3-loop splitting functions and perturbative matching of the TMDs

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in collaboration with

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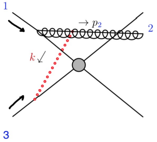
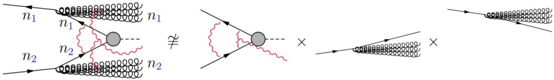
Related articles

- Published articles
  - 3-loop matching of the unpolarized quark and gluon TMDs
    - ★ "The Four Loop QCD Rapidity Anomalous Dimension", by I. Moult, H. X. Zhu and Y. J. Zhu: 2205.02249
    - ★ "Quark Transverse Parton Distribution at the Next-to-Next-to-Next-to-Leading Order", by M. X. Luo, T. Z. Yang, H. X. Zhu and Y. J. Zhu: PhysRevLett.124.092001
    - ★ "Unpolarized Quark and Gluon TMD PDFs and FFs at N^3LO", by M. X. Luo, T. Z. Yang, H. X. Zhu and Y. J. Zhu: JHEP 06, 115 (2021)
- In preparation
  - 3-loop matching of the polarized quark and gluon TMDs
    - ★ "Linearly polarized Gluon TMD PDFs and FFs at N^3LO"
    - ★ "The TMD helicity distribution and fragmentation functions at N^3LO"
    - ★ "The transversely polarized quark TMD PDFs and FFs at N^3LO"

Research backgrounds and significance

- Precision physics on the LHC and the EIC, TMDs are building blocks for TMD factorization, e.g.
  1. Transverse EEC [H. T. Li, I. Vitev , Y. J. Zhu (2021)]
  2. Di-lepton Rapidity Distribution [X. Chen, T. Gehrmann, N. Glover, A. Huss, T. Z. Yang and H. X. Zhu (2022)]
  3. Higgs transverse momentum distribution [D. Boer,. . . (2012)]
- Spin puzzles, small-x quark helicity distribution [Kovchegov,. . . (2017)]
- Gauge invariance and (non-)universality, factorization violation



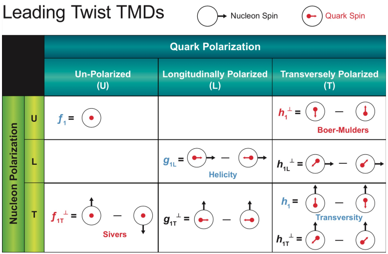


Outline

- Operator definitions
- Computational challenges and solutions
    1. Problem 1: Rapidity divergence and unconventional Feynman integrals
       Solution 1: Exponential regulator and generalized IBP identities
    2. Problem 2: γ_5 prescriptions (HVBM or Larin^+) and spurious anomaly
       Solution 2: Not clear for flavor exchanging channel
    3. Problem 3: Perturbative convergence
       Solution 3: Resummation of small-x double logarithms
- Results and symmetries
    1. Crossing symmetry, analytic continuation ($x \to 1/x$) from bare TMD beam functions to bare TMD fragmentation functions
    2. Reciprocity relation, linking space-like and time-like splitting functions
    3. Soft-rapidity correspondence ($\gamma^R(\epsilon^*) = \gamma^S$), the $\mathrm{N^4LO}$ rapidity anomalous dimension

TMD distributions as unintegrated parton distributions

- Collinear PDFs
  1. unpolarized PDF $f(x,\mu)$,parton densities with momentum fraction x
  2. helcity distribution $\Delta f(x,\mu)$, parton densities weighted with helicity
  3. transversity distribution $\delta f(x,\mu)$, asymmetries of transversely polarized partons densities in a fully transversely polarized target
- TMD PDFs $\int d^2k_T\, f^{\mathrm{TMD}}(x,k_T) = f(x)$ (naive)



Operator definitions and matching

- Collinear helicity PDFs

$$\lambda_{\text{targ}}\Delta\phi(x) = \frac{1}{2x(2\pi)^3}\int d^2k_T \langle P, \lambda_{\text{targ}}|b^\dagger_{k+}b_{k+} - b^\dagger_{k-}b_{k-}|P, \lambda_{\text{targ}}\rangle$$

$$= \int \frac{dw^-}{2\pi} e^{-ixP^+w^-}\langle P, \lambda_{\text{targ}}|\bar{\Psi}(w^-, 0_T)\frac{\gamma^+\gamma_5}{2}\Psi|P, \lambda_{\text{targ}}\rangle$$

- TMD helicity distribution as lightcone correlation functions

$$\lambda_{\text{targ}}\Delta f(x, k_T) = \frac{1}{2x(2\pi)^3}\langle P, \lambda_{\text{targ}}|b^\dagger_{k+}b_{k+} - b^\dagger_{k-}b_{k-}|P, \lambda_{\text{targ}}\rangle$$

$$= \int \frac{dw^- d^2w_T}{(2\pi)^3} e^{-ixP^+w^- + ik_T\cdot w_T}\langle P, \lambda_{\text{targ}}|\bar{\Psi}(w^-, w_T)\frac{\gamma^+\gamma_5}{2}\Psi|P, \lambda_{\text{targ}}\rangle$$

- Matching onto collinear PDFs

$$\Delta f_{i/N}(x, w_T) = \sum_i \int_x^1 \frac{d\xi}{\xi}\,\Delta\mathcal{C}_{ij}(\xi, w_T)\,\Delta\phi_{j/N}(x/\xi) + \mathcal{O}(w_T^2\Lambda^2_{\text{QCD}})$$

Symmetries and polarization dependence

- Consequence of parity symmetry

  matrix elements are linear in target helicity

- Proof by successive action of parity and rotation ($\hat{U} = \hat{R}\hat{P}$)

$$\hat{\mathcal{O}} = \boxed{b^\dagger_{k+} b_{k+} - b^\dagger_{k-} b_{k-}} \xrightarrow{\hat{P}} \boxed{b^\dagger_{\tilde{k}-} b_{\tilde{k}-} - b^\dagger_{\tilde{k}+} b_{\tilde{k}+}}$$

$$\swarrow \hat{R}$$

$$-\hat{\mathcal{O}} = \boxed{b^\dagger_{k-} b_{k-} - b^\dagger_{k+} b_{k+}}$$

so that

$$\langle P; +|\hat{\mathcal{O}}|P; +\rangle = \langle P; -|\hat{U}^\dagger \hat{\mathcal{O}} \hat{U}|P; -\rangle = -\langle P; -|\hat{\mathcal{O}}|P; -\rangle$$

The density matrix description

- the helicity density matrix

$$\hat{\Gamma}_{\rm targ} = \frac{1}{2}|P;+\rangle\langle P;+| - \frac{1}{2}|P;-\rangle\langle P;-|$$
$$\sim \begin{cases} 1/2(u_+\bar{u}_+ - u_-\bar{u}_-) = & -\not{P}\gamma_5/2 \quad \text{quark} \\ 1/2(\varepsilon_+^\mu \varepsilon_+^{*\nu} - \varepsilon_-^\mu \varepsilon_-^{*\nu}) = & -i\frac{\bar{n}_\rho P_\sigma}{2\bar{n}\cdot P}\varepsilon^{\rho\sigma\mu\nu} \quad \text{gluon} \end{cases}$$

- TMD helicity distribution

$$\overline{\Delta f(z,k_T)} = \mathrm{Tr}\left[\hat{\Gamma}_{\rm targ}\hat{\mathcal{O}}\right] = \frac{1}{2}(\langle+|\hat{\mathcal{O}}|+\rangle - \langle-|\hat{\mathcal{O}}|-\rangle) = \langle+|\hat{\mathcal{O}}|+\rangle$$

where

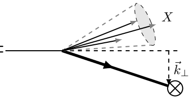
$$\mathcal{O}^q = \bar{\psi}\frac{\not{\bar{n}}}{2}\gamma_5\psi\,, \quad \mathcal{O}^g = \tilde{F}^{\mu+}F_{\mu+}\,, \quad \tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

The integrand

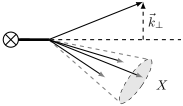
- Kinematics ($\tau = 1/\nu$, the inverse of rapidity scale)

$$\mathcal{B}_{j/i}^{\text{bare}} = \int \mathrm{dPS_X}\, e^{-b_0\tau \frac{P\cdot K}{P^+}} e^{-iK_\perp\cdot b_\perp}\delta(\frac{K^+}{P^+} - z_p)\Gamma^{aa'}\bar{\Gamma}^{bb'} \cdot \mathrm{Sp}^{\mathrm{X}}_{\mathrm{j_b^*}\leftarrow\mathrm{i_a}}\mathrm{Sp}^{\dagger\mathrm{X}}_{\mathrm{j_{b'}^*}\leftarrow\mathrm{i_{a'}}}$$

$$\mathrm{Sp}^{\mathrm{s.l.\,X}}_{\mathrm{j_b^*}\leftarrow\mathrm{i_a}} =$$



$$\mathrm{Sp}^{\mathrm{t.l.\,X}}_{\mathrm{j_b^*}\leftarrow\mathrm{i_a}} =$$



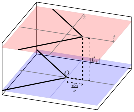
- Spin density matrix
  $\Gamma^{aa'}$ – specifying how initial states are prepared
  $\bar{\Gamma}^{bb'}$ – specifying how final states are measured

$$\Gamma_q = \{\frac{1}{2}\not{P}, \frac{1}{2}\not{P}\gamma_5, \frac{1}{2}\not{P}\gamma_\perp^\mu\gamma_5\} \quad \Gamma_g = \{-g_\perp^{\mu\nu}, i\varepsilon_\perp^{\mu\nu}\}$$

$$\bar{\Gamma}_q = \{\frac{1}{2}\bar{\not{n}}, \frac{1}{2}\bar{\not{n}}\gamma_5, \frac{1}{2}\bar{\not{n}}\gamma_\perp^\mu\gamma_5\} \quad \bar{\Gamma}_g = \{-g_\perp^{\mu\nu}, i\varepsilon_\perp^{\mu\nu}\}$$

The exponential rapidity regulator

- Soft sector [Y. Li, D. Neill and H. X. Zhu(2016)]



$$\mathcal{S}^{\text{bare}} \equiv \lim_{\tau\to 0} \text{Tr}\{\langle 0|[S_{\bar{n}}^\dagger S_n](-ib_0\tau, -ib_0\tau, b_\perp)[S_n^\dagger S_{\bar{n}}](0)|0\rangle\}|_{\tau=1/\nu}$$

- Collinear sector [Luo, Wang, Xu, Yang, Yang, H. X. Zhu (2019)]

$$\mathcal{B}_{q/N}^{\text{bare}}(x, b_\perp, \nu) \equiv \lim_{\tau\to 0} \int \frac{db_-}{2\pi} e^{-ixb_-P_+/2}$$
$$\times\{\langle N(P)|\bar{\psi}(-ib_0\tau, b_- - ib_0\tau, b_\perp)\bar{\Gamma}\psi(0)|N(P)\rangle|\}|_{\tau=1/\nu}$$

- Finite renormalized TMD distributions are regulator independent!

$$\partial_\nu[\text{TMDs}^{\text{R}}(x, b_\perp, \mu)] = \partial_\nu[\Delta\mathcal{C}_{ij}(x, b_\perp, \nu)\sqrt{\mathcal{S}(b_\perp, \nu)}] = 0$$

The IBP identities for unconventional Feynman integrals

- Generalized IBP equations [T. Z. Yang, H. X. Zhu , Y. J. Zhu (2020)]

$$0 = \int d^D q \frac{\partial}{\partial q^\mu}\left[e^{-b_0\tau\frac{P\cdot K}{P^+}} F(\{l\})\right]$$

$$= \begin{cases} \int d^D q e^{-b_0\tau\frac{P\cdot K}{P^+}}\left[-b_0\tau\frac{P_\mu}{P^+} + \frac{\partial}{\partial q^\mu}\right] F(\{l\}) & q = K \\ \int d^D q e^{-b_0\tau\frac{P\cdot K}{P^+}} \frac{\partial}{\partial q^\mu} F(\{l\}) & q \neq K \end{cases}$$

- Asymptotic expansion in the rapidity ($\tau = 1/\nu$)

$$f_i(z,\tau,\epsilon) \overset{\tau\to 0}{=} \sum_j \sum_{k=0} f_i^{(j,k)}(z,\epsilon)\tau^j \ln^k \tau$$

~~τ^ϵ~~ $\gamma_R(\epsilon^*) = \gamma_S$

- $\epsilon-$forms

$$\partial_x \vec{f}(x,\epsilon) = \epsilon M_x \vec{f}(x,\epsilon)$$

γ_5 in $D = 4 - 2\epsilon$ dimensions

- HVBM (or Larin) scheme [Larin, 1993]
  1. Evaluate integrals first
  a) Multilinear property $\mathrm{Tr}[\not{l}_1 \not{l}_2 \ldots \gamma_5] = l_1^{\mu_1} l_2^{\mu_2} \times \ldots \mathrm{Tr}[\gamma^{\mu_1}\gamma^{\mu_2} \ldots \gamma_5]$
  b) Evaluate tensor-like Feynman integrals and phase space integrals in D-dimensions
  2. Begin explicit definition of γ_5
  c) Replace the γ_5-matrix by

$$\gamma_\mu \gamma_5 = \frac{i}{6}\epsilon_{\mu\rho\sigma\tau}\gamma^\rho\gamma^\sigma\gamma^\tau \quad \text{or} \quad \gamma_5 = \frac{i}{24}\epsilon_{\mu\rho\sigma\tau}\gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\tau$$

  d) Compute trace of Dirac matrix in D dimensions
  e) Contract Levi-Civita tensors in four dimensions
- Larin^+ scheme [Reyes, Scimemi, Vladimirov (2017)]

$$\gamma^+\gamma_5 \to \frac{i\epsilon_\perp^{\alpha\beta}}{2}\gamma^+\gamma_\alpha\gamma_\beta\,, \quad \epsilon_\perp^{\alpha_1\beta_1}\epsilon_\perp^{\alpha_2\beta_2} = -g_\perp^{\alpha_1\alpha_2}g_\perp^{\beta_1\beta_2} + g_\perp^{\alpha_1\beta_2}g_\perp^{\beta_1\alpha_2}$$

γ_5 in $D = 4 - 2\epsilon$ dimensions

- Anti-commutative property of γ_5 and cyclicity of the trace can not be true at the same time ! HVBM scheme of γ_5 is not anti-commutative!

consequences: $\begin{cases} \text{1. The non-singlet axial-vector current not conserved} \\ \text{2. Violation of the Adler-Bardeen theorem} \end{cases}$

- Non-renormalization of massless non-singlet axial current

$$\partial_\mu [J^\mu_{5\,\mathrm{ns=+}}]_R = 0\,, \quad \Delta P^{\mathrm{ns=+}}_{qq}(N=1) = 0$$

- Adler-Bardeen theorem

$$\partial_\mu [J^\mu_{5\,\mathrm{s}}]_R = a_s n_f T_F [F\tilde{F}]_R$$

- Gauge invariance
- Lorentz symmetry

γ_5 in $D = 4 - 2\epsilon$ dimensions

- Refactorization (TMD distributions are physical and scheme irrelevant)

$$\mathcal{C}^{\mathrm{HVBM}} \otimes \phi^{\mathrm{HVBM}} = (\mathcal{C}^{\overline{\mathrm{MS}}} \otimes Z_5^{-1}) \otimes (Z_5 \otimes \phi^{\overline{\mathrm{MS}}})$$

where

$$Z_5 = \left\{ \begin{array}{cc} z_{qq} & 0 \\ 0 & 1 \end{array} \right\}, \quad z_{qq} = 1 + as^1 z_{\mathsf{ns}}^{(1)} + as^2 (z_{\mathsf{ps}}^{(2)} + z_{\mathsf{ns}}^{(2)}) + \dots$$

- Anti-commutative γ_5 for the non-singlet sector

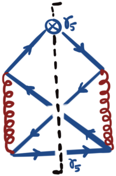


Fig 1: connected fermion line

$$Z^{\text{ns},\pm}(N) \equiv \frac{\mathcal{I}^{(\mp)}_{\text{unpol.}}}{\Delta\mathcal{I}^{(\pm)}_{\texttt{HVBM}}} = 1 + \sum_k \left(\frac{\alpha_s}{4\pi}\right)^k z^{(k)}_{\text{ns},\pm}(N)$$

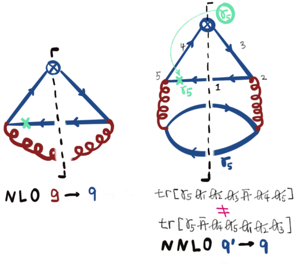
$$\boxed{\Delta P^{\text{ns}=+}_{\text{anti5}}(N=1) = 0}$$

[Larin (1993)] $Z^{\text{ns},+}|_{N=1} = 1 - \left(\frac{\alpha_s}{4\pi}\right) 4\, C_F + \left(\frac{\alpha_s}{4\pi}\right)^2 \left[22\, C_F - \frac{107}{9}\, C_F C_A + \frac{2}{9}\, C_F N_f\right] + \left(\frac{\alpha_s}{4\pi}\right)^3 (\ldots)$

γ_5 in $D = 4 - 2\epsilon$ dimensions

- 'Reading point scheme' of γ_5 for pure singlet [Kreimer (1993)]:

  first anticommutate γ_5 to another vertex in the diagram and then calculate loop integration in D dimension



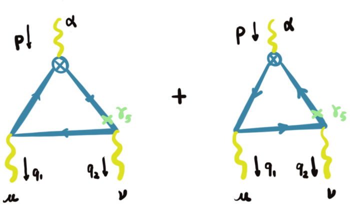
a) vanishing anomaly: $\Delta P^{\text{singlet}}_{\text{anti5}}(N=1) = 0$

b) [Y. Matiounine, J. Smith and W. L. van Neerven (1998)]

$$z^{(2)}_{\text{ps}}(N) = \Delta\mathcal{C}^{(2)}_{\overline{\text{MS}}} - \Delta\mathcal{C}^{(2)}_{\text{HVBM}} = -4C_f N_f \frac{N^3 + N^2 - 3N - 2}{N^3(N+1)^3}$$

- one-loop AVV vertex in 'reading point' scheme

$$\boxed{p_\alpha M_5^{\alpha\mu\nu} = 0} \Longleftrightarrow \boxed{\Delta P_{\text{anti5}}^{\text{singlet}}(N=1) = 0}$$



$$\boxed{q_{2\mu} M_5^{\alpha\mu\nu} = \frac{1}{2\pi^2}\epsilon^{\alpha\nu\rho\sigma} q_1^\rho q_2^\sigma}$$

Symmetries in TMDs

Numerics tells the values, but analytical expressions tells the symmetries.

- Crossing symmetry, relating time-like fragmentation functions to space-like beam functions
- Reciprocity relations of splitting functions
- Soft rapidity correspondence

Crossing symmetries

- Beam and fragmentation functions are cross-section level quantities, not analytic!
- Holomorphic part of the beam and fragmentation functions

$$\text{Holomorphic}=\int [\mathrm{d\,phase}] \times \text{loop amplitudes} \times \text{tree}^*$$

- analytic continuation

$$(-1)^{i_F}(1-z_p)^{1-2\epsilon}\widetilde{\mathcal{B}}^{(n)}_{q/i}(\frac{z_p e^{i\pi}}{1-z_p},\widetilde{K}_\perp) = \widetilde{\mathcal{F}}^{(n)}_{\bar{\imath}\leftarrow\bar{q}}(z_p,\widetilde{K}_\perp)$$

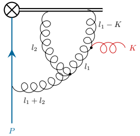
$$(-1)^{1+i_F}(1-z_p)^{1-2\epsilon}\widetilde{\mathcal{B}}^{(n),\mu\nu}_{g/i}(\frac{z_p e^{i\pi}}{1-z_p},\widetilde{K}_\perp) = \widetilde{\mathcal{F}}^{(n),\mu\nu}_{\bar{\imath}\leftarrow g}(z_p,\widetilde{K}_\perp)$$

where $z_p = 1 - z$, and z is the usual momentum fraction

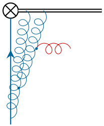
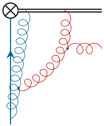
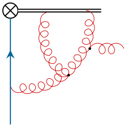
Analytic Continuation by regions

- Typical integral for VVR (space-like) contribution

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1-K)^2\,\bar{n}\cdot(K-l_1)\,l_2^2\,\bar{n}\cdot(K-l_1-l_2)\,(l_1+l_2)^2\,(l_1+l_2+P)^2}$$

$\lim\limits_{K\to 0}$ 

$$z_p \equiv \frac{\bar{n}\cdot K}{\bar{n}\cdot P}\,, \quad \text{crossing:} \quad P \to -P\,,\ z_p \to e^{i\pi} z_p$$

$=$  $+$  $+$  $+$ subleading

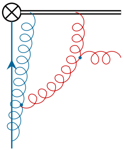
$$= e^{i2\pi\epsilon}\left[z_p^{1+2\epsilon} \sum_{i=0}^{\infty} {z_p}^i C_{\mathrm{c}}^i(\epsilon) + z_p^{\epsilon} e^{i\pi\epsilon} \sum_{i=0}^{\infty} {z_p}^i C_{\mathrm{cs}}^i(\epsilon) + z_p^{-1}\, e^{2i\pi\epsilon} \sum_{i=0}^{\infty} {z_p}^i C_{\mathrm{s}}^i(\epsilon) \right]$$

The Method of Regions ($K_\perp^2 = 1, z_p = 1 - z = \frac{\bar{n}\cdot K}{\bar{n}\cdot P}$)

- Leading collinear-soft region ($l_1 \sim K\,, l_2 \parallel P$)

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1-K)^2\,\bar{n}\cdot(K-l_1)\,l_2^2\,\bar{n}\cdot(K-l_1-l_2)\,(l_1+l_2)^2\,(l_1+l_2+P)^2}$$

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1-K)^2\,\bar{n}\cdot(K-l_1)\,l_2^2\,\bar{n}\cdot(-l_2)\,(l_2+\bar{n}\,P\cdot l_1)^2\,(l_2+P+\bar{n}\,P\cdot l_1)^2}$$



$$= z_p^\epsilon e^{i\pi\epsilon}\left[-3\epsilon^{-3}+3\epsilon^{-2}-3\epsilon^{-1}+3+32\zeta_3+\Big(-3-32\zeta_3+45\zeta_4\Big)\epsilon\right]$$

Fragmentation function from analytic continuation

- It's legal to perform analytic continuation of vRR and RRR at cross section (Re[$|$RRR$|^2$ +2 VRR $\times$ RR*]) level

Building blocks	Lowest order asymptotic behavior
VVR $\times$ R*	$c_2(\epsilon)z_p^{2\epsilon} + c_1(\epsilon)e^{i\pi\epsilon}z_p^{\epsilon} + c_0(\epsilon)e^{i2\pi\epsilon}z_p^{-1}$
VRR $\times$ RR*	$\left[c_1^0(\epsilon) + c_1^1(\epsilon)\ln\tau\right]z_p^{\epsilon} + \left[c_0^0(\epsilon) + c_0^1(\epsilon)\ln\tau\right]\frac{e^{i\pi\epsilon}}{z_p}$
$\|$RRR$\|^2$	$\left[c_0^0(\epsilon) + c_0^1(\epsilon)\ln\tau + c_0^2(\epsilon)\ln\tau^2\right]z_p^{-1}$

c_i^j are real numbers, crossing to time-like $z \to -\frac{e^{i\pi}}{z}$

- Alphabet
  1. $|$RRR$|^2$ or VRR $\times$ RR*: $\{z, \quad 1-z, \quad 1+z, \quad 2-z, \quad z^2-z+1\}$
  2. Re[$|$RRR$|^2$ +2 VRR $\times$ RR*]: $\{z, \quad 1-z, \quad 1+z\}$
  3. VVR $\times$ R* : $\{z, \quad 1-z, \quad 1+z\}$

Symmetries of splitting functions

- Agree with [Almasy, Moch and Vogt, 2011] except $P_{qg}^{T(2)}$

$$P_{qg}^{T,(2)}\big|_{\text{CLYZZ}} - P_{qg}^{T,(2)}\big|_{\text{AMV}} = \\ -2\zeta_2(C_A - C_F)\beta_0\big[-4 + 8z + z^2 + 6(1 - 2z + 2z^2)\ln z\big]$$

$$[2\gamma_{\pm}^{\text{S}}(N,\alpha_s) - \ 2\gamma_{\pm}^{\text{T}}(N + 2\gamma_{\pm}^{\text{S}}(N,\alpha_s),\alpha_s)]_{\text{CLYZZ}} = \alpha_s^3 \times (\dots) = 0$$
$$[2\gamma_{\pm}^{\text{S}}(N,\alpha_s) - \ 2\gamma_{\pm}^{\text{T}}(N + 2\gamma_{\pm}^{\text{S}}(N,\alpha_s),\alpha_s)]_{\text{AMV}} = \alpha_s^3 \times (\dots) \neq 0$$

- Reciprocity relation for eigenvalues of singlet splitting functions

$$2\gamma_{\pm}^{\text{S}}(N,\alpha_s) = 2\gamma_{\pm}^{\text{T}}(N + 2\gamma_{\pm}^{\text{S}}(N,\alpha_s),\alpha_s)$$
$$2\gamma_{\pm}^{\text{T}}(N,\alpha_s) = 2\gamma_{\pm}^{\text{S}}(N - 2\gamma_{\pm}^{\text{T}}(N,\alpha_s),\alpha_s)$$

where $\widehat{\gamma}(N,\alpha_s) = -\int_0^1 dx\, x^{N-1}\widehat{P}(x,\alpha_s)$.

Reciprocity relation

- Reciprocity in $\mathcal{N}=4$ [B. Basso and G. P. Korchemsky (2007)]

$$2\gamma_S^{\mathcal{N}=4}(N,\alpha_s) = 2\gamma_T^{\mathcal{N}=4}(N+2\gamma_S^{\mathcal{N}=4},\alpha_s)$$

- RG for EEC jet function [L. J. Dixon, I. Moult and H. X. Zhu (2019)]

$$\frac{d\vec{J}(\ln\frac{zQ^2}{\mu^2},\mu)}{d\ln\mu^2} = \int_0^1 dy\, y^2 \vec{J}(\ln\frac{zy^2Q^2}{\mu^2},\mu)\cdot\widehat{P}_T(y,\mu)$$

Solution in CFT limit: $J(zQ^2,\mu) = C_J(\alpha_s)\left(\frac{zQ^2}{\mu^2}\right)^{\gamma_J^{\mathcal{N}=4}(\alpha_s)}$, where

$$\boxed{2\gamma_J^{\mathcal{N}=4}(\alpha_s) = 2\gamma_T^{\mathcal{N}=4}(1+2\gamma_J^{\mathcal{N}=4},\alpha_s)} \Longrightarrow \boxed{\gamma_J^{\mathcal{N}=4}(\alpha_s) = \gamma_S^{\mathcal{N}=4}(1,\alpha_s)}$$

Soft rapidity correspondence

- 4-loop rapidity anomalous dimension from CFT relations

$$\gamma_S^{\mathsf{CFT}} = \gamma_R^{\mathsf{CFT}} = \gamma_R^{\mathsf{QCD}}(\epsilon^*)\,, \quad \beta^*(\epsilon^*) \equiv \frac{1}{\alpha_s}\beta(\alpha_s) - 2\epsilon^* = 0$$

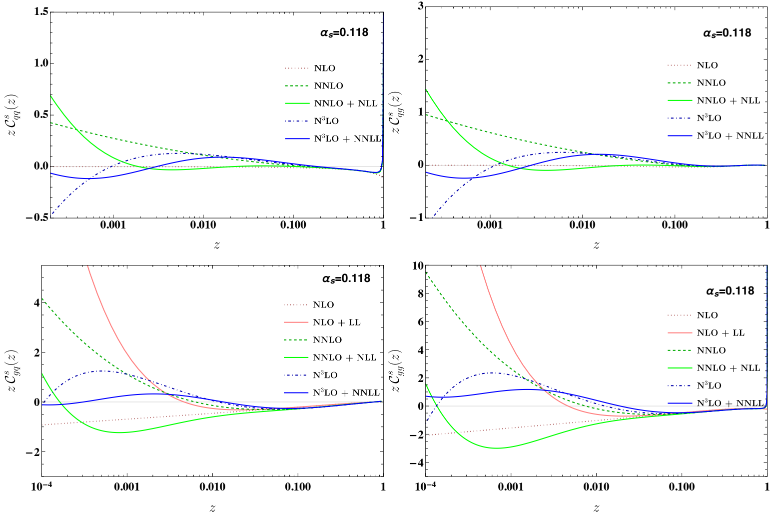
- EEC in the back-to-back limit under CFT limit [I. Moult and H. X. Zhu (2018)]

$$\frac{d\sigma}{dz}|_{\mathsf{CFT}} = \frac{1}{4}\int_0^\infty db\, bJ_0(bQ\sqrt{1-z})H(Q,\mu_h)j^q_{\mathsf{EEC}}(b,b_0/b,Q)j^{\bar{q}}_{\mathsf{EEC}}(b,b_0/b,Q)S_{\mathsf{EEC}}(b,\mu_s,\nu_s)$$
$$\exp\left[-\frac{1}{2}\Gamma^{\mathsf{cusp}}\log^2\left(\frac{b^2Q^2}{b_0^2}\right) + 2B_\delta\log\left(\frac{b^2Q^2}{b_0^2}\right) + (\gamma^R - \gamma_S)\log\left(\frac{b^2Q^2}{b_0^2}\right)\right]$$

- Using the representation of EEC as four point Wightman function in CFT [Korchemsky(2019)]

$$\frac{d\sigma}{dz} = \frac{1}{4}\int_0^\infty db\, bJ_0(bQ\sqrt{1-z})H(Q,\mu_h)j^q_{\mathsf{EEC}}(b,b_0/b,Q)j^{\bar{q}}_{\mathsf{EEC}}(b,b_0/b,Q)S_{\mathsf{EEC}}(b,\mu_s,\nu_s)$$
$$\exp\left[-\frac{1}{2}\Gamma_{\mathsf{cusp}}\log^2\left(\frac{b^2Q^2}{b_0^2}\right) + 2B_\delta\log\left(\frac{b^2Q^2}{b_0^2}\right)\right]$$

Resummation of small-x double logarithms for TMD FFs



Summary and future directions

- working status
  - 3-loop twist-2 matching of the space-like and time-like TMDs ✓
  - 3-loop polarized and unpolarized splitting functions ✓
  - small-x resummation for time-like TMDs and DGLAP kernels ✓
  - 4-loop rapidity anomalous dimensions ✓
- future directions
  - Small-x resummation for helicity splitting functions and it's contributions to the proton's missing spin ('spin puzzles') [Y. V. Kovchegov, D. Pitonyak and M. D. Sievert(2017)]
  - Soft rapidity correspondence for multi-leg processes [Ø. Almelid, C. Duhr and E. Gardi(2015)]

$$\Gamma_S = \sum_{i<j} \mathbf{T}_i \cdot \mathbf{T}_j \gamma_{\rm cusp} \ln \frac{\nu^2\, n_i \cdot n_j}{2\mu^2} - \sum_i \frac{c_i}{2} \gamma_s \mathbf{1} - \gamma_{\rm quad}$$

Appendix: Flavor decomposition

- Decompose $\mathbb{S}_N \times \mathbb{Z}_2$ into irreducible components

$$\mathbb{S}_N \times \mathbb{Z}_2 = \begin{cases} + = [N-1, +1] & (q_i + \bar{q}_i) - (q_k + \bar{q}_k) \\ - = [N-1, -1] & (q_i - \bar{q}_i) - (q_k - \bar{q}_k) \\ \mathrm{v} = [1, -1] & \sum_i q_i - \bar{q}_i \\ \mathrm{s} = [1, +1] & \sum_i q_i + \bar{q}_i \end{cases}$$

- Non-singlet DGLAP ($\mathrm{ns}=\{+,-,\mathrm{v},\mathrm{s}\}$)

$$\frac{dq_{\mathrm{ns}}}{d\ln\mu^2} = P_{\mathrm{ns}} \otimes q_{\mathrm{ns}}$$

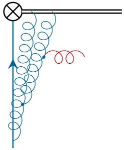
Appendix: The Method of Regions $(K_\perp^2 = 1, z_p = 1 - z = \frac{\bar{n}\cdot K}{\bar{n}\cdot P})$

- Leading double collinear region $(l_1 \parallel P\,, l_2 \parallel P)$

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1 - K)^2\,\bar{n}\cdot(K - l_1)\, l_2^2\, \bar{n}\cdot(K - l_1 - l_2)\,(l_1 + l_2)^2\,(l_1 + l_2 + P)^2}$$

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1 - \bar{n}\,P\cdot K)^2\,\bar{n}\cdot(-l_1)\, l_2^2\, \bar{n}\cdot(-l_1 - l_2)\,(l_1 + l_2)^2\,(l_1 + l_2 + P)^2}$$

$$(l_1 - K)^2 \simeq l_1^2 - 2\, l_1\cdot\bar{n}\;\; P\cdot K = (l_1 - \bar{n}\,P\cdot K)^2$$



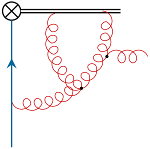
$$= z_p^{1+2\epsilon}\left[-\frac{3}{4}\epsilon^{-2} + \frac{3}{4}\epsilon^{-1} + \frac{3}{4} + \frac{3}{4}\zeta_2 + \left(-\frac{21}{4} - \frac{3}{4}\zeta_2 + 8\zeta_3\right)\epsilon\right] + \mathcal{O}(\epsilon^2)$$

Appendix:The Method of Regions($K_\perp^2 = 1, z_p = 1 - z = \frac{\bar{n}\cdot K}{\bar{n}\cdot P}$)

- Leading soft region ($l_1 \sim K\,, l_2 \sim K$)

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1-K)^2\,\bar{n}\cdot(K-l_1)\,l_2^2\,\bar{n}\cdot(K-l_1-l_2)\,(l_1+l_2)^2\,(l_1+l_2+P)^2}$$

$$\int \frac{d^D l_1}{i\pi^{2-\epsilon}} \frac{d^D l_2}{i\pi^{2-\epsilon}} \frac{1}{l_1^2\,(l_1-K)^2\,\bar{n}\cdot(K-l_1)\,l_2^2\,\bar{n}\cdot(K-l_1-l_2)\,(l_1+l_2)^2\,(2P)\cdot(l_1+l_2)}$$



$$= e^{i2\pi\epsilon} z_p^{-1}\left[-\epsilon^{-4} - \frac{5}{2}\zeta_2\epsilon^{-2} + \frac{25}{6}\zeta_3\epsilon^{-1} - \frac{17}{2}\zeta_4\right] + \mathcal{O}(\epsilon)$$