A subtraction scheme for jet cross sections at NNLO

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Outline

- Motivations
- Production rates at NNLO
- Subtraction at NNLO
- Constructing the approximate cross sections
  - Matching of limits
  - Momentum mappings
  - True subtraction terms
- Conclusions

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- Motivations
- Production rates at NNLO
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- Method is very algorithmic as is to be expected in PT

Precision QCD

- Within SM, precise determination of
  - strong coupling constant α_{s}
  - parton density functions
  - LHC parton luminosity
  - electroweak parameters
- Beyond SM, accurate predictions for
  - Higgs production
  - New Physics production
  - their backgrounds
- LO predictions: order of magnitude estimates (strong dependence on unphysical renormalization and factorization scales)
- . . . so at least NLO corrections must be included (reduced scale dependence)

Beyond NLO – motivations

- NNLO corrections may be relevant if:
  - the NLO corrections are large
    $\Longrightarrow$ Higgs production in gluon fusion (NLO corrections may be larger than 100%)
  - the NLO error bands are too large to test theory *vs.* data
    $\Longrightarrow$ open b-quark production in hadron collisions
  - the main source of uncertainty in extracting info from data is due to NLO theory
    $\Longrightarrow$ measurement of $\alpha_{\rm s}$ S. Bethke, 2006

$$\alpha_{\rm s}(M_Z) = 0.121 \pm 0.001(\text{experiment}) \pm 0.005(\text{theory})$$

  - NLO calculation is effectively LO
    $\Longrightarrow$ energy distribution in jet cones
  - ...

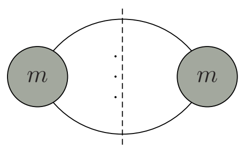
Production rates at NNLO

- The formal loop expansion for a producution rate to NNLO accuracy reads

$$\sigma = \sigma^{\mathrm{LO}} + \sigma^{\mathrm{NLO}} + \sigma^{\mathrm{NNLO}} + \ldots$$

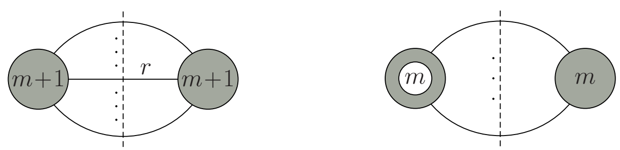
- Consider m-jet production

  - LO



$$\sigma^{\mathrm{LO}} = \sigma_m^{\mathrm{B}} = \int \mathrm{d}\phi_m |\mathcal{M}_m^{(0)}|^2 J_m$$

  - NLO



$$\sigma^{\mathrm{NLO}} = \sigma_{m+1}^{\mathrm{R}} + \sigma_m^{\mathrm{V}} = \int \mathrm{d}\phi_{m+1} |\mathcal{M}_{m+1}^{(0)}|^2 J_{m+1} + \int \mathrm{d}\phi_m 2\mathrm{Re}\langle \mathcal{M}_m^{(0)} | \mathcal{M}_m^{(1)} \rangle J_m$$

  - NNLO

$$\sigma^{\mathrm{NNLO}} = \sigma_{m+2}^{\mathrm{RR}} + \sigma_{m+1}^{\mathrm{RV}} + \sigma_m^{\mathrm{VV}} = \int \mathrm{d}\phi_{m+2} |\mathcal{M}_{m+2}^{(0)}|^2 J_{m+2} +$$

$$+ \int \mathrm{d}\phi_{m+1} 2\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)} | \mathcal{M}_{m+1}^{(1)} \rangle J_{m+1} + \int \mathrm{d}\phi_m \left[|\mathcal{M}_m^{(1)}|^2 + 2\mathrm{Re}\langle \mathcal{M}_m^{(0)} | \mathcal{M}_m^{(2)} \rangle \right] J_m$$

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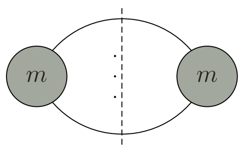
- Consider m-jet production

  - ☐ LO



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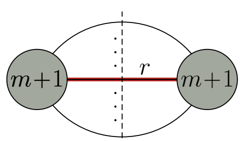
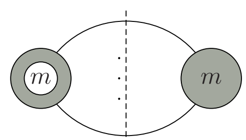
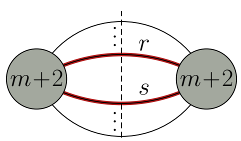
  - ☐ NLO

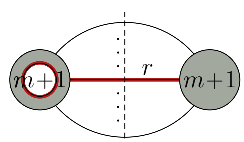


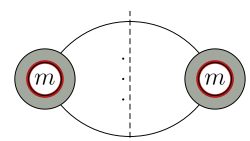


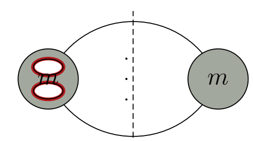
$$\sigma^{\mathrm{NLO}} = \sigma_{m+1}^{\mathrm{R}} + \sigma_m^{\mathrm{V}} = \int \mathrm{d}\phi_{m+1} |\mathcal{M}_{m+1}^{(0)}|^2 J_{m+1} + \int \mathrm{d}\phi_m 2\mathrm{Re}\langle \mathcal{M}_m^{(0)} | \mathcal{M}_m^{(1)} \rangle J_m$$

  - ☐ NNLO









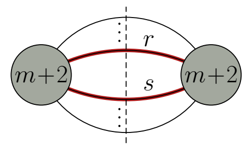
$$\sigma^{\mathrm{NNLO}} = \sigma_{m+2}^{\mathrm{RR}} + \sigma_{m+1}^{\mathrm{RV}} + \sigma_m^{\mathrm{VV}} = \int \mathrm{d}\phi_{m+2} |\mathcal{M}_{m+2}^{(0)}|^2 J_{m+2} +$$

$$+ \int \mathrm{d}\phi_{m+1} 2\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)} | \mathcal{M}_{m+1}^{(1)} \rangle J_{m+1} + \int \mathrm{d}\phi_m \left[|\mathcal{M}_m^{(1)}|^2 + 2\mathrm{Re}\langle \mathcal{M}_m^{(0)} | \mathcal{M}_m^{(2)} \rangle \right] J_m$$

Subtraction at NNLO

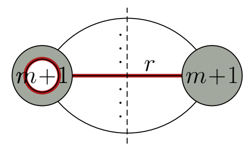
$$\sigma^{\mathrm{NNLO}} = \sigma^{\mathrm{RR}}_{m+2} + \sigma^{\mathrm{RV}}_{m+1} + \sigma^{\mathrm{VV}}_{m} =$$

$$= \int_{m+2} \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2}$$



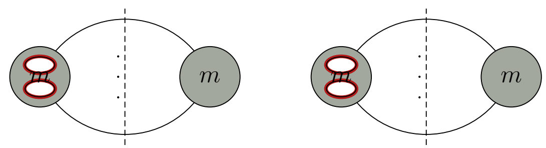
Implicit IR pole from PS integral

$$+ \int_{m+1} \mathrm{d}\sigma^{\mathrm{RV}}_{m+1} J_{m+1}$$



{ Explicit IR pole from loop integral
Implicit IR pole from PS integral

$$+ \int_{m} \mathrm{d}\sigma^{\mathrm{VV}}_{m} J_{m}$$



Explicit IR pole from loop integral

- The three terms are separately IR divergent, but their sum is finite for IR safe observables
- General strategy of subtraction: use approximate cross sections to redistribute the singularities among the contributions
  $\Longrightarrow$ construction of approx. cross sections made possible by universal IR structure

Subtraction at NNLO

$$\sigma^{\rm NNLO} = \sigma^{\rm NNLO}_{m+2} + \sigma^{\rm NNLO}_{m+1} + \sigma^{\rm NNLO}_{m} =$$

$$= \int_{m+2} \Big\{ {\rm d}\sigma^{\rm RR}_{m+2} J_{m+2} - {\rm d}\sigma^{\rm RR,A_2}_{m+2} J_m - \Big({\rm d}\sigma^{\rm RR,A_1}_{m+2} J_{m+1} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} J_m \Big) \Big\}$$

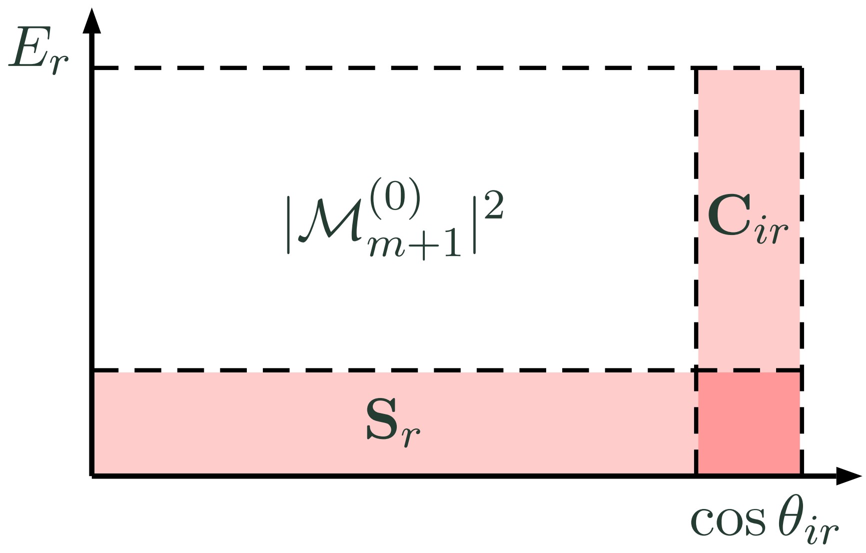
$$+ \int_{m+1} \Big\{ \Big({\rm d}\sigma^{\rm RV}_{m+1} + \int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big) J_{m+1} - \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] J_m \Big\}$$

$$+ \int_m \Big\{ {\rm d}\sigma^{\rm VV}_m + \int_2 \Big({\rm d}\sigma^{\rm RR,A_2}_{m+2} - {\rm d}\sigma^{\rm RR,A_{12}}_{m+2} \Big) + \int_1 \Big[{\rm d}\sigma^{\rm RV,A_1}_{m+1} + \Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1} \Big] \Big\} J_m$$

- The approximate cross sections ${\rm d}\sigma^{\rm RR,A_1}_{m+2}$ and ${\rm d}\sigma^{\rm RR,A_2}_{m+2}$ regularize the singly- and doubly-unresolved limits of ${\rm d}\sigma^{\rm RR}_{m+2}$, their overlap is added back in ${\rm d}\sigma^{\rm RR,A_{12}}_{m+2}$
- The approximate cross sections ${\rm d}\sigma^{\rm RV,A_1}_{m+1}$ and $\Big(\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2} \Big)^{\rm A_1}$ regularize the singly-unresolved limits of ${\rm d}\sigma^{\rm RV}_{m+1}$ and $\int_1 {\rm d}\sigma^{\rm RR,A_1}_{m+2}$
- Each integral on the r.h.s. is finite in $d = 4$ provided J is IR safe

Devising approximate cross sections

- Use known IR limits of squared matrix elements



$$\mathbf{C}_{ir}|\mathcal{M}^{(0)}_{m+1}|^2 \propto \frac{1}{s_{ir}}\hat{P}_{ir} \otimes |\mathcal{M}^{(0)}_{m}|^2$$

$$\mathbf{S}_r|\mathcal{M}^{(0)}_{m+1}|^2 \propto \sum_{i,k} \frac{s_{ik}}{s_{ir}s_{kr}}|\mathcal{M}^{(0)}_{m;(i,k)}|^2$$

- We face two difficulties
  - The various IR regions of the PS and thus the various IR limits overlap
    $\Longrightarrow$ the overlaps must be disentangled: "matching of limits"
  - The IR factorization formulae are only defined in the strict limits
    $\Longrightarrow$ give unambiguous meaning away from the limits: "extension"

Matching the singly-unresolved limits

- Only two types of limits
  - Collinear limit

$$\mathbf{C}_{ir}|\mathcal{M}_{m+2}^{(0)}|^2 \propto \frac{1}{s_{ir}}\hat{P}_{ir} \otimes |\mathcal{M}_{m+1}^{(0)}|^2$$

  - Soft limit

$$\mathbf{S}_r|\mathcal{M}_{m+2}^{(0)}|^2 \propto \sum_{i\neq k} \mathcal{S}_{ik}(r)|\mathcal{M}_{m+1;(i,k)}^{(0)}|^2$$

Matching the singly-unresolved limits

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  - Collinear limit
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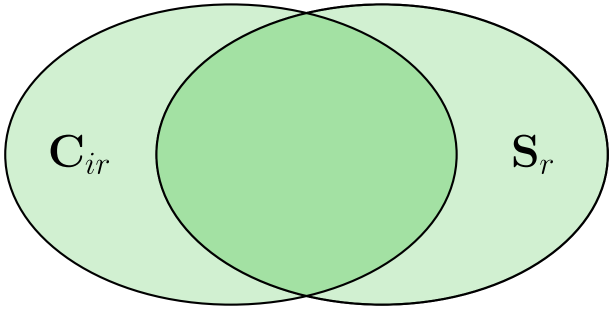


- The formal operator

$$\mathbf{A}_1 = \sum_r \left[\sum_{i \neq r} \frac{1}{2} \mathbf{C}_{ir} + \mathbf{S}_r - \sum_{i \neq r} \mathbf{C}_{ir} \mathbf{S}_r \right]$$

counts each unresolved limit precisely once (it is free of double subtractions), so

$$\mathbf{A}_1 |\mathcal{M}_{m+2}^{(0)}|^2 \;\; , \;\; \mathbf{A}_1 2\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)} | \mathcal{M}_{m+1}^{(1)} \rangle \;\; , \;\; \ldots$$

has the same singly-unresolved singularity structure as

$$|\mathcal{M}_{m+2}^{(0)}|^2 \;\; , \;\; 2\mathrm{Re}\langle \mathcal{M}_{m+1}^{(0)} | \mathcal{M}_{m+1}^{(1)} \rangle \;\; , \;\; \ldots$$

Matching the doubly-unresolved limits

- Four different types of limits
  - Triple collinear

$$\mathbf{C}_{irs}|\mathcal{M}_{m+2}^{(0)}|^2 \propto \frac{1}{s_{irs}^2}\hat{P}_{irs} \otimes |\mathcal{M}_m^{(0)}|^2$$

  - Doubly single collinear

$$\mathbf{C}_{ir;js}|\mathcal{M}_{m+2}^{(0)}|^2 \propto \frac{1}{s_{ir}s_{js}}\hat{P}_{ir}\hat{P}_{js} \otimes |\mathcal{M}_m^{(0)}|^2$$

  - Doubly soft-collinear

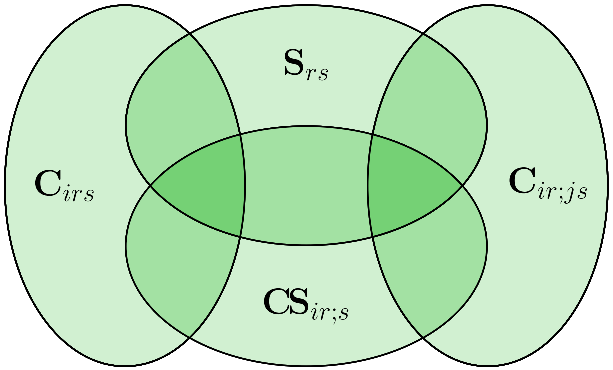
$$\mathbf{CS}_{ir;s}|\mathcal{M}_{m+2}^{(0)}|^2 \propto \frac{1}{s_{ir}}\mathcal{S}_{jl}(s)\hat{P}_{ir} \otimes |\mathcal{M}_{m;(j,l)}^{(0)}|^2$$

  - Double soft

$$\mathbf{S}_{rs}|\mathcal{M}_{m+2}^{(0)}|^2 \propto \mathcal{S}_{ik}(r)\mathcal{S}_{jl}(s)|\mathcal{M}_{m;(i,k)(j,l)}^{(0)}|^2 - 2C_{\mathrm{A}}\mathcal{S}_{ik}(r,s)|\mathcal{M}_{m;(i,k)}^{(0)}|^2$$

Matching the doubly-unresolved limits

- Four different types of limits
  - Triple collinear
  - Doubly single collinear
  - Doubly soft-collinear
  - Double soft



- The formal operator $\mathbf{A}_2$ counts each unresolved limit precisely once. . .

$$
\begin{aligned}
\mathbf{A}_3 \quad = \quad & \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \left[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s} \right] + \mathbf{S}_{rs} \\
- \quad & \sum_{i\neq r,s} \left[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs} \right. \\
+ \quad & \left. \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs} \right] + \sum_{i\neq r,s} \left[\mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \sum_{j\neq i,r,s} \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} \right] \Bigg\}
\end{aligned}
$$

Matching the doubly-unresolved limits

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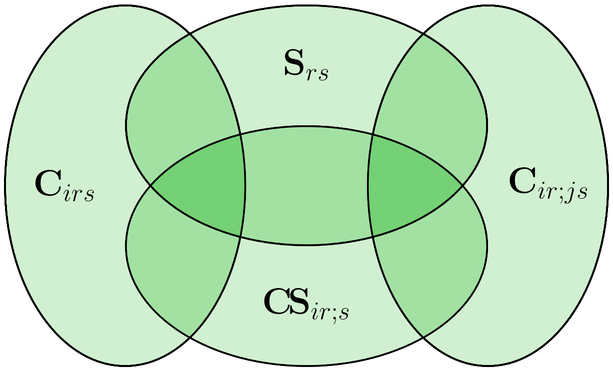

- The formal operator $\mathbf{A}_2$ counts each unresolved limit precisely once...

$$\mathbf{A}_2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \left[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s} \right] + \mathbf{S}_{rs}$$

$$- \sum_{i\neq r,s} \left[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs} \right.$$

$$\left. + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs} \right] + \sum_{i\neq r,s} \left[\mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} + \sum_{j\neq i,r,s} \mathbf{C}_{ir;js}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} \right] \Bigg\}$$

Matching the doubly-unresolved limits

- Four different types of limits
  - Triple collinear
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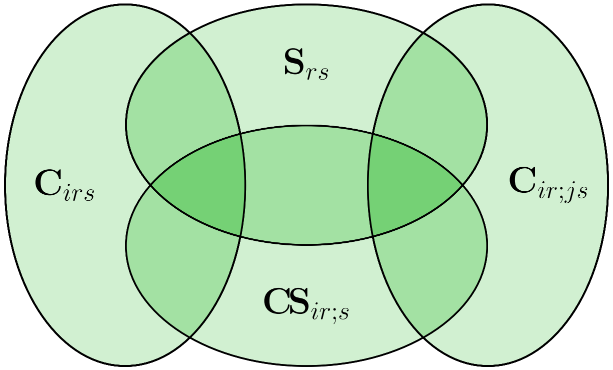

- The formal operator $\mathbf{A}_2$ counts each unresolved limit precisely once. . .

$$\mathbf{A}_2 = \sum_r \sum_{s\neq r} \Bigg\{ \sum_{i\neq r,s} \Bigg[\frac{1}{6}\mathbf{C}_{irs} + \sum_{j\neq i,r,s} \frac{1}{8}\mathbf{C}_{ir;js} + \frac{1}{2}\mathbf{CS}_{ir;s} \Bigg] + \mathbf{S}_{rs}$$
$$- \sum_{i\neq r,s} \Bigg[\frac{1}{2}\mathbf{C}_{irs}\mathbf{CS}_{ir;s} + \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{CS}_{ir;s} + \frac{1}{2}\mathbf{C}_{irs}\mathbf{S}_{rs} + \mathbf{CS}_{ir;s}\mathbf{S}_{rs}$$
$$- \sum_{j\neq i,r,s} \frac{1}{2}\mathbf{C}_{ir;js}\mathbf{S}_{rs} - \mathbf{C}_{irs}\mathbf{CS}_{ir;s}\mathbf{S}_{rs} \Bigg] \Bigg\}$$

Matching the doubly-unresolved limits

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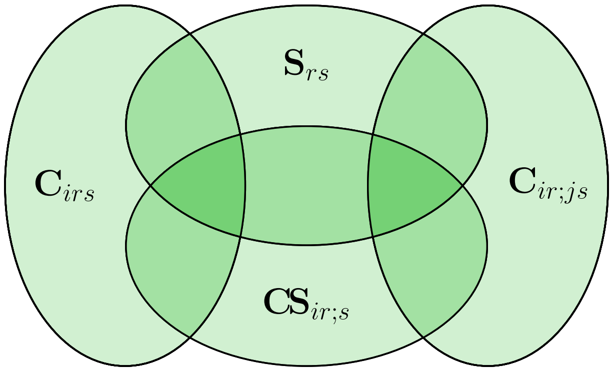


- The formal operator $\mathbf{A}_2$ counts each unresolved limit precisely once...
- ...and thus

$$\mathbf{A}_2 |\mathcal{M}_{m+2}^{(0)}|^2$$

has the same doubly-unresolved singularity structure as

$$|\mathcal{M}_{m+2}^{(0)}|^2$$

Overlap of the singly- and doubly-unresolved limits

- The singly- and doubly-unresolved limits overlap
  $\Longrightarrow$ need $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ to avoid double subtraction
- The role of $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$ is delicate
  - in doubly-unresolved limits
    $\Longrightarrow$ it needs to regularize $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{1}}$
  - in singly-unresolved limits
    $\Longrightarrow$ it needs to regularize $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{2}}$ and spurious singularities in $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{1}}$
- We find that

$$(\mathbf{A}_1 + \mathbf{A}_2 - \mathbf{A}_1\mathbf{A}_2)|\mathcal{M}_{m+2}^{(0)}|^2$$

has the same singularity structure as

$$|\mathcal{M}_{m+2}^{(0)}|^2$$

in all singly- and doubly-unresolved limits and is free of multiple subtractions

Extending the candidate subtraction terms

- The action of the formal operators $\mathbf{A}_1$ and $\mathbf{A}_2$ defines candidate subtraction terms that are however only well defined in the strict IR limits
  $\Longrightarrow$ extend these candidate terms over the full PS
- The extension requires the specification of

Single unresolved

- single momentum mapping
$$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_{m+1}$$
- momentum conservation
$$\sum_{i=1}^{m+2} p_i = \sum_{i=1}^{m+1} \tilde{p}_i$$
- PS factorization
$$\mathrm{d}\phi_{m+2} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]$$

Double unresolved

- double momentum mapping
$$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_{m}$$
- momentum conservation
$$\sum_{i=1}^{m+2} p_i = \sum_{i=1}^{m} \tilde{p}_i$$
- PS factorization
$$\mathrm{d}\phi_{m+2} = \mathrm{d}\phi_{m}[\mathrm{d}p_2]$$

Momentum mappings (for final state radiation)

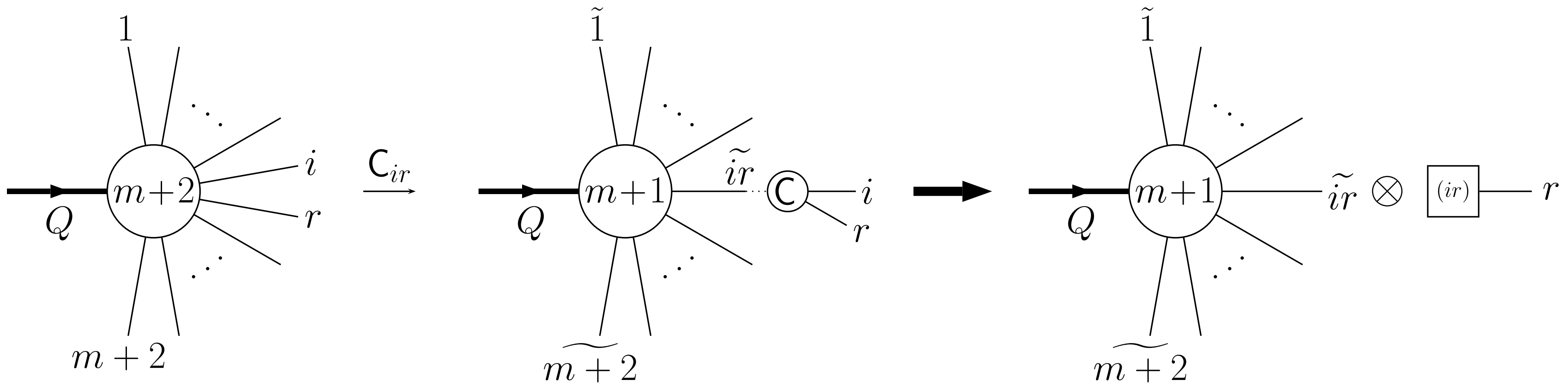
- Two types of single mappings (corresponding to two basic limits: collinear, soft)

Momentum mappings (for final state radiation)

- Two types of single mappings (corresponding to two basic limits: collinear, soft)
- Collinear mapping

$$\tilde{p}^{\mu}_{ir} = \frac{1}{1-\alpha_{ir}}(p^{\mu}_i + p^{\mu}_r - \alpha_{ir} Q^{\mu}) \,, \qquad \tilde{p}^{\mu}_n = \frac{1}{1-\alpha_{ir}} p^{\mu}_n \,, \quad n \neq i, r$$

$$\alpha_{ir} = \frac{1}{2}\left[y_{(ir)Q} - \sqrt{y^2_{(ir)Q} - 4y_{ir}}\right]$$

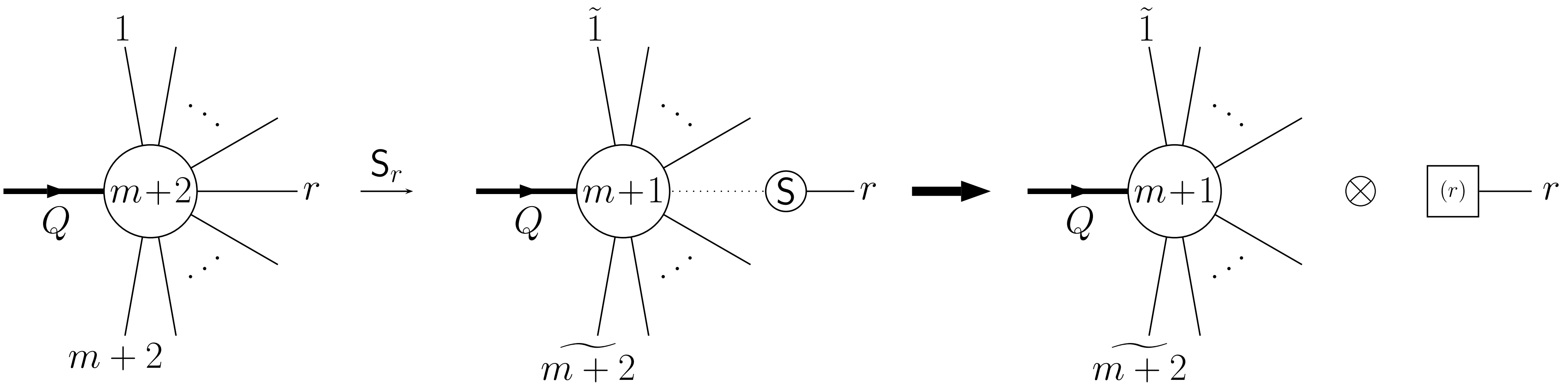


Momentum mappings (for final state radiation)

- Two types of single mappings (corresponding to two basic limits: collinear, soft)
- Soft mapping

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q,(Q-p_r)/\lambda_r](p_n^\nu/\lambda_r)\,, \quad n \neq r\,, \qquad \lambda_r = \sqrt{1-y_{rQ}}\,,$$

$$\Lambda^\mu_\nu[K,\widetilde{K}] = g^\mu_\nu - \frac{2(K+\widetilde{K})^\mu(K+\widetilde{K})_\nu}{(K+\widetilde{K})^2} + \frac{2K^\mu \widetilde{K}_\nu}{K^2}$$



Momentum mappings (for final state radiation)

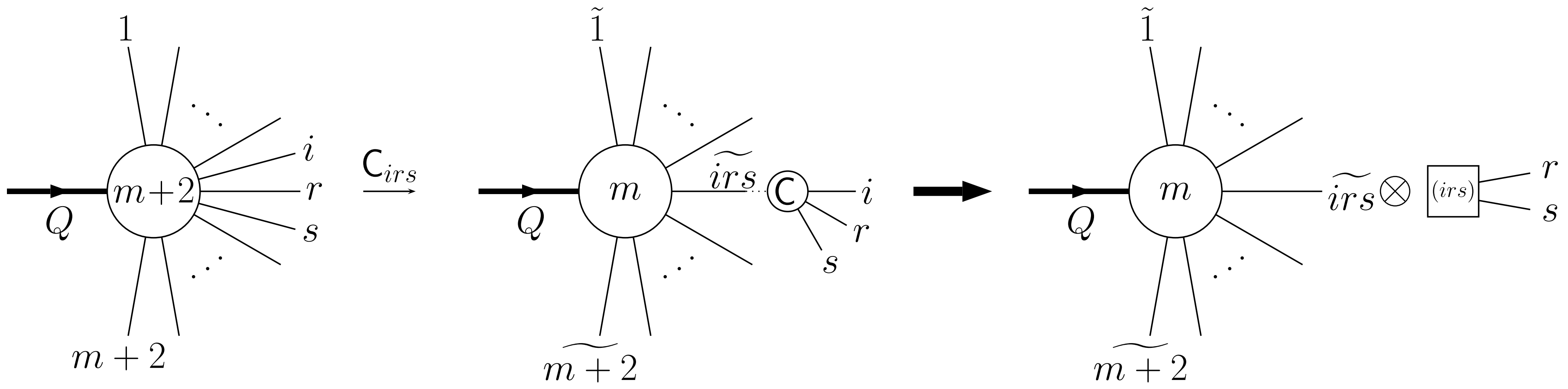
- Four types of double mappings (corresponding to four basic limits)

Momentum mappings (for final state radiation)

- Four types of double mappings (corresponding to four basic limits)
- Triple collinear mapping

$$\tilde{p}^\mu_{irs} = \frac{1}{1-\alpha_{irs}}(p^\mu_i + p^\mu_r + p^\mu_s - \alpha_{irs} Q^\mu)\,, \qquad \tilde{p}^\mu_n = \frac{1}{1-\alpha_{irs}} p^\mu_n\,, \quad n \neq i, r, s$$

$$\alpha_{irs} = \frac{1}{2}\left[y_{(irs)Q} - \sqrt{y^2_{(irs)Q} - 4y_{irs}}\,\right]$$

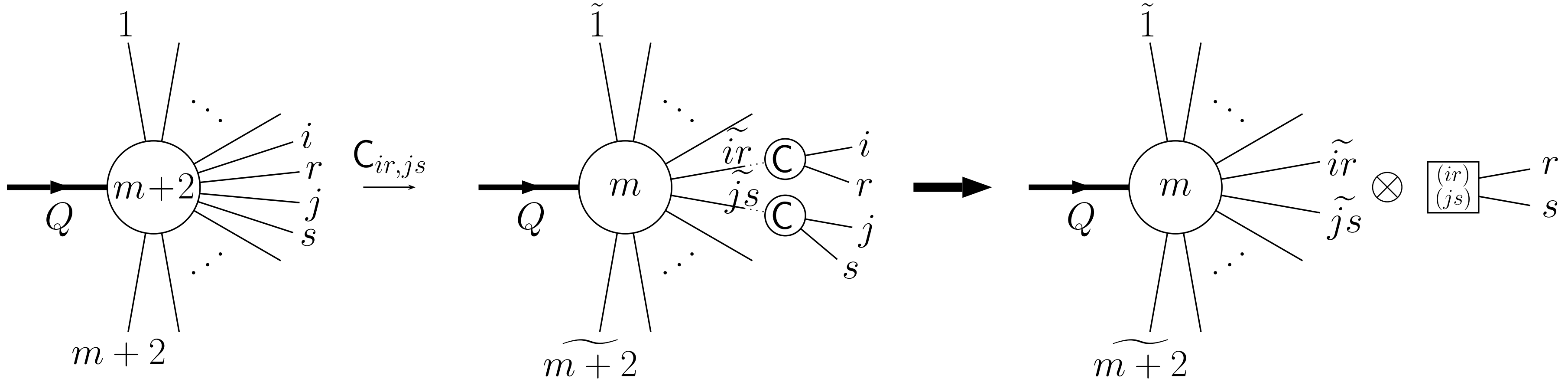


Momentum mappings (for final state radiation)

- Four types of double mappings (corresponding to four basic limits)
- Doubly single collinear mapping

$$\tilde{p}^\mu_{ir} = \frac{p^\mu_i + p^\mu_r - \alpha_{ir} Q^\mu}{1 - \alpha_{ir} - \alpha_{js}}\,, \qquad \tilde{p}^\mu_{js} = \frac{p^\mu_j + p^\mu_s - \alpha_{js} Q^\mu}{1 - \alpha_{ir} - \alpha_{js}}\,,$$

$$\tilde{p}^\mu_n = \frac{1}{1 - \alpha_{ir} - \alpha_{js}} p^\mu_n\,, \qquad \alpha_{kl} = \frac{1}{2}\left[y_{(kl)Q} - \sqrt{y^2_{(kl)Q} - 4 y_{kl}}\,\right]$$

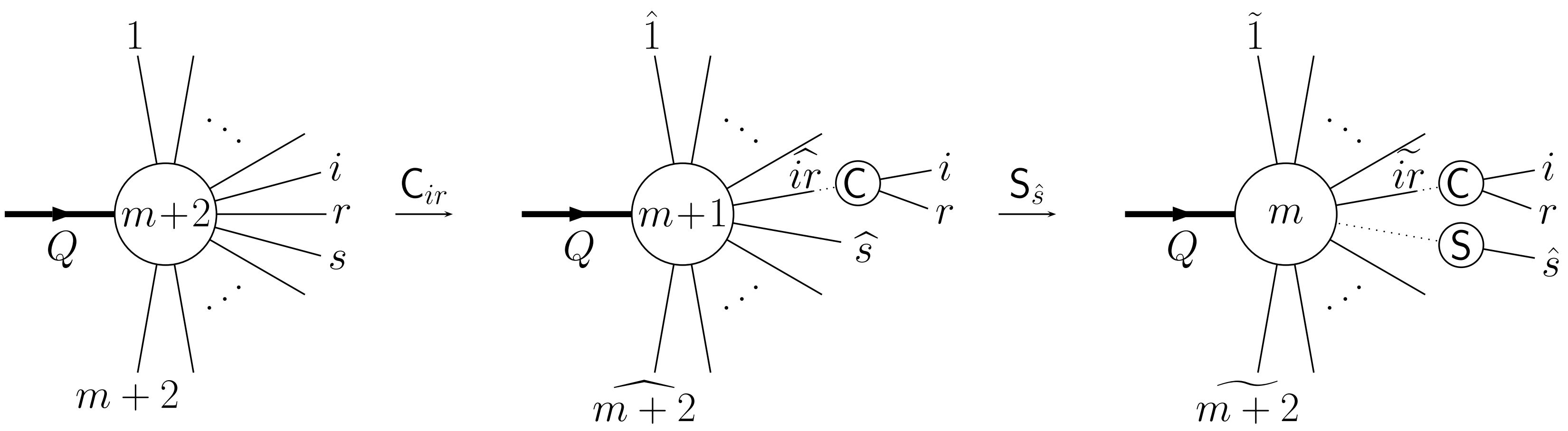


Momentum mappings (for final state radiation)

- Four types of double mappings (corresponding to four basic limits)
- Double soft-collinear mapping

$$\tilde{p}_n^\mu = \Lambda_\nu^\mu[Q,(Q-\hat{p}_s)/\lambda_{\hat{s}}](\hat{p}_n^\nu/\lambda_{\hat{s}})\,, \quad n \neq r,s\,, \qquad \lambda_{\hat{s}} = \sqrt{1-y_{\hat{r}Q}}\,,$$

$$\hat{p}_{ir}^\mu = \frac{1}{1-\alpha_{ir}}(p_i^\mu + p_r^\mu - \alpha_{ir}Q^\mu)\,, \qquad \hat{p}_n^\mu = \frac{1}{1-\alpha_{ir}}p_n^\mu\,,$$



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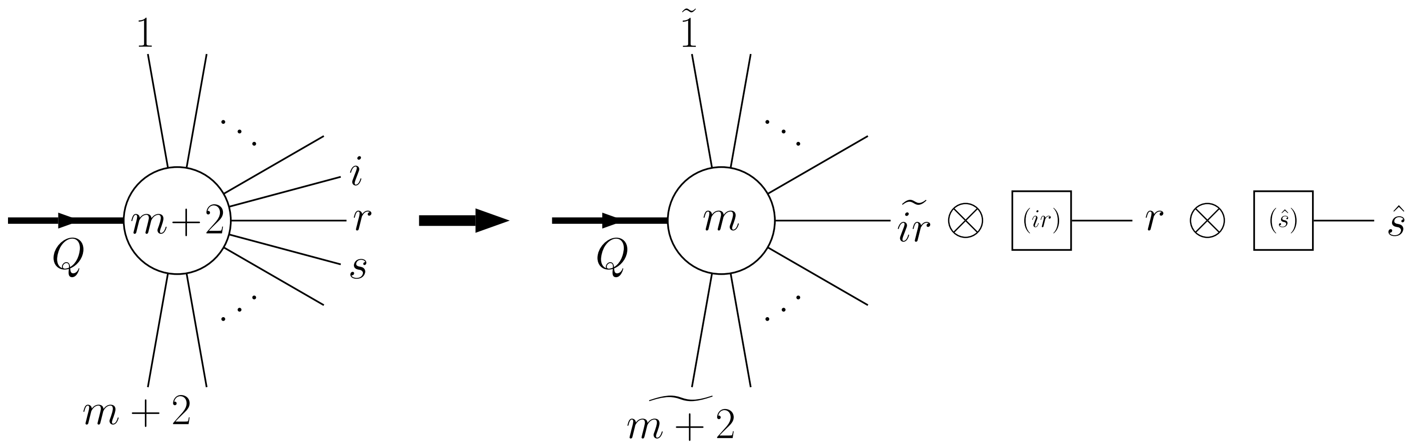

Momentum mappings (for final state radiation)

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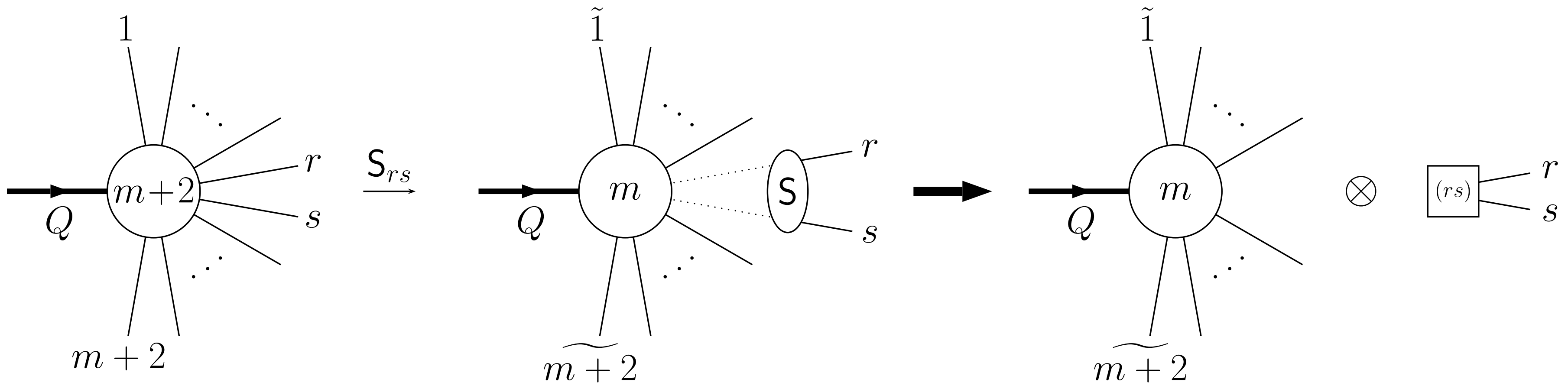
- Double soft mapping

$$\tilde{p}_n^\mu = \Lambda^\mu_\nu[Q,(Q-p_r-p_s)/\lambda_{rs}](p_n^\nu/\lambda_{rs})\,, \qquad n \neq r,s\,,$$

$$\lambda_{rs} = \sqrt{1-\Big(y_{(rs)Q} - y_{rs}\Big)}$$



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- The momentum mappings define extensions of the limit formulae
  $\Longrightarrow$ these extensions define true subtraction terms

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$$\mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} = \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2} J_m - \left(\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}} J_m\right)$$

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Limit	Mapping	PS fact
$\mathbf{A}_1 \lvert\mathcal{M}^{(0)}_{m+2}\rvert^2$	$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_{m+1}$	$\mathrm{d}\phi_{m+2} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]$

$$\downarrow$$

$$\boldsymbol{\mathcal{A}}_1 \lvert\mathcal{M}^{(0)}_{m+2}\rvert^2$$

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Limit

$$\mathbf{A}_1 |\mathcal{M}^{(0)}_{m+2}|^2$$

$$\downarrow$$

$$\boldsymbol{\mathcal{A}}_1 |\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow$$

Mapping

$$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_{m+1}$$

PS fact

$$\mathrm{d}\phi_{m+2} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1]$$

$$\downarrow$$

$$\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} = \mathrm{d}\phi_{m+1}[\mathrm{d}p_1] \boldsymbol{\mathcal{A}}_1 |\mathcal{M}^{(0)}_{m+2}|^2$$

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Limit

$$\mathbf{A}_2 |\mathcal{M}^{(0)}_{m+2}|^2$$

$$\boldsymbol{\mathcal{A}}_2 |\mathcal{M}^{(0)}_{m+2}|^2$$

Mapping

$$\{p\}_{m+2} \longrightarrow \{\tilde{p}\}_m$$

PS fact

$$\mathrm{d}\phi_{m+2} = \mathrm{d}\phi_m [\mathrm{d}p_2]$$

$$\mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} = \mathrm{d}\phi_m [\mathrm{d}p_2] \boldsymbol{\mathcal{A}}_2 |\mathcal{M}^{(0)}_{m+2}|^2$$

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$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} = \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR,A}_2}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR,A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} J_m \right)$$

Limit

Mapping

PS fact

$$\mathbf{A}_1 \mathbf{A}_2 |\mathcal{M}^{(0)}_{m+2}|^2 \qquad \{p\}_{m+2} \to \{\hat{p}\}_{m+1} \to \{\tilde{p}\}_m \qquad \mathrm{d}\phi_{m+2} = \mathrm{d}\phi_m [\mathrm{d}p_1][\mathrm{d}p_1]$$

$$\boldsymbol{\mathcal{A}}_{12} |\mathcal{M}^{(0)}_{m+2}|^2 \longrightarrow \mathrm{d}\sigma^{\mathrm{RR,A}_{12}}_{m+2} = \mathrm{d}\phi_m [\mathrm{d}p_1][\mathrm{d}p_1] \boldsymbol{\mathcal{A}}_{12} |\mathcal{M}^{(0)}_{m+2}|^2$$

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- The momentum mappings define extensions of the limit formulae
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$$\mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} = \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_2} J_m - \left(\mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_{12}} J_m \right)$$

- Real-virtual subtraction terms

$$\mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} = \left(\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} \right) J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV,A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR,A}_1} \right)^{\mathrm{A}_1} \right] J_m$$

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  $\Longrightarrow$ these extensions define true subtraction terms
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$$\mathrm{d}\sigma_{m+2}^{\mathrm{NNLO}} = \mathrm{d}\sigma_{m+2}^{\mathrm{RR}} J_{m+2} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2} J_m - \left(\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1} J_{m+1} - \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}} J_m\right)$$

- Real-virtual subtraction terms

$$\mathrm{d}\sigma_{m+1}^{\mathrm{NNLO}} = \left(\mathrm{d}\sigma_{m+1}^{\mathrm{RV}} + \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right) J_{m+1} - \left[\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1} + \left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}\right] J_m$$

  - Integrating $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$ over the factorized phase space $\Longrightarrow \int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}$

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$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+2} = \mathrm{d}\sigma^{\mathrm{RR}}_{m+2} J_{m+2} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_2}_{m+2} J_m - \left(\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} J_{m+1} - \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_{12}}_{m+2} J_m\right)$$

- Real-virtual subtraction terms

$$\mathrm{d}\sigma^{\mathrm{NNLO}}_{m+1} = \left(\mathrm{d}\sigma^{\mathrm{RV}}_{m+1} + \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right) J_{m+1} - \left[\mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} + \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)^{\mathrm{A}_1}\right] J_m$$

  - Integrating $\mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$ over the factorized phase space $\Longrightarrow \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}$
  - We use the same construction as in the RR case to define

$$\begin{aligned} 2\mathrm{Re}\langle \mathcal{M}^{(0)}_{m+1} | \mathcal{M}^{(1)}_{m+1} \rangle \quad &\Longrightarrow \quad \mathrm{d}\sigma^{\mathrm{RV},\mathrm{A}_1}_{m+1} \\ \int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2} \quad &\Longrightarrow \quad \left(\int_1 \mathrm{d}\sigma^{\mathrm{RR},\mathrm{A}_1}_{m+2}\right)^{\mathrm{A}_1} \end{aligned}$$

Remarks

- All approximate cross sections explicitly defined for final state radiation (i.e. for $e^+e^- \to m$ jets for any m)
  - they are fully local: all colour and azimuthal correlations correctly accounted for
  - they have been checked for $e^+e^- \to 3$ jets: the regularized RR and RV pieces (i.e. $\mathrm{d}\sigma_5^{\mathrm{NNLO}}$ and $\mathrm{d}\sigma_4^{\mathrm{NNLO}}$) are finite

n	$\langle(1-t)^n\rangle_{\mathrm{RV}}/10^1$	$\langle C^n\rangle_{\mathrm{RV}}/10^1$	$\langle(1-t)^n\rangle_{\mathrm{RR}}$	$\langle C^n\rangle_{\mathrm{RR}}$
1	123 ± 1	433 ± 5	-92.7 ± 3.4	-344 ± 14
2	25.5 ± 0.2	325 ± 2	-3.07 ± 0.43	-142 ± 3
3	4.79 ± 0.03	180 ± 1	2.01 ± 0.12	6.29 ± 1.87

- No new concepts needed to include initial state radiation
  - cross the limit formulae (only collinear formulae change)
  - generalize the momentum mappings

Conclusions

- Set up a subtraction scheme for computing NNLO corrections to jet cross sections: the calculation is organised into 3 pieces: RR, RV and VV
- Approximate cross sections defined in a two step process
  - carefully match limits (as embodied in the formal operators $\mathbf{A}_1$ and $\mathbf{A}_2$)
  - extend formulae over full phase space (mom. mappings and PS fact.)
- Constructed all approximate cross sections for final state radiation explicitly
  - counterterms fully local
  - RR and RV contributions are finite as required for $e^+e^- \to 3$ jets
- To do
  - Integrate $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_2}$, $\mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_{12}}$, $\mathrm{d}\sigma_{m+1}^{\mathrm{RV},\mathrm{A}_1}$ and $\left(\int_1 \mathrm{d}\sigma_{m+2}^{\mathrm{RR},\mathrm{A}_1}\right)^{\mathrm{A}_1}$ analytically and combine with two-loop squared matrix element to get VV piece
  - Approximate cross sections for initial state radiation