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Quantum information and CP measurement in $H \to \tau^+\tau^-$ at future high energy lepton colliders

arXiv:2211.10513

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Bohr vs EPR

"Niels Bohr: argued that reality or the state of a particle at the fundamental level was not only unknown but was unknowable until it was measured."

"If, without in any way disturbing a system, we can predict with certainty (i.e. with probability equal to unity) the value of a physical quantity, then there exist an element of physical reality corresponding to this physical quantity"

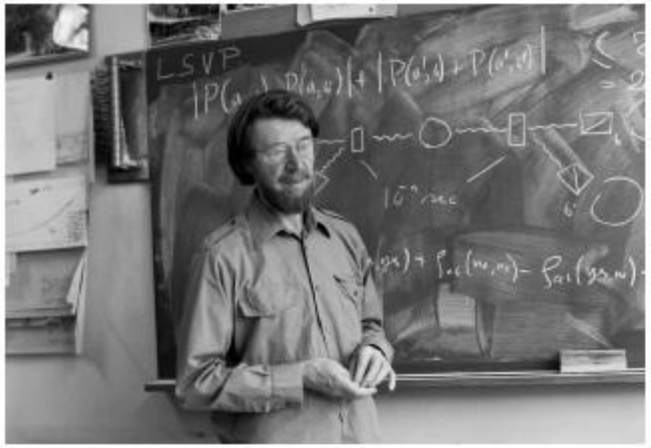
Einstien, Podolski and Rosen, 1935

QM violates both local and real requirements (i.e. entanglement violate Locality).
And QM already tested by Stern Gerlach Experiment.

As per EPR, the QM behavior could be explained by additional variables called Local Hidden variables (LHV). These would restore locality and causality to the theory (and they demonstrated it for the Stern Gerlach experimental observations).

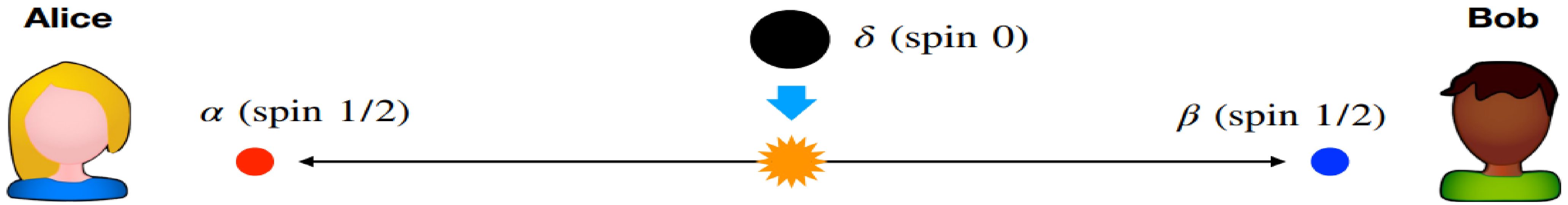
It seems difficult to experimentally discriminate QM and general hidden variable theories.

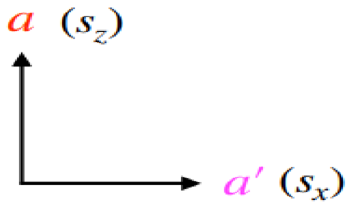
John Bell (1964) derived simple inequalities that can discriminate QM from any local-real hidden variable theories: **Bell inequalities**



CHSH inequality

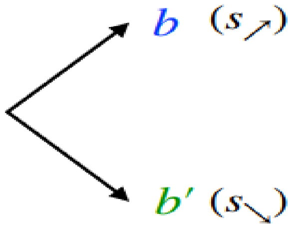
[Clauser, Horne, Shimony, Holt, 1969]





The experiment consists of 4 sessions:

1) Alice and Bob measure $s_a[\alpha]$ and $s_b[\beta]$, respectively. Repeat the measurement many times and calculate $\langle s_a \cdot s_b \rangle$.
2) Repeat (1) but for a and b'.
3) Repeat (1) but for a' and b.
4) Repeat (1) but for a' and b'.



Finally, we construct

$$R_{\mathrm{CHSH}} \equiv \frac{1}{2} \left| \langle s_a s_b \rangle - \langle s_a s_{b'} \rangle + \langle s_{a'} s_b \rangle + \langle s_{a'} s_{b'} \rangle \right|$$

CHSH inequality in LHV theories

$$R_{\mathrm{CHSH}} \equiv \frac{1}{2}\left|\langle s_a s_b\rangle - \langle s_a s_{b'}\rangle + \langle s_{a'} s_b\rangle + \langle s_{a'} s_{b'}\rangle\right| \leq 1$$

$$\begin{aligned}
|\langle ab\rangle - \langle ab'\rangle| &= \left|\int d\lambda\,(ab - ab')\,P\right| \\
&= \int d\lambda\,|ab(1 \pm a'b')P - ab'(1 \pm a'b)P| \\
&\leq \int d\lambda\Big(|ab||1 \pm a'b'|P + |ab'||1 \pm a'b|P\Big) \\
&= \int d\lambda\Big[(1 \pm a'b')P + (1 \pm a'b)P\Big] \\
&= 2 \pm (\langle a'b'\rangle + \langle a'b\rangle)
\end{aligned}$$

$\pm\, aba'b'P - (\pm\, aba'b'P) = 0$

$|ab| = |ab'| = 1$

$|1 \pm a'b'|, |1 \pm a'b| \geq 0$

$a = s_a$
$b = s_b$
$\vdots$

$$\boxed{\langle ab\rangle = \int a(\lambda)b(\lambda)P(\lambda)d\lambda}$$

$$\boxed{\int P(\lambda)d\lambda = 1}$$

$$\Rightarrow \quad \tilde{R}_{\mathrm{CHSH}} = \frac{1}{2}\left(|\langle ab\rangle - \langle ab'\rangle| + |\langle a'b\rangle + \langle a'b'\rangle|\right) \leq 1$$

$$\max_{(\vec{a},\vec{b},\vec{a}',\vec{b}')}(R_{\mathrm{CHSH}}) = \max_{(\vec{a},\vec{b},\vec{a}',\vec{b}')}\left(\tilde{R}_{\mathrm{CHSH}}\right)$$

CHSH inequality in QM

- Lets consider an QM wavefunction of singlet state of two spin ½ particles

$$|\psi^{(0,0)}\rangle = \frac{|+-\rangle_z - |-+\rangle_z}{\sqrt{2}}$$

one can show

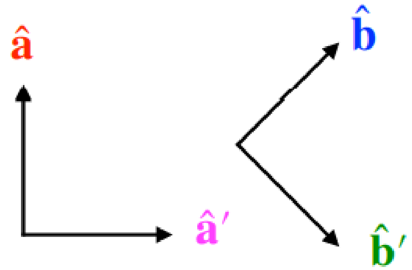
$$\langle s_a s_b \rangle = \langle \Psi^{(0,0)} | s_a s_b | \Psi^{(0,0)} \rangle = (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})$$

therefore

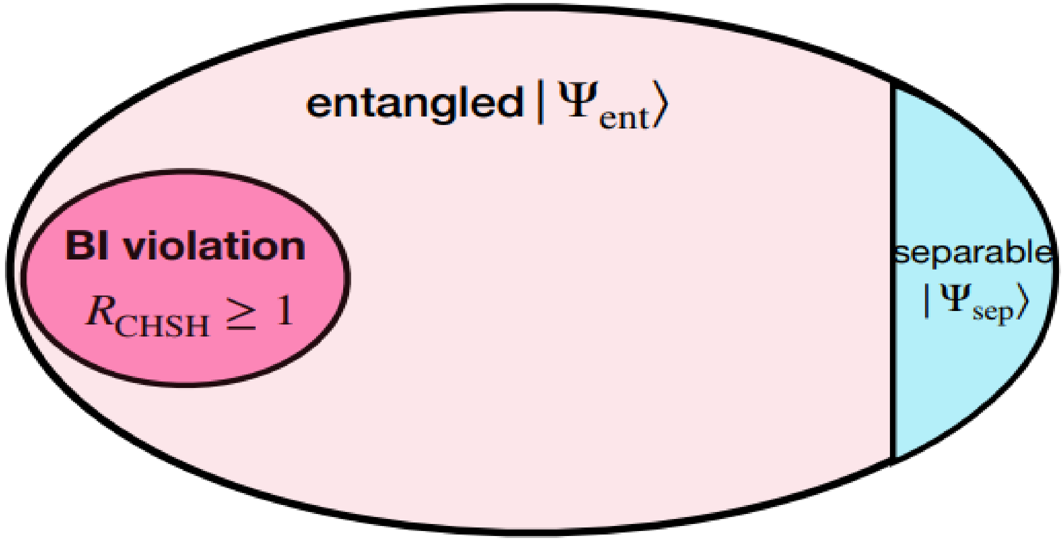
violates the upper bound of hidden variable theories!

$$R_{\mathrm{CHSH}} = \frac{1}{2} \left| \langle s_a s_b \rangle - \langle s_a s_{b'} \rangle + \langle s_{a'} s_b \rangle + \langle s_{a'} s_{b'} \rangle \right|$$

$$= \frac{1}{2} \left| \underbrace{(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})}_{\frac{1}{\sqrt{2}}} - \underbrace{(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}')}_{-\frac{1}{\sqrt{2}}} + \underbrace{(\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}})}_{\frac{1}{\sqrt{2}}} + \underbrace{(\hat{\mathbf{a}}' \cdot \hat{\mathbf{b}}')}_{\frac{1}{\sqrt{2}}} \right| = \sqrt{2}$$



$$R_{\mathrm{CHSH}} \le \begin{cases} 1 & \text{(HV theories)} \\ \sqrt{2} & \text{(QM)} \end{cases}$$



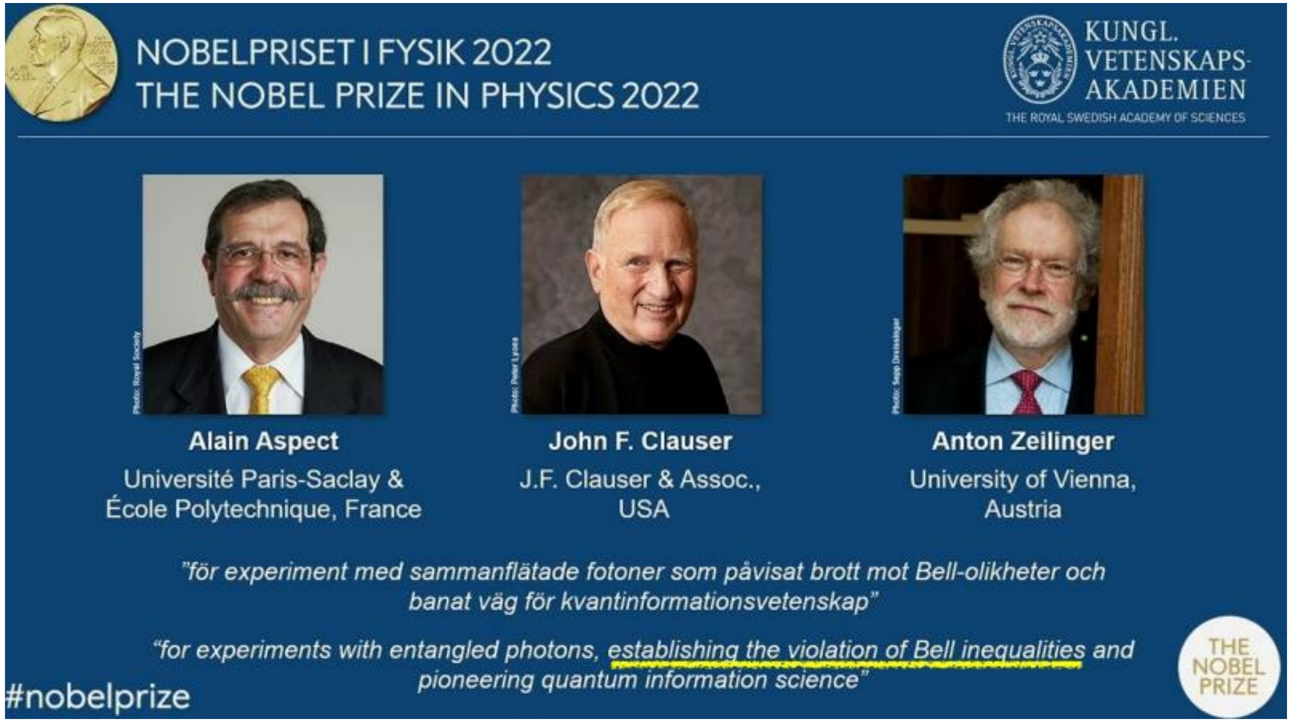
Q: Could we check this experimentally?

A: We already has been observed Bell inequality violation ($R_{CHSH} \geq 1$) in low energy experiments:

- Entangled photon pairs (from decays of Calcium atoms)
  Crauser, Horne, Shimony, Holt (1969), Freedman and Clauser (1972), A. Aspect et. al. (1981, 1982), Y. H. Shih, C. O. Alley (1988), L. K. Shalm et al. (2015) [5σ]

- Entangled proton pairs (from decays of 2_{He})
  M. M. Lamehi-Rachti, W. Mitting (1972), H. Sakai (2006)

- $K^0\overline{K}^0, B^0\overline{B}^0$ flavour oscillation
  CPLEAR (1999), Belle (2004, 2007)



Can we test Bell inequality violation ($R_{CHSH} \geq 1$) and entanglement at High Energy Colliders?

- Entanglement in pp → $t\bar{t}$ @ LHC – Y. Afik, J. R. M. de Nova (2020)
- Bell inequality test in pp → $t\bar{t}$ @ LHC M. Fabbrichesi, R. Floreanini, G. Panizzo (2021) C. Severi, C. D. Boschi, F. Maltoni, M. Sioli (2021) J. A. Aguilar-Saavedra, J. A. Casas (2022)
- Bell inequality test in H → WW* @ LHC – A.J. Barr(2021)

We are interested in study of Quantum property test in H → τ τ @ high energy colliders e+ e−

Density Operator

probability of having $|\Psi_1\rangle$

- For a statistical ensemble $\{\{p_1 : |\Psi_1\rangle\}, \{p_2 : |\Psi_2\rangle\}, \{p_3 : |\Psi_3\rangle\}, \cdots\}$, we define the **density operator/matrix**

$$\hat{\rho} \equiv \sum_k p_k |\Psi_k\rangle\langle\Psi_k| \qquad \rho_{ab} \equiv \langle e_a|\hat{\rho}|e_b\rangle$$

$$0 \leq p_k \leq 1$$

$$\sum_k p_k = 1$$

$$\langle e_a|e_b\rangle = \delta_{ab}$$

- Probability and expectation values:

$$\hat{A}\,|\,a\rangle = a\,|\,a\rangle$$

$$P(a\,|\,\hat{A},\hat{\rho}) = \langle a\,|\,\rho\,|\,a\rangle$$

Probability for outcome a when $\hat{A}$ is measured on the state $\hat{\rho}$

$$\langle\hat{A}\rangle_\rho = \mathrm{Tr}\left[\hat{A}\hat{\rho}\right]$$

Expectation value for $\hat{A}$ on the state $\hat{\rho}$

Spin ½ biparticle system

- The spin system of α and β particles has 4 independent bases:

$$\left(\,|e_1\rangle,\, |e_2\rangle,\, |e_3\rangle,\, |e_4\rangle \,\right) \;=\; \left(\,|++\rangle,\, |+-\rangle,\, |-+\rangle,\, |--\rangle \,\right)$$

- ==> ρ_{ab} is a 4 x 4 matrix (hermitian, Tr=1). It can be expanded as

$$\rho = \frac{1}{4}\left(\mathbf{1}_4 + B_i \cdot \sigma_i \otimes \mathbf{1} + \overline{B}_i \cdot \mathbf{1} \otimes \sigma_i + C_{ij} \cdot \sigma_i \otimes \sigma_j\right)$$

3 x 3 matrix

$$B_i, \overline{B}_i, C_{ij} \in \mathbb{R}$$

- For the spin operators $\hat{s}^\alpha$ and $\hat{s}^\beta$,

$$\langle \hat{s}_i^\alpha \rangle = \mathrm{Tr}\left[\hat{s}_i^\alpha \hat{\rho}\right] = B_i \qquad \langle \hat{s}_i^\beta \rangle = \mathrm{Tr}\left[\hat{s}_i^\beta \hat{\rho}\right] = \overline{B}_i$$

spin-spin correlation

$$\boxed{\langle \hat{s}_i^\alpha \hat{s}_j^\beta \rangle = \mathrm{Tr}\left[\hat{s}_i^\alpha \hat{s}_j^\beta \hat{\rho}\right] = C_{ij}}$$

$H \to \tau^+\tau^-$

$$\mathscr{L}_{\rm int} = -\frac{m_\tau}{v_{\rm SM}}\kappa H \bar{\psi}_\tau(\cos\delta + i\gamma_5 \sin\delta)\,\psi_\tau$$

SM: $(\kappa, \delta) = (1,0)$

$H \to \tau^+\tau^-$

$$\mathscr{L}_{\text{int}} = -\frac{m_\tau}{v_{\text{SM}}}\kappa H \bar{\psi}_\tau(\cos\delta + i\gamma_5 \sin\delta)\psi_\tau$$

SM: $(\kappa, \delta) = (1,0)$

$$\rho_{mn,\bar{m}\bar{n}} = \frac{\mathcal{M}^{*n\bar{n}}\mathcal{M}^{m\bar{m}}}{\sum_{m\bar{m}}|\mathcal{M}^{m\bar{m}}|^2}$$

$$\mathcal{M}^{m\bar{m}} = c\,\bar{u}^m(p)(\cos\delta + i\gamma_5\sin\delta)v^{\bar{m}}(\bar{p})$$

$$\rho_{mn,\bar{m}\bar{n}} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & e^{-i2\delta} & 0 \\ 0 & e^{i2\delta} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$H \to \tau^+\tau^-$

$$\mathscr{L}_{\text{int}} = -\frac{m_\tau}{v_{\text{SM}}}\kappa H \bar{\psi}_\tau(\cos\delta + i\gamma_5 \sin\delta)\psi_\tau$$

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$$\rho = \frac{1}{4}\big(\mathbf{1}_4 + B_i \cdot \sigma_i \otimes \mathbf{1} \\ \bar{B}_i \cdot \mathbf{1} \otimes \sigma_i + C_{ij} \cdot \sigma_i \otimes \sigma_j\big)$$

$$B_i = \bar{B}_i = 0$$

$$C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$H \to \tau^+\tau^-$

$$\mathscr{L}_{\rm int} = -\frac{m_\tau}{v_{\rm SM}}\kappa H \bar{\psi}_\tau(\cos\delta + i\gamma_5 \sin\delta)\,\psi_\tau$$

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$$\rho = \frac{1}{4}\left(\mathbf{1}_4 + B_i \cdot \sigma_i \otimes \mathbf{1} \right.$$
$$\left. \bar{B}_i \cdot \mathbf{1} \otimes \sigma_i + C_{ij} \cdot \sigma_i \otimes \sigma_j\right)$$

$$B_i = \bar{B}_i = 0$$

$$C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$|\Psi_{H\to\tau\tau}(\delta)\rangle \propto |+-\rangle + e^{i2\delta}|-+\rangle$$

$\delta = 0$ (CP even)

$\delta = \pi/2$ (CP odd)

$$|\Psi^{(1,m)}\rangle \propto \begin{pmatrix} |++\rangle \\ \boxed{|+-\rangle + |-+\rangle} \\ |--\rangle \end{pmatrix} \qquad |\Psi^{(0,0)}\rangle \propto \boxed{|+-\rangle - |-+\rangle}$$

Parity: $P = (\eta_f \eta_{\bar{f}}) \cdot (-1)^l$ with $\eta_f\eta_{\bar{f}} = -1$:

$$J^P = \begin{cases} 0^+ \implies -\, l = s = 1 \\ 0^- \implies \quad l = s = 0 \end{cases}$$

Separable state (compliment of entangled state):

$$0 \le p_k \le 1 \qquad \sum_k p_k = 1$$

$$P(a,b\,|\,A,B) = \sum_k p_k \langle a\,|\,\rho_k^{\alpha}\,|\,a\rangle \cdot \langle b\,|\,\rho_k^{\beta}\,|\,b\rangle \qquad \rho = \sum_k p_k\, \rho_k^{\alpha} \otimes \rho_k^{\beta}$$

Non-positive definite

State is entangled

$$\rho^{\mathbf{T}_\beta} \equiv \sum_{\mathbf{k}} \mathbf{p_k} \rho_{\mathbf{k}}^{\alpha} \bigotimes [\rho_{\mathbf{k}}^{\beta}]^{\mathbf{T}}$$

If it is still Physics density matrix with Tr=1 and Positive definite

State is separable

Peres-Horodecki (1996, 1997)

$$\hat{A}\,|\,a\rangle = a\,|\,a\rangle$$

$$P(a\,|\,\hat{A},\hat{\rho}) = \langle a\,|\,\rho\,|\,a\rangle$$

Probability for outcome a when $\hat{A}$ is measured on the state $\hat{\rho}$

$0 \leq p_k \leq 1 \quad \sum_k p_k = 1$

Separable state (compliment of entangled state):

$$P(a, b \,|\, A, B) = \sum_k p_k \langle a \,|\, \rho_k^\alpha \,|\, a \rangle \cdot \langle b \,|\, \rho_k^\beta \,|\, b \rangle \longleftarrow \rho = \sum_k p_k \, \rho_k^\alpha \otimes \rho_k^\beta$$

Hidden Variable state (complement of Bell nonlocal state):

$$P(a, b \,|\, A, B) = \sum_\lambda p(\lambda) \, \underset{\substack{\uparrow}}{P_\alpha(a \,|\, A, \lambda)} \cdot \underset{\substack{\uparrow}}{P_\beta(b \,|\, B, \lambda)}$$

arbitrary conditional probabilities

$\hat{A} \,|\, a \rangle = a \,|\, a \rangle$

$P(a \,|\, \hat{A}, \hat{\rho}) = \langle a \,|\, \rho \,|\, a \rangle$ Probability for outcome a when $\hat{A}$ is measured on the state $\hat{\rho}$

$$0 \le p_k \le 1 \qquad \sum_k p_k = 1$$

Separable state (compliment of entangled state):

$$P(a, b \,|\, A, B) = \sum_k p_k \langle a \,|\, \rho_k^{\alpha} \,|\, a \rangle \cdot \langle b \,|\, \rho_k^{\beta} \,|\, b \rangle \quad \longleftarrow \quad \rho = \sum_k p_k \, \rho_k^{\alpha} \otimes \rho_k^{\beta}$$

Un-steerable state (not-steerable by Alice):

[Jones, Wiseman, Doherty 2007]

$$P(a, b \,|\, A, B) = \sum_k p_k \, P_{\alpha}(a \,|\, A, k) \cdot \langle b \,|\, \rho_k^{\beta} \,|\, b \rangle \quad \longleftarrow$$

If this description is possible, Alice cannot influence (`steer") Bob's local state

Hidden Variable state (complement of Bell nonlocal state):

$$P(a, b \,|\, A, B) = \sum_{\lambda} p(\lambda) \, P_{\alpha}(a \,|\, A, \lambda) \cdot P_{\beta}(b \,|\, B, \lambda)$$

arbitrary conditional probabilities

$$\hat{A} \,|\, a \rangle = a \,|\, a \rangle$$

$$P(a \,|\, \hat{A}, \hat{\rho}) = \langle a \,|\, \rho \,|\, a \rangle$$

Probability for outcome a when $\hat{A}$ is measured on the state $\hat{\rho}$

$0 \le p_k \le 1 \quad \sum_k p_k = 1$

Separable state (compliment of entangled state):

$$P(a,b\,|\,A,B) = \sum_k p_k \langle a\,|\,\rho_k^{\alpha}\,|\,a\rangle \cdot \langle b\,|\,\rho_k^{\beta}\,|\,b\rangle \longleftarrow \rho = \sum_k p_k\, \rho_k^{\alpha} \otimes \rho_k^{\beta}$$

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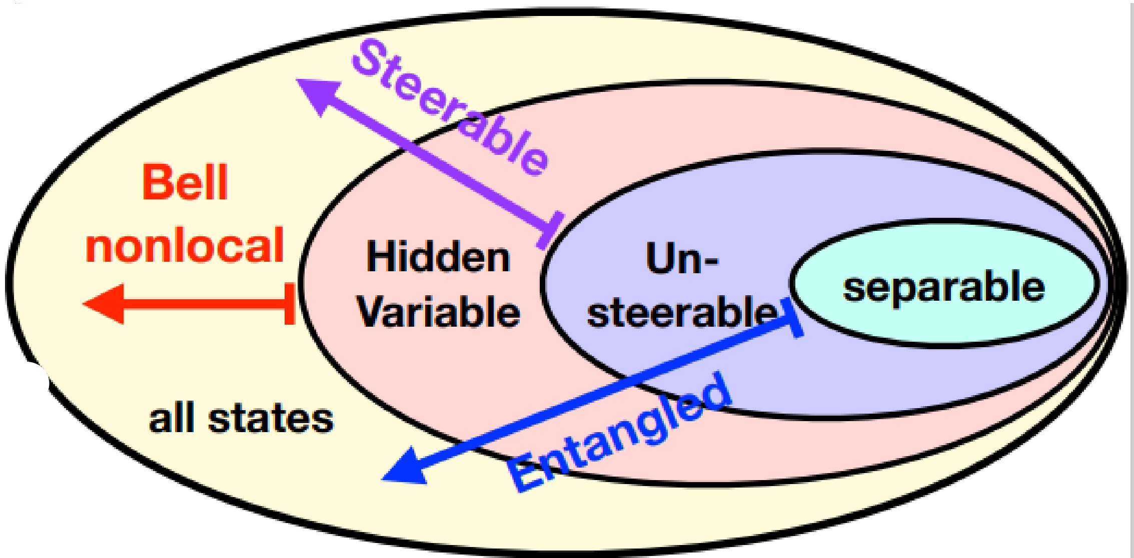
$$P(a,b\,|\,A,B) = \sum_k p_k\, P_{\alpha}(a\,|\,A,k) \cdot \langle b\,|\,\rho_k^{\beta}\,|\,b\rangle \longleftarrow$$

If this description is possible, Alice cannot influence (`steer") Bob's local state

Hidden Variable state (complement of Bell nonlocal state):

$$P(a,b\,|\,A,B) = \sum_\lambda p(\lambda)\, P_{\alpha}(a\,|\,A,\lambda) \cdot P_{\beta}(b\,|\,B,\lambda)$$



Bell-nonlocality:

$$R_{\mathrm{CHSH}} \equiv \frac{1}{2}\left|\boxed{\langle s_a s_b\rangle} - \boxed{\langle s_a s_{b'}\rangle} + \boxed{\langle s_{a'} s_b\rangle} + \boxed{\langle s_{a'} s_{b'}\rangle}\right| > 1$$

[Clauser, Horne, Shimony, Holt, 1969]

$$R_{\mathrm{CHSH}}(H \to \tau^+\tau^-) = \sqrt{2}$$

Steerability: (assuming $B_i = \overline{B}_i = 0$)

$$\mathcal{S}[\rho] > 1$$

$$\mathcal{S}[\rho] \equiv \frac{1}{2\pi}\int d\Omega_{\mathbf{n}} \sqrt{\mathbf{n}^T \boxed{C^T C}\, \mathbf{n}}$$

[Jevtic, Hall, Anderson, Zwierz, Wiseman 2015]

$$\mathcal{S}[\rho](H \to \tau^+\tau^-) = 2$$

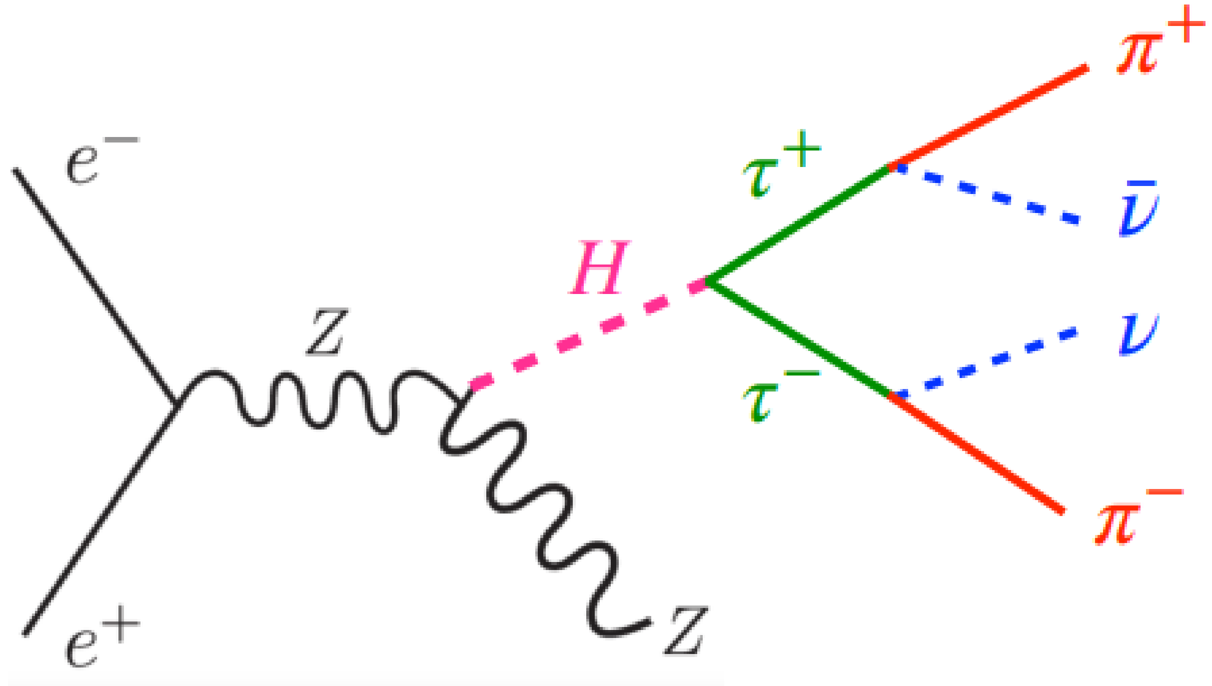
Entanglement:

$$\mathrm{E} > 1$$

$$\mathrm{E} \equiv \boldsymbol{max_i}\{|\boldsymbol{Tr}(\boldsymbol{C}) - \boldsymbol{C_{ii}}| - \boxed{\boldsymbol{C_{ii}}}\}$$

$$\mathrm{E}(H \to \tau^+\tau^-) = 3$$

$$\langle \hat{s}_i^\alpha \hat{s}_j^\beta \rangle = \mathrm{Tr}\left[\hat{s}_i^\alpha \hat{s}_j^\beta \hat{\rho}\right] = C_{ij}$$



e^-
e^+
Z
Z
H
τ^+
τ^-
π^+
$\bar{\nu}$
ν
π^-

Spin correlation of $\tau^-\tau^+$ in term of angular correlation b/w $\tau^-\tau^+$ decay product

❖ The conditional probability that the decay product, d, takes the direction u (at the rest frame of τ^-) , when the tau spin is polarised into s direction, is given by

$$P(u|s) = 1 + \boxed{\alpha_{f,d}}\, u.s$$

spin analysing power which is maximum $(1\ or\ -1)\ for\ \tau^{\pm} \rightarrow \vartheta\, \pi^{\pm}$

❖ Using, join probability P(s| $\bar{s}$) that τ^- and τ^+ are polarised into s and $\bar{s}$, we can write both tau spin correlation and pion momentum correlation as

$$\langle s_a \bar{s}_b \rangle = \int \frac{d\Omega_{\mathbf{s}}}{4\pi} \frac{d\Omega_{\bar{\mathbf{s}}}}{4\pi} (\mathbf{a}\cdot\mathbf{s})(\mathbf{b}\cdot\bar{\mathbf{s}}) P(\mathbf{s},\bar{\mathbf{s}})$$

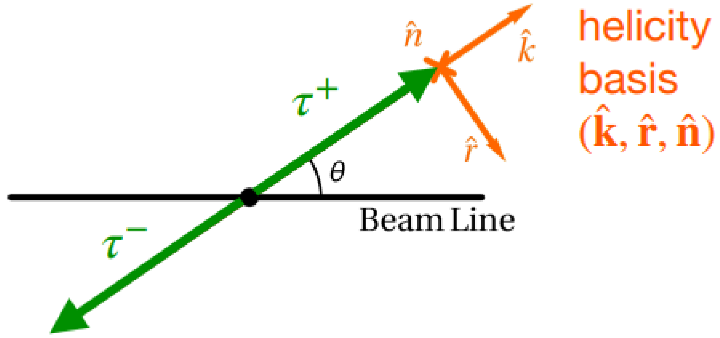
$$\langle u_a \bar{u}_b \rangle = \int \frac{d\Omega_{\mathbf{u}}}{4\pi} \frac{d\Omega_{\bar{\mathbf{u}}}}{4\pi} \frac{d\Omega_{\mathbf{s}}}{4\pi} \frac{d\Omega_{\bar{\mathbf{s}}}}{4\pi} (\mathbf{a}\cdot\mathbf{u})(\mathbf{b}\cdot\bar{\mathbf{u}}) \times P(\mathbf{u}|\mathbf{s}) P(\bar{\mathbf{u}}|\bar{\mathbf{s}}) P(\mathbf{s},\bar{\mathbf{s}})\,.$$

$$\boxed{\langle u_a \bar{u}_b \rangle = \frac{\alpha_{f,d}\alpha_{f',d'}}{9} \langle s_a \bar{s}_b \rangle}$$

CHSH inequality in terms of angular correlation b/w decay products

$$R_{\mathrm{CHSH}} \equiv \frac{1}{2}\left|\langle s_a s_b\rangle - \langle s_a s_{b'}\rangle + \langle s_{a'} s_b\rangle + \langle s_{a'} s_{b'}\rangle\right|$$

$$= \frac{9}{2|x_\alpha x_\beta|}\left|\left\langle(\hat{\mathbf{l}}_\alpha)_a(\hat{\mathbf{l}}_\beta)_b\right\rangle - \left\langle(\hat{\mathbf{l}}_a)(\hat{\mathbf{l}}_\beta)_{b'}\right\rangle + \left\langle(\hat{\mathbf{l}}_\alpha)_{a'}(\hat{\mathbf{l}}_\beta)_b\right\rangle + \left\langle(\hat{\mathbf{l}}_\alpha)_{a'}(\hat{\mathbf{l}}_\beta)_{b'}\right\rangle\right|$$



R_{CHSH} can be directly calculated once the unit vectors $(\hat{\mathbf{a}}, \hat{\mathbf{a}}', \hat{\mathbf{b}}, \hat{\mathbf{b}}')$ are fixed.

r ≡ (h − k cos θ)/ sin θ

- we define helicity basis at the Higgs rest frame.

- In the $\tau^{+(-)}$ rest frame, we measure the direction of $\pi^{+(-)}$, $\hat{\mathbf{l}}^+$ and $\hat{\mathbf{l}}^-$, and calculate R_{CHSH} directly with

$$(\hat{\mathbf{a}}, \hat{\mathbf{a}}', \hat{\mathbf{b}}, \hat{\mathbf{b}}') = \left(\hat{\mathbf{k}}, \hat{\mathbf{r}}, \frac{1}{\sqrt{2}}(\hat{\mathbf{k}} + \hat{\mathbf{r}}), \frac{1}{\sqrt{2}}(\hat{\mathbf{k}} - \hat{\mathbf{r}})\right)$$

and measure C_{ij}

Simulation

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	0.83×10^{-4}
beam resolution e^- (%)	0.27	0.83×10^{-4}
$\sigma(e^+e^- \to HZ)$ (fb)	240.1	240.3
# of signal ($\sigma \cdot \mathrm{BR} \cdot L \cdot \epsilon$)	385	663
# of background ($\sigma \cdot \mathrm{BR} \cdot L \cdot \epsilon$)	20	36

- Main background

$$e^+e^- \to Z\,\boxed{\tau^+\tau^-} \quad \gamma^*/Z^*$$

- Event selection

$$|m_{recoil} - M_H| < 5\ GeV$$

- Generate the SM events (κ, δ) = (1,0) with **MadGraph5_aMC@NLO.** And use **TauDecay** Package for τ decays.

$$e^+e^- \to H\,Z, \qquad Z \to f\,\bar{f}(f\,\bar{f} = q\bar{q}, e^+e^-, \mu^+\mu^-), \qquad \tau^\pm \to \vartheta\,\pi^\pm\ (Br(\tau^\pm \to \vartheta\,\pi^\pm) = 0.109)$$

- Incorporate the detector effects by **smearing energies** of all visible final state particles with

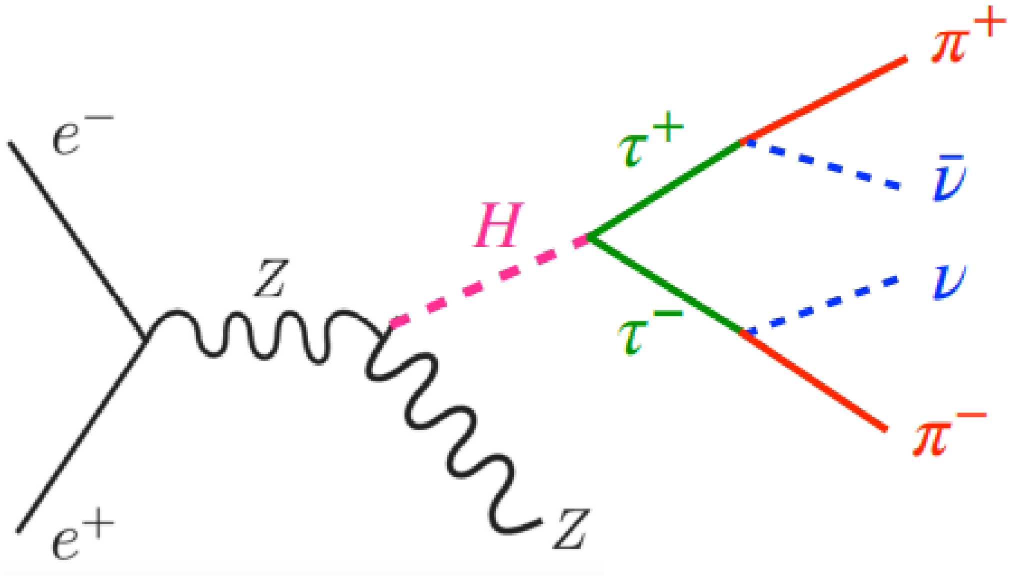
$$E^{true} \to E^{obs} = (1 + \sigma_E.\,\omega).\,E^{true}$$

random number from the normal distribution.

Energy resolution $\sigma_E = 0.03$ for both ILC and FCC-ee.

- 100 **pseudo-experiments** to estimate the statistical uncertainties.

- To determine the tau momenta, we have to reconstruct the unobserved neutrino momenta $(p_x^{\nu}, p_y^{\nu}, p_z^{\nu})$, $(p_x^{\bar{\nu}}, p_y^{\bar{\nu}}, p_z^{\bar{\nu}})$.



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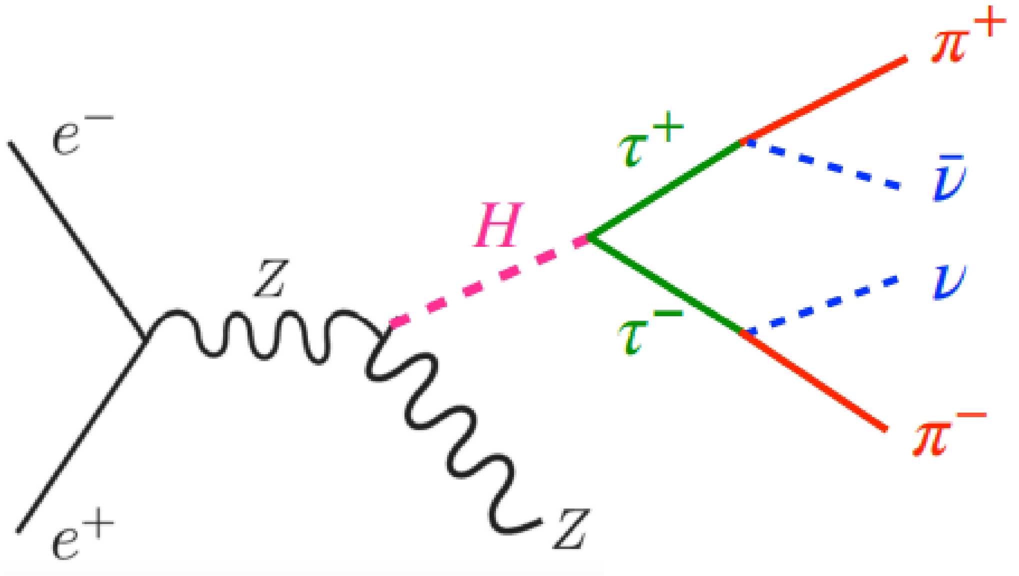
- **6** unknowns can be constrained by **2** mass-shell conditions and **4** energy-momentum conservation.

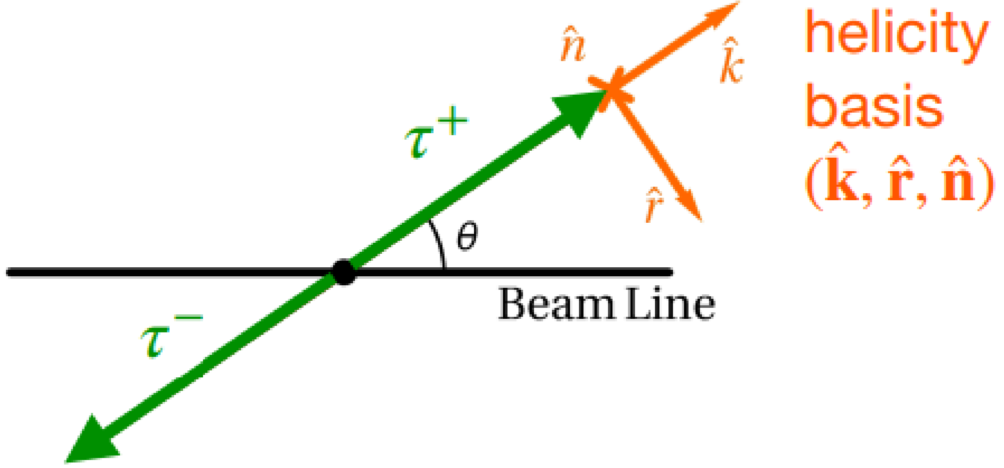
$$m_\tau^2 = (p_{\tau^+})^2 = (p_{\pi^+} + p_{\bar{\nu}})^2$$

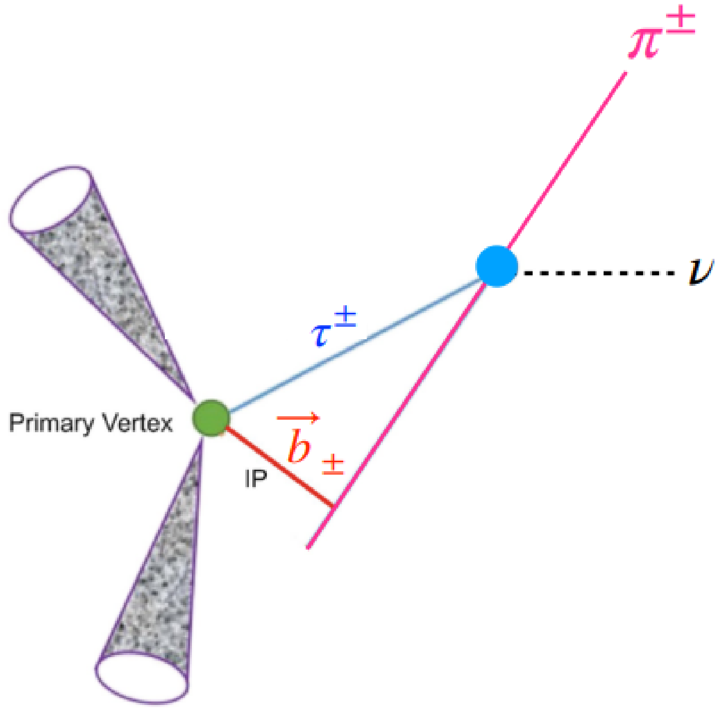
$$m_\tau^2 = (p_{\tau^-})^2 = (p_{\pi^-} + p_{\nu})^2$$

$$(p_{ee} - p_Z)^\mu = p_H^\mu = \left[(p_{\pi^-} + p_\nu) + (p_{\pi^+} + p_{\bar{\nu}})\right]^\mu$$

- With the reconstructed momenta, we define $(\hat{\mathbf{k}}, \hat{\mathbf{r}}, \hat{\mathbf{n}})$ basis at the Higgs rest frame.







Use impact parameter information

- We use the information of impact parameter $\vec{b}_\pm$ measurement of $\pi^\pm$ to "correct" the observed energies of $\tau^\pm$ and Z decay products
- We check whether the reconstructed τ momenta are consistent with the measured impact parameters.
- We construct the likelihood function and search for the most likely τ momenta.

$$\boxed{E_\alpha(\delta_\alpha) = (1 + \sigma_\alpha^E \cdot \delta_\alpha) \cdot E_\alpha^{\rm obs}}$$

$$\vec{b}_+ = |\vec{b}_+| \left(\sin^{-1}\Theta_+ \cdot \vec{e}_{\tau^+} - \tan^{-1}\Theta_+ \cdot \vec{e}_{\pi^+}\right)$$

$$\vec{\Delta}^i_{b_+}(\{\delta\}) \equiv \vec{b}_+ - |\vec{b}_+| \left(\sin^{-1}\Theta^i_+(\{\delta\}) \cdot \vec{e}^{\,i}_{\tau^+}(\{\delta\}) - \tan^{-1}\Theta^i_+(\{\delta\}) \cdot \vec{e}_{\pi^+}\right)$$

$$L^i_\pm(\{\delta\}) = \frac{[\Delta^i_{b_\pm}(\{\delta\})]^2_x + [\Delta^i_{b_\pm}(\{\delta\})]^2_y}{\sigma^2_{b_T}} + \frac{[\Delta^i_{b_\pm}(\{\delta\})]^2_z}{\sigma^2_{b_z}}$$

$$\boxed{L^i(\{\delta\}) = L^i_+(\{\delta\}) + L^i_-(\{\delta\})}$$

Results

	ILC	FCC-ee
C_{ij}	$\begin{pmatrix} 0.830 \pm 0.176 & 0.020 \pm 0.146 & -0.019 \pm 0.159 \\ -0.034 \pm 0.160 & 0.981 \pm 0.1527 & -0.029 \pm 0.156 \\ -0.001 \pm 0.158 & -0.021 \pm 0.155 & -0.729 \pm 0.140 \end{pmatrix}$	$\begin{pmatrix} 0.925 \pm 0.109 & -0.011 \pm 0.110 & 0.038 \pm 0.095 \\ -0.009 \pm 0.110 & 0.929 \pm 0.113 & 0.001 \pm 0.115 \\ -0.026 \pm 0.122 & -0.019 \pm 0.110 & -0.879 \pm 0.098 \end{pmatrix}$
E_k	$2.567 \pm 0.279 \sim 5\sigma$	$2.696 \pm 0.215 \sim 5\sigma$
$\mathcal{S}[\rho]$	$1.760 \pm 0.161 \sim 4\sigma$	$1.851 \pm 0.111 \sim 5\sigma$
R^*_{CHSH}	1.103 ± 0.163	$1.276 \pm 0.094 \sim 3\sigma$

SM values: $$C_{ij} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

$E = 3$ **Entanglement** $\Longrightarrow E > 1$

$\mathcal{S}[\rho] = 2$ **Steerablity** $\Longrightarrow \mathcal{S}[\rho] > 1$

$R_{\mathrm{CHSH}} = \sqrt{2} \simeq 1.414$ **Bell-nonlocal** $\Longrightarrow R_{\mathrm{CHSH}} > 1$

Superiority of FCC-ee over ILC is due to a better beam resolution

	ILC	FCC-ee
energy (GeV)	250	240
luminosity (ab^{-1})	3	5
beam resolution e^+ (%)	0.18	$0.83 \cdot 10^{-4}$
beam resolution e^- (%)	0.27	$0.83 \cdot 10^{-4}$

CP measurement

- Under CP, the spin correlation matrix transforms: $C \overset{CP}{\longrightarrow} C^T$
- This can be used for a *model-independent* test of CP violation. We define:

$$A \equiv (C_{rn} - C_{nr})^2 + (C_{nk} - C_{kn})^2 + (C_{kr} - C_{rk})^2 \;\; \geq 0$$

- Observation of $A \neq 0$ immediately confirms CP violation.
- From our simulation, we observe

$$A = \begin{cases} 0.168 \pm 0.131 & (\text{ILC}) \\ 0.081 \pm 0.060 & (\text{FCC-ee}) \end{cases} \longleftarrow$$

consistent with absence of CPV

- This model independent bounds can be translated to the constraint on the CP-phase δ

$$\mathcal{L}_{\text{int}} \propto H \bar{\psi}_\tau(\cos\delta + i\gamma_5 \sin\delta)\psi_\tau \;\Rightarrow\; C_{ij} = \begin{pmatrix} \cos 2\delta & \sin 2\delta & 0 \\ -\sin 2\delta & \cos 2\delta & 0 \\ 0 & 0 & -1 \end{pmatrix} \;\Rightarrow\; A(\delta) = 4\sin^2 2\delta$$

CP measurement

- Focusing on the region near $|\delta| = 0$, we find the 1-σ bounds:

$$|\delta| < \begin{cases} 7.9^o & \text{(ILC)} \\ 5.4^o & \text{(FCC-ee)} \end{cases}$$

- Other studies:

$$\Delta\delta \sim 11.5^o \quad \text{(HL-LHC)}$$ [Hagiwara, Ma, Mori 2016]

$$\Delta\delta \sim 4.3^o \quad \text{(ILC)}$$ [Jeans and G. W. Wilson 2018]

Summary

- High energy tests of entanglement and Bell inequality has recently attracted an attention.
- $\tau^+\tau^-$ pairs from $H \to \tau^+\tau^-$ form the EPR triplet state $|\Psi^{(1,0)}\rangle = \dfrac{|+,-\rangle + |-,+\rangle}{\sqrt{2}}$, and maximally entangled.
- We investigated feasibility of quantum property tests @ ILC and FCC-ee.
- Quantum test requires to a precise reconstruction of the tau rest frames and IP information is crucial to achieve this.
- Spin correlation is sensitive to CP-phase and we can measure the CP-phase as a byproduct of the quantum property measurement.

	Entanglement	Steering	Bell-nonlocality	CP-phase
ILC	~ 5σ	~ 4σ		7.9°
FCC-ee	~ 5σ	~ 5σ	~ 3σ	5.4°

Thank you for the attention!