B_2(d)}$] … Contaminations including the non-trivial size effect

$\boldsymbol{N_t N_p / N_d^2} \sim \int_x \langle n(x) n(0) \rangle / \langle n \rangle^2$… proportional to critical correlation

K.-J. Sun, PLB **774**, 103 (2017), etc.

Realistic collision setup ≠ uniform/equilibrium

Various effects on the background *f*() in collision process



- **Expansion** → coordinate-momentum correlation
- **Initial event-by-event fluctuations**
  → Non-critical long range correlations
- **Non-flow effects (jets, hadron rescattering and decays)**
  → *f*() may not be in equilibrium or a fn of (T, μ, u^μ, …)
- **Finite size and finite time (Kibble-Zurek)**
  → Upper bound in critical correlation development

Q. How is $N_t N_p / N_d^2 \sim \int_x \langle n(x)n(0)\rangle / \langle n \rangle^2$ **affected?**

1. Non-trivial background distribution in a single event

$$f(\boldsymbol{r},\boldsymbol{p}) \equiv f_0(\boldsymbol{r},\boldsymbol{p}).$$

Assumption 1: No (critical) fluctuations
Assumption 2: isospin symmetry

2. Event-by-event fluctuations

$$f^{(1)}(\boldsymbol{r},\boldsymbol{p}), f^{(2)}(\boldsymbol{r},\boldsymbol{p}), f^{(3)}(\boldsymbol{r},\boldsymbol{p}), \ldots.$$

$$N_A^{(1)}, N_A^{(2)}, N_A^{(3)}, \ldots \to \langle N_A^{(e)} \rangle$$

Event average

using **Toy model**

1. NON-TRIVIAL BACKGROUND DISTRIBUTION IN A SINGLE EVENT

S. WU, K. MURASE, S. TANG AND H. SONG, PHYS. REV. C 106 (2022), 034905.

Example: Simple Gaussian *f*()

K.-J. Sun et al, PLB792, 132

1. (input) Distribution function

$$f(\boldsymbol{r},\boldsymbol{p}) = \frac{\rho_0}{(2\pi mT)^{3/2}} \exp\Big(-\frac{\boldsymbol{r}^2}{2R_s^2}\Big) \exp\Big(-\frac{\boldsymbol{p}^2}{2mT}\Big)$$

2. (result) Integrate the coalescence formula: $N_A \sim \int W_A \, \Pi^A f_i$

$$N_d = g_d N_p^2 \Big[\Big(R_s^2 + \frac{\sigma_d^2}{2}\Big)\Big(mT + \frac{1}{2\sigma_d^2}\Big)\Big]^{-\frac{3}{2}}$$

$$N_t = g_t N_p^3 \Big[\Big(R_s^2 + \frac{\sigma_t^2}{2}\Big)\Big(mT + \frac{1}{2\sigma_t^2}\Big)\Big]^{-3}.$$

Note: light-nuclei sizes σ_d=2.26, σ_t=1.59 fm

If the size difference is negligible:

$$N_t N_p / N_d^2 = g_t / g_d^2 = 4/9$$

The effects from Gaussian profile exactly cancels!

→ **Investigate effect of general *f*() with non-Gaussian components**

Idea for *non-Gaussian effect:* Expand *f()* in terms of *cumulants*

Input

$$\frac{f(\boldsymbol{z}_i)}{N_p} = \rho(\boldsymbol{z}_i) = \int \frac{d^6\boldsymbol{t}_i}{(2\pi)^6} e^{-\mathrm{i}\boldsymbol{t}_i\cdot\boldsymbol{z}_i} \exp\left[\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^6} \frac{\mathcal{C}_{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}(\mathrm{i}\boldsymbol{t}_i)^{\boldsymbol{\alpha}}\right]$$

Expand in "phase-space cumulants" C_α [$z \sim (r, p)$]

Cumulants up to 2nd order → generate Gaussian distribution

Higher order cumulants → the deviation from the Gaussian?

Result for the yield N_A for general A

Lowest order (2nd order cumulants)

$$N_A^{(0)} \equiv g_A N_p^A 8^{A-1} [\det(\mathcal{C}_2 + \mathcal{I}_6)]^{-(A-1)/2}$$

independent of A

Higher order may be taken into account by the perturbations

$$H_A \equiv \frac{N_A}{N_A^{(0)}} = 1 + \int \left\{ \prod_{i=1}^{A-1} \frac{d^6\boldsymbol{T}_i \exp[-\frac{1}{2}\boldsymbol{T}_i(\mathcal{C}_2 + \mathcal{I}_6)\boldsymbol{T}_i]}{\sqrt{(2\pi)^6 \det(\mathcal{C}_2 + \mathcal{I}_6)^{-1}}} \right\} \times \sum_{m=1}^{\infty} \frac{[\mathcal{H}(\{\boldsymbol{T}_i\}_{i=1}^{A-1})]^m}{m!} = 1 + \mathcal{O}(\{\mathcal{C}_{\boldsymbol{\alpha}}\}_{|\boldsymbol{\alpha}|\geq 3}),$$

$$\mathcal{H}(\{\boldsymbol{T}_i\}_{i=1}^{A-1}) \equiv \sum_{k=3}^{\infty} \mathcal{H}_k(\{\boldsymbol{T}_i\}_{i=1}^{A-1}) \equiv \sum_{k=3}^{\infty} \sum_{i=1}^{A} \sum_{|\boldsymbol{\alpha}|=k} \frac{\mathcal{C}_{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} [\mathrm{i}(O^{\mathrm{T}}\boldsymbol{T})_i]^{\boldsymbol{\alpha}} \Big|_{\boldsymbol{T}_A=0}$$

Generalized yield ratio

Generalization of $N_t N_p / N_d^2$

$$R_{A_1,\ldots,A_{n_k}}^{p_1,\ldots,p_{n_k}} \equiv N_p^{-\sum_{k=1}^{n_k} p_k} \prod_{k=1}^{n_k} N_{A_k}^{p_k}$$

$$= \left(\prod_{k=1}^{n_k} g_{A_k}^{p_k}\right)[1 + \mathcal{O}(\{\mathcal{C}_{\boldsymbol{\alpha}}\}_{|\boldsymbol{\alpha}|\geq 3})]$$

subject to

$$\sum_{k=1}^{n_k} p_k(A_k - 1) = 0$$

+ (additional constraints without isospin sym)

E.g. $n_k = 2$ (two nuclei)

$$R_{A,B}^{1-B,A-1} = \frac{N_p^{B-A} N_B^{A-1}}{N_A^{B-1}} = \frac{g_B^{A-1}}{g_A^{B-1}}[1 + \mathcal{O}(\{\mathcal{C}_{\boldsymbol{\alpha}}\}_{|\alpha|\geq 3})].$$

Background effects cancel up to the 2nd order of cumulants

Note: Effect of matter expansions

→ Coordinate-momentum (r-p) correlations in $f()$

→ Canceled in the ratios up to 2nd order

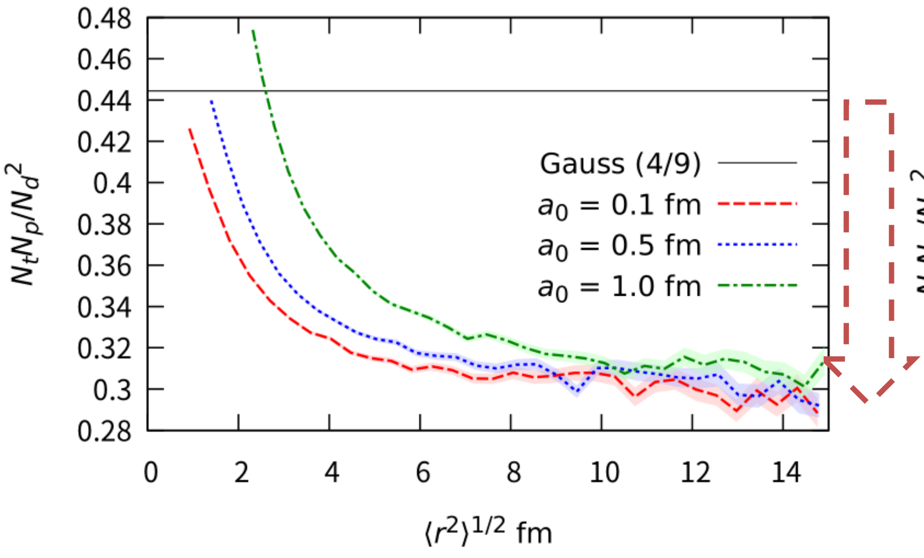
$$\mathcal{C}_2 = 2\begin{pmatrix} \frac{\langle \boldsymbol{r}\boldsymbol{r}^{\mathrm{T}}\rangle}{\sigma^2} & \langle \boldsymbol{r}\boldsymbol{p}^{\mathrm{T}}\rangle \\ \langle \boldsymbol{p}\boldsymbol{r}^{\mathrm{T}}\rangle & \sigma^2\langle \boldsymbol{p}\boldsymbol{p}^{\mathrm{T}}\rangle \end{pmatrix}$$

Case 1: Woods-Saxon (non-Gaussian case)

Input

$$f(\boldsymbol{r},\boldsymbol{p}) = \frac{\rho_{\rm WS}}{1+\exp\frac{r-R_0}{a_0}} \cdot \frac{1}{(2\pi mT)^{3/2}} \exp\left(-\frac{\boldsymbol{p}^2}{2mT}\right)$$

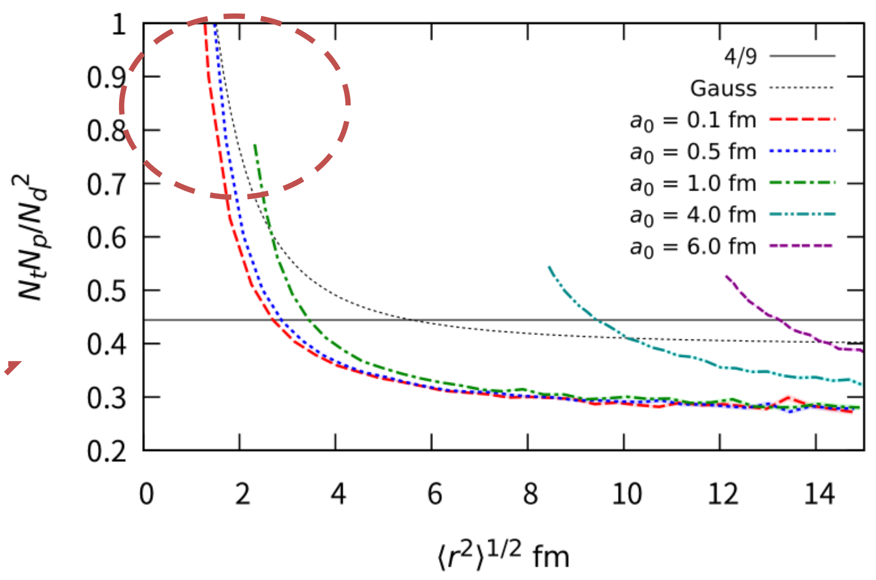
Result (common *σ* = 1.59 fm)



The Woods-Saxon distribution decreases the ratio

Effect of different nucleus-size

(physical *σ* = 2.23, 1.59 for *d*, *t*)



The different nucleus-size effect is significant when fireball size is close to *σ*

✔ *spatial structure decreases the ratio*

Case 2: Anisotropic flow (Blast-wave)

Input

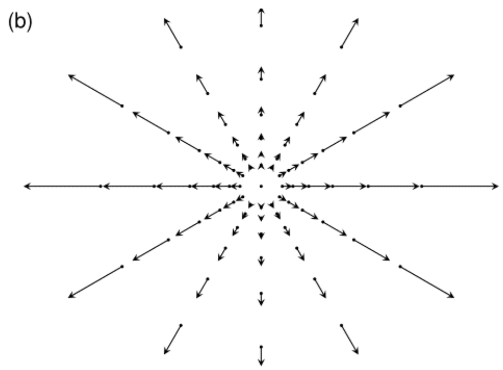
$$f(\boldsymbol{r},\boldsymbol{p}) = \frac{\rho_0}{(2\pi m T)^{3/2}} e^{-\frac{\boldsymbol{r}^2}{2R_s^2}} \exp\left(-\frac{m}{2T}\left[\frac{\boldsymbol{p}}{m} - \boldsymbol{v}(\boldsymbol{r})\right]^2\right),$$

$$\boldsymbol{v}(\boldsymbol{r}) = \frac{1}{R_s}(r_x, r_y, 0)^{\mathrm{T}}(1 + 2u_n \cos n\phi_s)$$

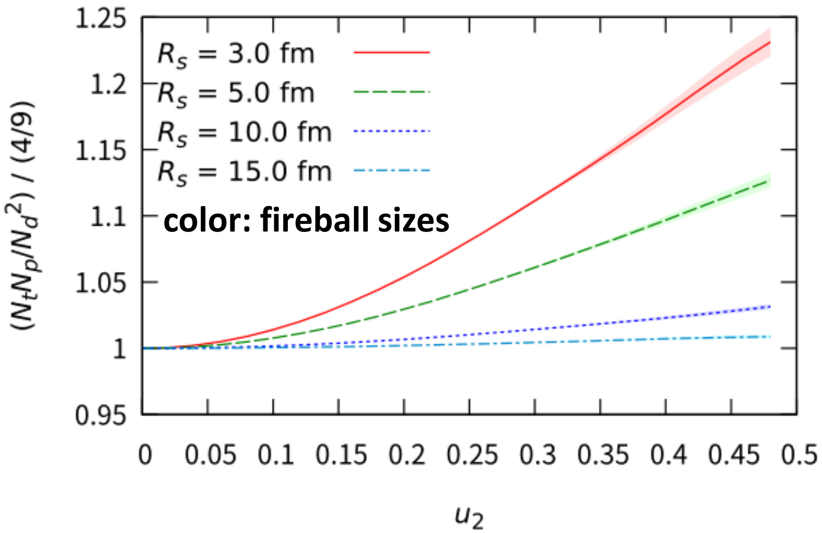
anisotropy param (u_n)

Blast-wave flow P. Huovinen et al, PLB 503, 58 (2001)

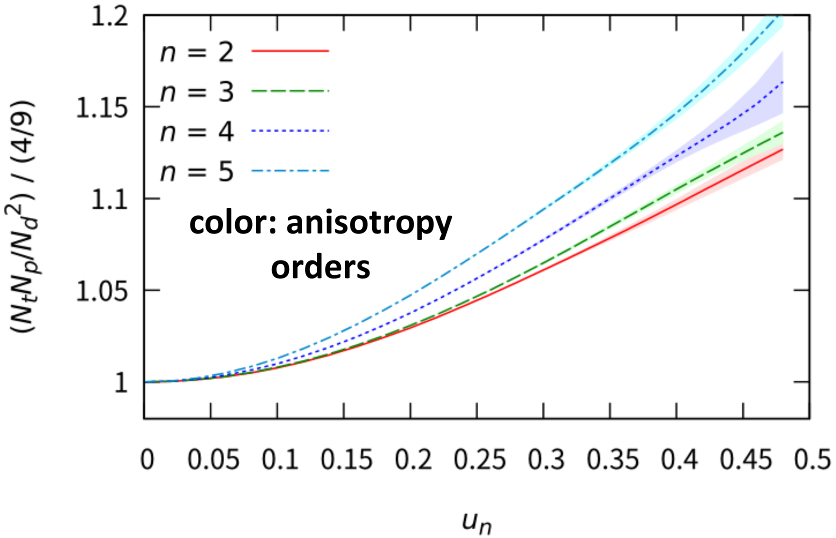
Flow profile for *n* = 2



Ratio vs anisotropy parameter (*n*=2)



Ratio vs anisotropy parameter (R_s=5)



The effect is negligible when the fireball size is large

Centrality dependence?

Momentum anisotropy tends to increase the ratio

2. EVENT-BY-EVENT FLUCTUATIONS

K. MURASE, S. WU, IN PREP.

Model for event-by-event distributions

Single event: Sum of *n-Gaussian "hot spots"*

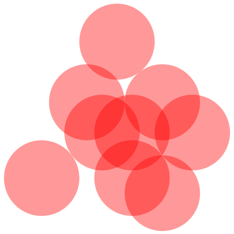
$$f(\boldsymbol{z}) = \frac{1}{n}\sum_{h=1}^{n} c_h G(\boldsymbol{z}-\boldsymbol{a}_h; \mathcal{C}_2^{\mathrm{hs}})$$

n Number of hot spots

$\mathcal{C}_2^{\mathrm{hs}}$ Size/Shape of hot spots in phase space (6x6)

$\boldsymbol{a}_h$ Center of hot spot *h*

c_h Magnitude of hot spot *h*



Gaussian

$$G(\boldsymbol{z}-\boldsymbol{a}; \mathcal{C}_2^{\mathrm{hs}}) := \frac{1}{\sqrt{\det(2\pi\mathcal{C}_2^{\mathrm{hs}})}} \exp\left[-\frac{1}{2}(\boldsymbol{z}-\boldsymbol{a})^{\mathrm{T}}(\mathcal{C}_2^{\mathrm{hs}})^{-1}(\boldsymbol{z}-\boldsymbol{a})\right].$$

$z \sim \sqrt{2}(r/\sigma, p\sigma)$
phase space

Event-by-event distribution of distribution function by **fluctuating** (c_i, a_i).

Gaussian

$$\Pr(\{c_h, \boldsymbol{a}_h\}_{h=1}^{n}) = \prod_{h=1}^{n} p(c_h) \frac{e^{-\frac{1}{2}\boldsymbol{a}_h^{\mathrm{T}}(\mathcal{C}_2^{\mathrm{hc}})^{-1}\boldsymbol{a}_h}}{\sqrt{\det(2\pi\mathcal{C}_2^{\mathrm{hc}})}}$$

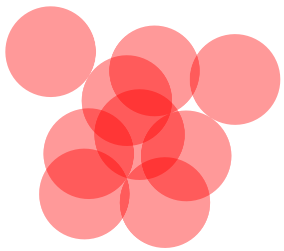
$\mathcal{C}_2^{\mathrm{hc}}$ Covariance of hot-spot centers

Result: Single-event yield

Single event yield by ∫ f^A W $d^A z$

$$N_A^f = g_A 8^{A-1} \det(\mathcal{C}_2^{\mathrm{hs}}+1)^{-(A-1)/2} n^{-A} \times \sum_{h_1,\ldots,h_A=1}^{n} \left(\prod_{i=1}^{A} c_{h_i}\right) e^{-\frac{1}{2}\sum_{i=1}^{A-1} \boldsymbol{A}_i^{\mathrm{T}}(\mathcal{C}_2^{\mathrm{hs}}+1)^{-1}\boldsymbol{A}_i}$$

gaussian form



$$\boldsymbol{A}_i(h_1,\ldots,h_A) := \sum_{j=1}^{A} O_{ij}\boldsymbol{a}_{h_j}$$

~ Gaussian as a function of hot-spot positions *a*

→ Perform event average with probability density

N_A = ∫ d*a*d*c* Pr(*a*,*c*) × N_a^f

$$\Pr(\{c_h, \boldsymbol{a}_h\}_{h=1}^n) = \prod_{h=1}^{n} p(c_h) \frac{e^{-\frac{1}{2}\boldsymbol{a}_h^{\mathrm{T}}(\mathcal{C}_2^{\mathrm{hc}})^{-1}\boldsymbol{a}_h}}{\sqrt{\det(2\pi\mathcal{C}_2^{\mathrm{hc}})}}$$

Result: Yield

Yield (Event averaged)

$$N_A = g_A 8^{A-1} \det(\mathcal{C}_2^{\mathrm{hs}} + 1)^{-(A-1)/2}$$
$$\times \sum_{\substack{m_1 \geq \cdots \geq m_n \geq 0 \\ \sum_{h=1}^{n} m_h = A}} \frac{n!A!}{n^A S_{m_1,\ldots,m_n}} \frac{I_{m_1} \cdots I_{m_n}}{\prod_\lambda \left(\sum_{h=1}^{n} \frac{m_h}{A} \frac{1}{1+m_h\lambda}\right)^{1/2}},$$

$$I_m = \frac{1}{\prod_\lambda (1+m\lambda)^{1/2}} \int dc\, p(c) c^m$$ moments of *c*

$$S_{m_1,\ldots,m_n} = \prod_m a_m! (m!)^{a_m}$$ symmetry factor

λ eigenvalues of $\mathcal{C}_2^{\mathrm{hc}} (\mathcal{C}_2^{\mathrm{hs}} + 1)^{-1}$

yield is determined by this factor

$$\sum_{\substack{m_1 \geq \cdots \geq m_n \geq 0 \\ \sum_{h=1}^{n} m_h = A}}$$ sum for classification of *A* items into *n*

Result: Yield

Yield (Event averaged)

$$N_A = g_A 8^{A-1} \det(\mathcal{C}_2^{\mathrm{hs}} + 1)^{-(A-1)/2} \times \sum_{\substack{m_1 \geq \cdots \geq m_n \geq 0 \\ \sum_{h=1}^{n} m_h = A}} \frac{n! A!}{n^A S_{m_1,\ldots,m_n}} \frac{I_{m_1} \cdots I_{m_n}}{\prod_\lambda \left(\sum_{h=1}^{n} \frac{m_h}{A} \frac{1}{1+m_h \lambda} \right)^{1/2}},$$

$$I_m = \frac{1}{\prod_\lambda (1+m\lambda)^{1/2}} \int dc\, p(c) c^m$$ moments of c

$$\lambda \sim \frac{\text{Cov[fireball]}}{\text{Cov[nucleus]} + \text{Cov[hotspot]}}$$

$S_{m_1,\ldots,m_n} = \prod_m a_m(m)\ldots$ symmetry factor

λ eigenvalues of $\mathcal{C}_2^{\mathrm{hc}} (\mathcal{C}_2^{\mathrm{hs}} + 1)^{-1}$

yield is determined by this factor

$\sum_{\substack{m_1 \geq \cdots \geq m_n \geq 0 \\ \sum_{h=1}^{n} m_h = A}}$ sum for classification of A items into n

Result: Yield

Examples A=2 (deuteron) & A=3 (tritons)

$$N_d = \frac{8g_d}{\det(\mathcal{C}_2^{\rm hs}+1)^{1/2}}\frac{1}{n}\left[\langle c^2\rangle + \frac{(n-1)\langle c\rangle^2}{\prod_\lambda(1+\lambda)^{1/2}}\right]$$

$$N_t = \frac{8^2 g_d}{\det(\mathcal{C}_2^{\rm hs}+1)}\frac{1}{n^2}\left[\langle c^3\rangle + \frac{3(n-1)\langle c^2\rangle\langle c\rangle^2}{\prod_\lambda(1+\frac{4}{3}\lambda)^{1/2}} + \frac{(n-1)(n-2)\langle c\rangle^3}{\prod_\lambda(1+\lambda)}\right].$$

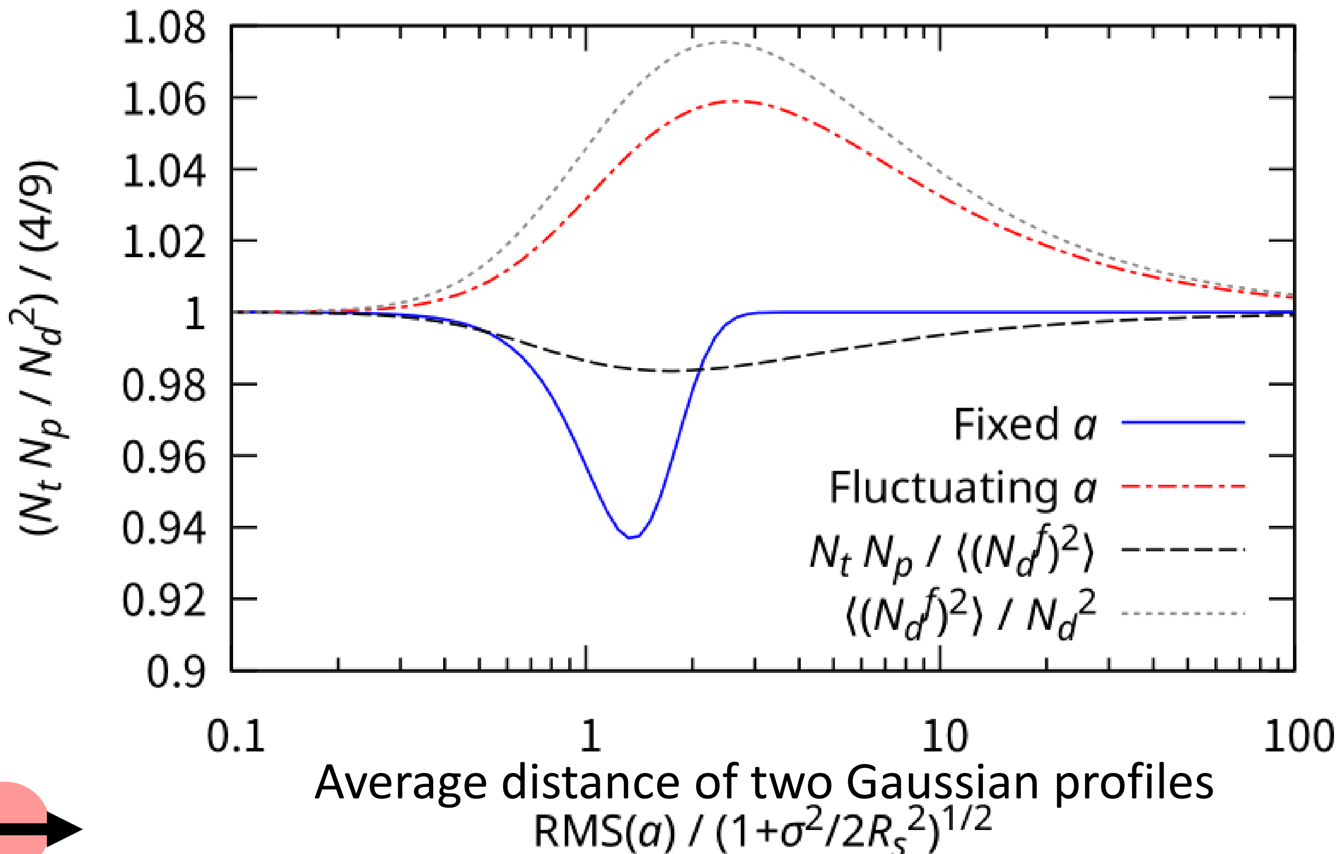
Calculate yield ratio N_pN_t/N_d^2
→ Change parameters
→ Check qualitative behavior

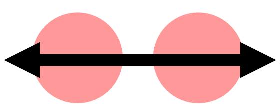
N_tN_p/N_d^2 with two Gaussian in 1d

***n*=1: single Gaussian Ratio = 4/9**

***n*=∞: infinite number of hot spots Ratio = 4/9** (fluctuations smeared out)

***n*=2: double Gaussian case** (no fluctuations in magnitude *c*)



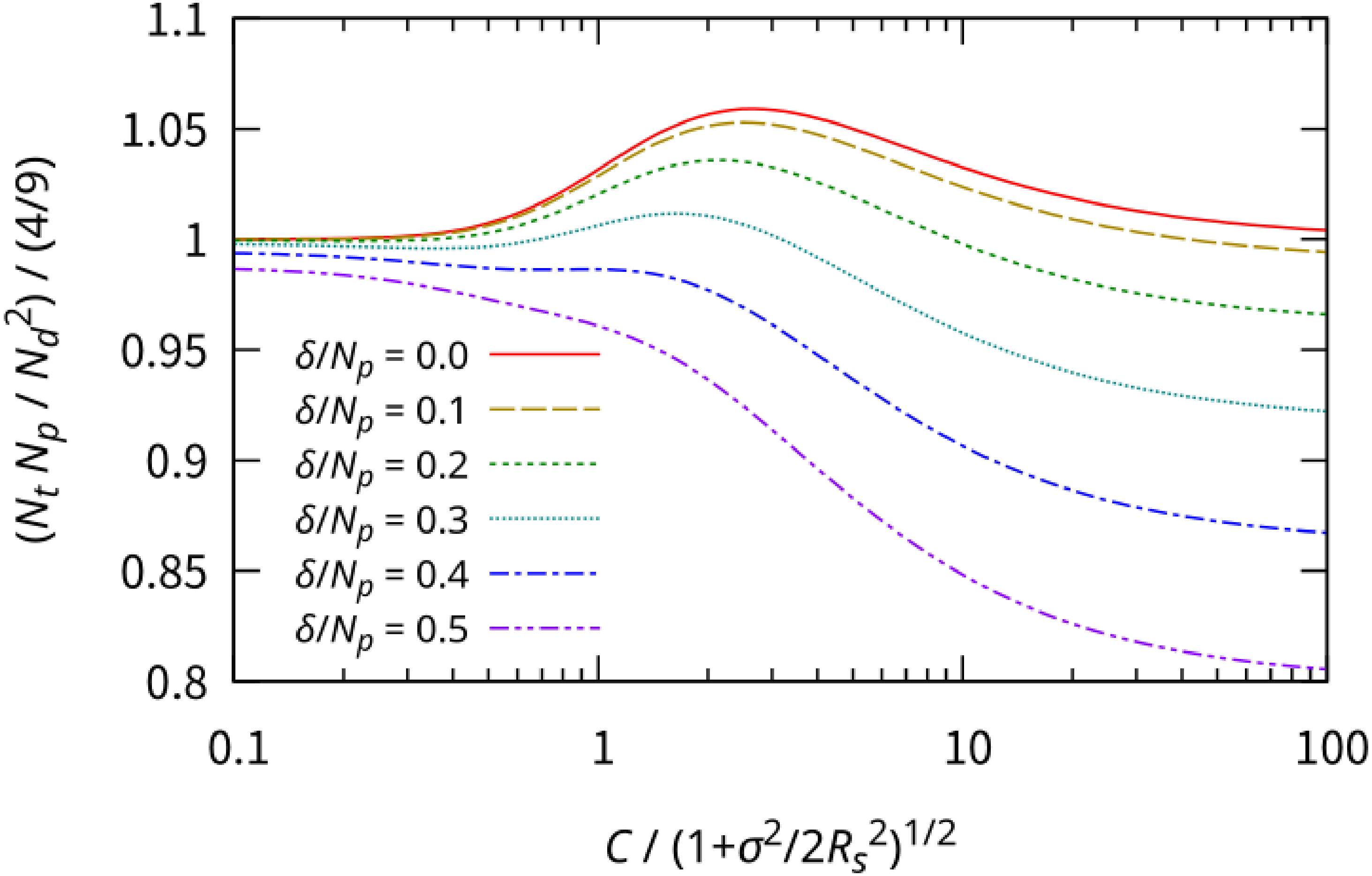


✔ *Position fluctuations increase the ratio*

$N_t N_p / N_d^2$ with two Gaussian in 1d

n*=2: with magnitude fluctuations in *c

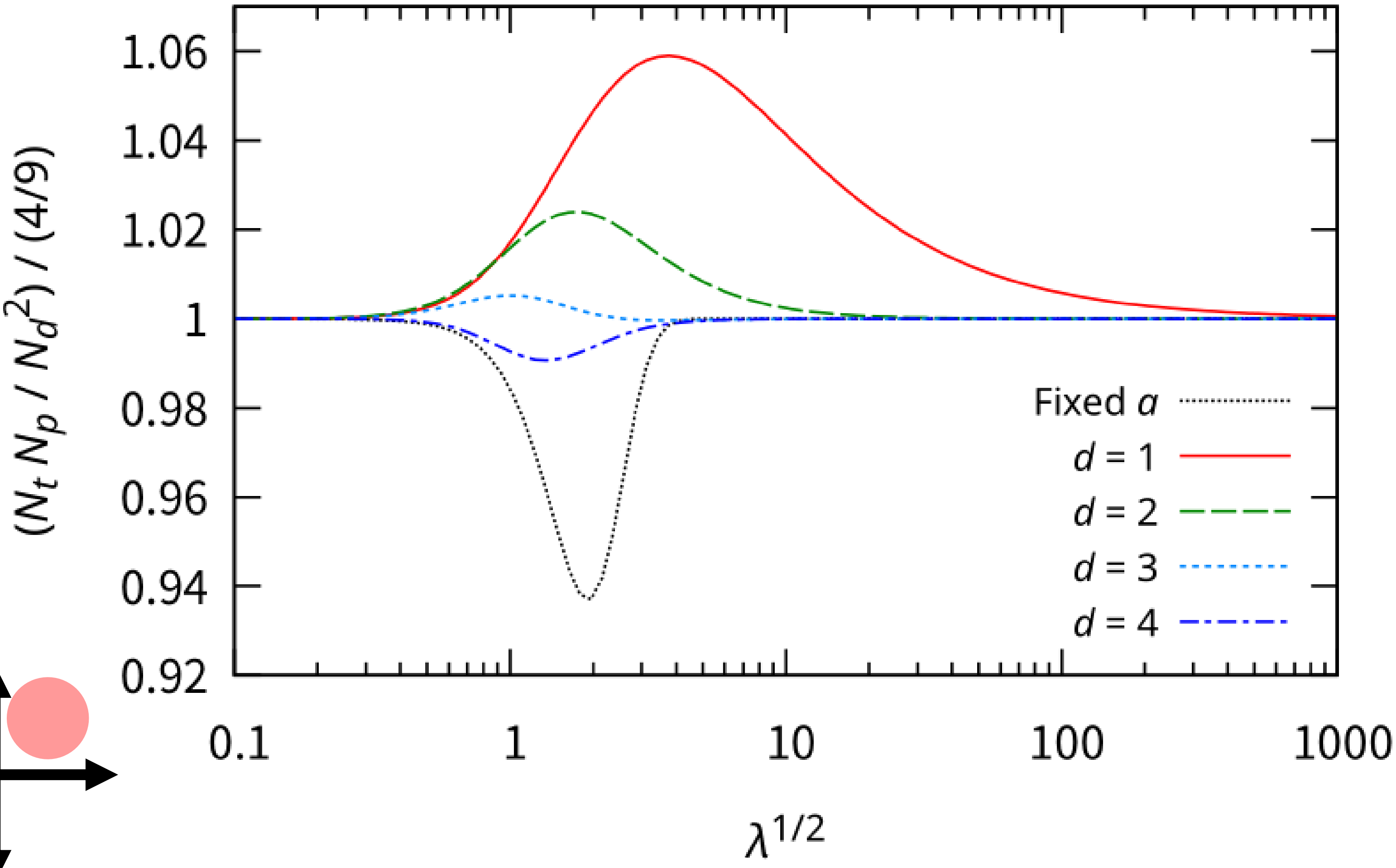
c: uniform distribution in $[N_p - \delta, N_p + \delta]$

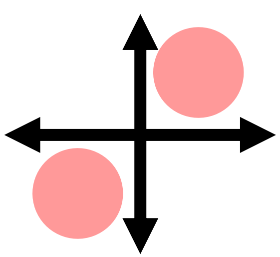


✔ *Magnitude fluctuations decrease the ratio*

$N_t N_p / N_d^2$ with two Gaussian

n*=2: dimensionality of fluctuation directions *d (rank of C_2^{hc})

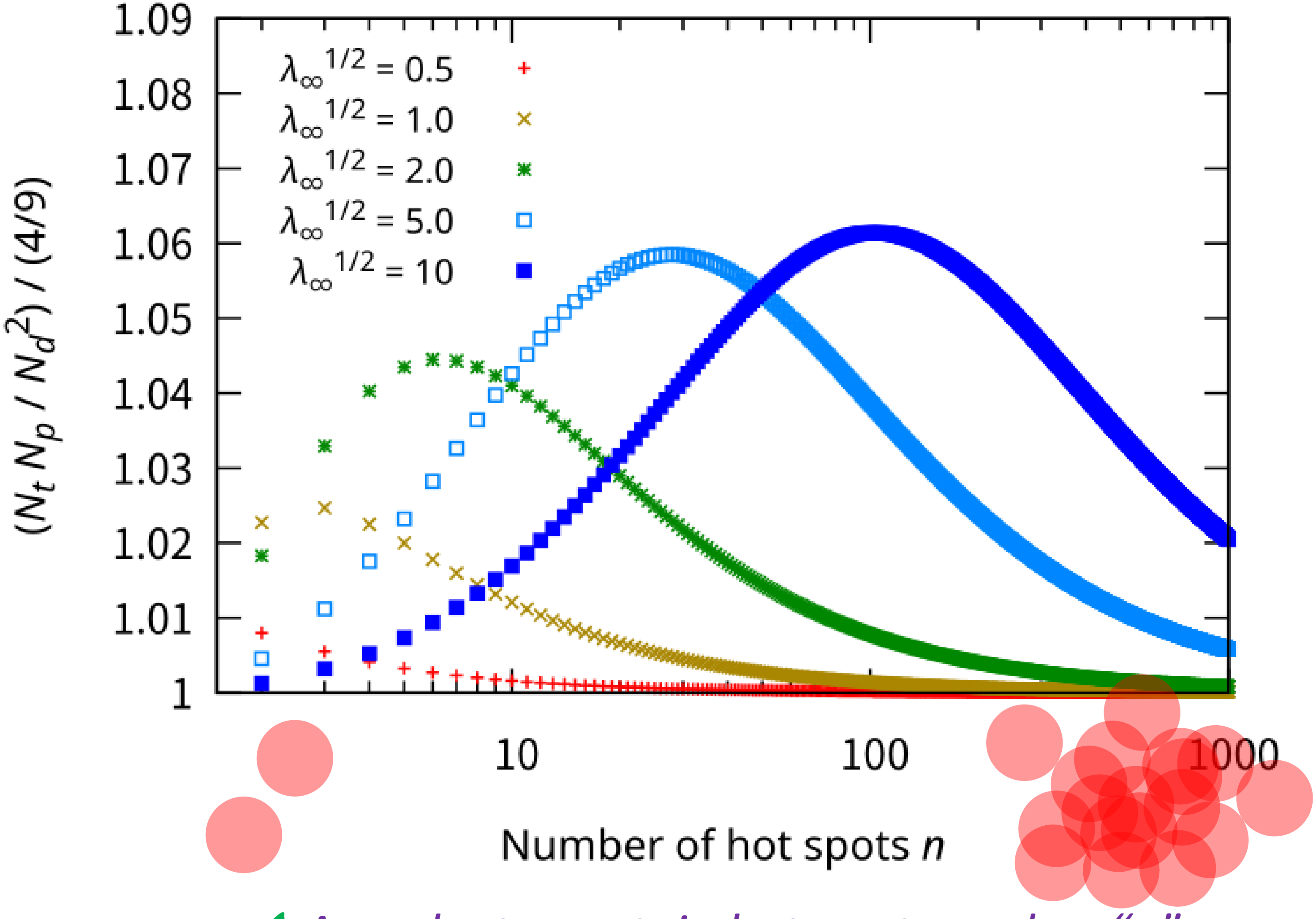




✔ *The increase largely depends on the dimensionality*

N_tN_p/N_d^2 with n Gaussian

Change n ~ fluctuation granularity (λ adjusted to fix the fireball size as λ_∞)



✔ *A peak at a certain hot-spot number "n"*

N_tN_p/N_d^2 with two fluctuation sources

f **~ sum of two fluctuations sources with different hot-spot sizes C^{hs1} & C^{hs2}**

$$f = f_1 + f_2,$$

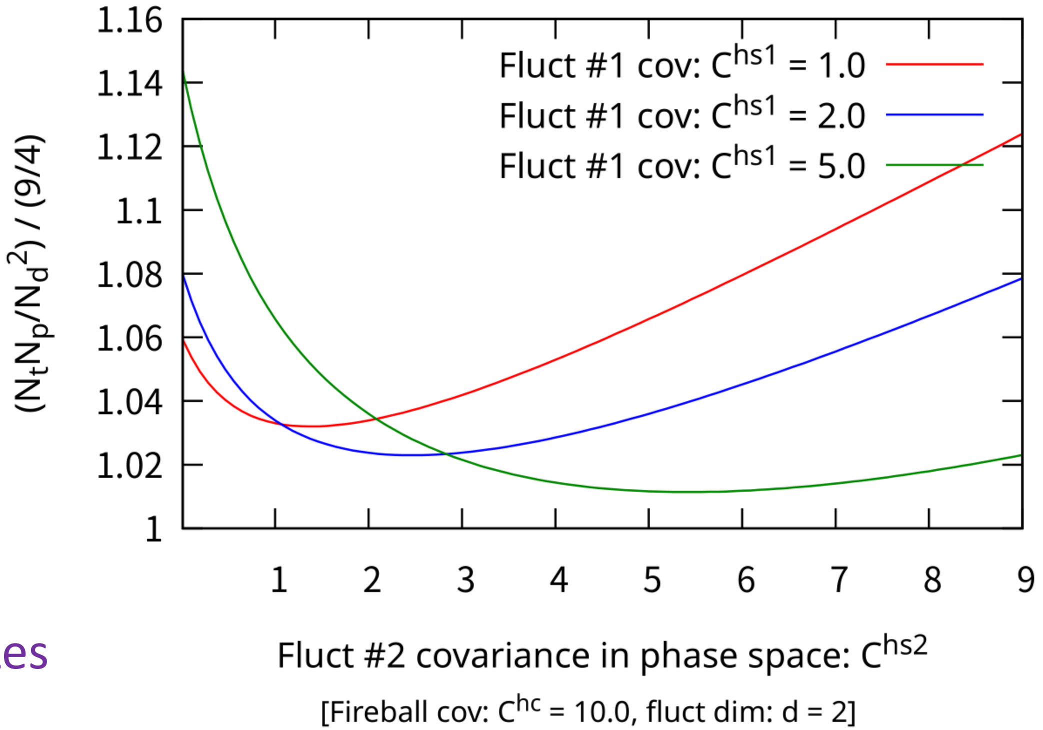
$$f_1 = \frac{1}{n_1}\sum_{h=1}^{n_1} c_h G_1(\boldsymbol{z} - \boldsymbol{a}_h; \mathcal{C}^{\text{hs1}}),$$

$$f_2 = \frac{1}{n_2}\sum_{h=1}^{n_2} c'_h G_2(\boldsymbol{z} - \boldsymbol{a}'_h; \mathcal{C}^{\text{hs2}}),$$

E.g. critical fluctuations *vs* shorter-scale thermal fluctuations

Insufficient scale separation in HIC
→ Can we differentiate them?

✔ The effect becomes larger when two sources have different sizes



Summary

- Yield ratio in heavy-ion is a candidate observable for the CP search
- Toy models with coalescence for a qualitative understanding
- Non-trivial background effect with phase-space cumulants
- Event-by-event fluctuations by *n*-Gaussian distribution function
  - Analytic formulae for yields
  - Position fluctuations increases the ratio.
  - Magnitude fluctuations decrease the ratio
  - The ratio takes a peak at a certain hot-spot number *n*
  - Different sizes of two fluctuation sources increase the ratio

Outlook

- Critical effect with critical correlators based on the Ising mapping
- HBT, Interpretation of the results in dynamical models, etc.