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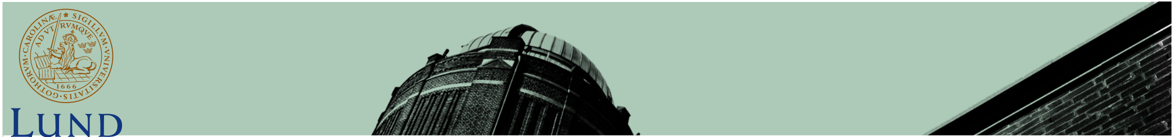


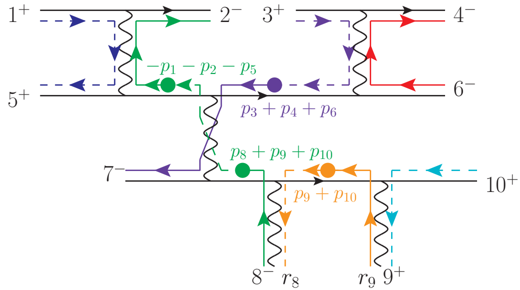
Speeding up SM Amplitude Calculations with Chirality Flow

MCNET MEETING 2022 23 SEPTEMBER 2022 - ANDREW LIFSON

BASED ON HEP-PH:2003.05877 (EPJC), HEP-PH:2011.10075 (EPJC), AND HEP-PH:2203.13618 (EPJC)

IN COLLABORATION WITH JOAKIM ALNEFJORD, CHRISTIAN REUSCHLE, MALIN SJÖDAHL, AND ZENNY WETTERSTEN

Our Main Analytical Result



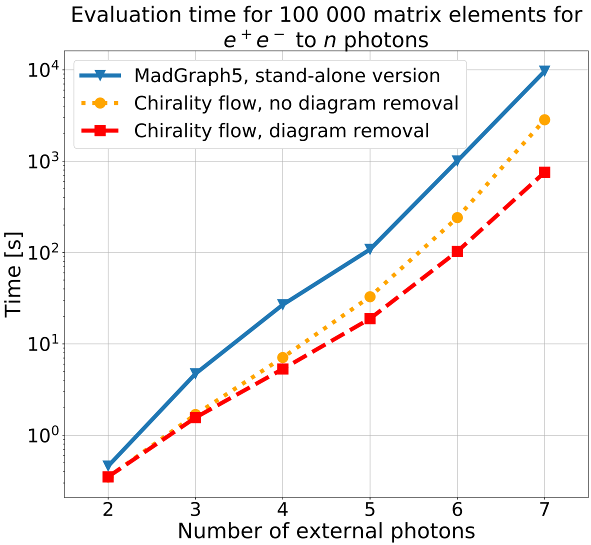
10-particle Feynman Diagram calculated in single slide

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{1\,2}\; s_{3\,4}\; s_{7\,8\,9\,10}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{1\,2\,5}\; s_{3\,4\,6}\; s_{8\,9\,10}\; s_{9\,10}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} [15]\langle 64\rangle[10\;9]$$

$$\times \Big(\langle r_9 9\rangle[9r_8] + \langle r_9 10\rangle[10 r_8]\Big)\Big(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle\Big)$$

$$\times \Big(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\;1]\langle 12\rangle - \langle 8\,10\rangle[10\;5]\langle 52\rangle\Big)$$

Our Main Numerical Result (so far) (hep-ph:2203.13618)



Spinor-Helicity: its Building Blocks

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Spinors (use chiral basis):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}$$

- Amplitude written in terms of Lorentz-invariant spinor inner products

$$\langle ij\rangle = -\langle ji\rangle \equiv \langle i||j\rangle \text{ and } [ij] = -[ji] \equiv [i||j]$$

- These are well known complex numbers, $\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor-Helicity: Vectors and Removing μ Indices

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Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

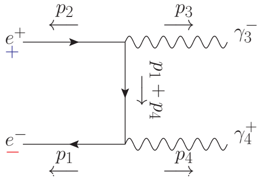
Remove vector indices with e.g.

$$\underbrace{\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]}_{\text{Fierz identity}}, \qquad \underbrace{\sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p|}_{\text{Contraction with Pauli}}$$

Polarisation vectors ($r \equiv$ gauge choice, $r^2 = 0$, $r \cdot p \neq 0$):

$$\not{\epsilon}_+(p,r) = \frac{|p]\langle r|}{\langle rp\rangle}, \qquad \not{\epsilon}_-(p,r) = \frac{|r]\langle p|}{[pr]}$$

An Illuminating Example: $e^+e^- \to \gamma\gamma$



- $|p\rangle \equiv$ right-chiral spinor
- $|p] \equiv$ left-chiral spinor
- $\tau^\mu, \bar{\tau}^\mu \equiv$ Pauli matrices
- $\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$

Spinor helicity: analytic

$$\sim \langle p_1|\bar{\tau}^\mu \underbrace{\left(|p_1]\langle p_1| + |p_4]\langle p_4|\right)}_{\not{p}_1+\not{p}_4} \bar{\tau}^\nu |p_2] \underbrace{\frac{\langle r_3|\bar{\tau}_\nu|p_3]}{\langle r_3 3\rangle}}_{\epsilon_3^-} \underbrace{\frac{[r_4|\tau_\mu|p_4\rangle}{[4 r_4]}}_{\epsilon_4^+}$$

$$= \frac{\left(\langle p_1|\bar{\tau}^\mu|p_1] + \langle p_1|\bar{\tau}^\mu|p_4]\right)[r_4|\tau_\mu|p_4\rangle \left(\langle p_1|\bar{\tau}^\nu|p_2] + \langle p_4|\bar{\tau}^\nu|p_2]\right)[p_3|\tau_\nu|r_3\rangle}{\langle r_3 3\rangle[4 r_4]}$$

$$= \underbrace{\frac{\langle 1 r_4\rangle\left([41]\langle 13\rangle + [44]\langle 43\rangle\right)[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{\text{Fierz identities like } \langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]} = \underbrace{\frac{\langle 1 r_4\rangle[41]\langle 13\rangle[r_3 2]}{\langle r_3 3\rangle[4 r_4]}}_{[ii]=0}$$

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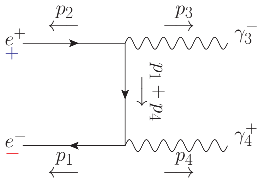

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Spinor helicity: explicit matrix multiplication

$$\sim \left[\bar{u}^-(p_1)\gamma^\mu\epsilon_\mu^+(p_4)\,(p_1^\nu + p_4^\nu)\,\gamma_\nu\gamma^\rho\epsilon_\rho^-(p_3)v^+(p_2)\right]$$

- Also cache and recycle various components
- Most common numerical method

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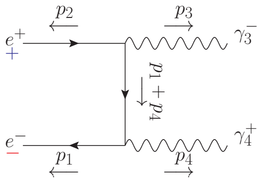

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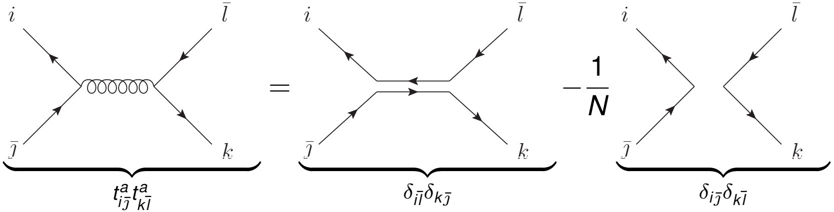
- Also cache and recycle various components
- Most common numerical method

Can we systematically remove need for algebra or matrix multiplication?

Colour Flow: a Quick Introduction

Standard method in $SU(N)$-colour calculations:

Write all objects in terms of $\delta_{i\bar{\jmath}} \equiv$ flows of colour (for simplicity $\mathrm{T_R} = 1$)
Calculations done pictorially, not via indices



Chirality Flow Building Blocks

Key idea: su(2) = su(N) (hep-ph:2003.05877)

Draw & connect lines to directly obtain inner products $\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$
Removes need to do algebra or matrix multiplication

- Define spinors as lines

$$\bar{u}_i^- = \bar{v}_i^+ = \langle i|^{\alpha} =$$ 

, $$u_j^+ = v_j^- = |j\rangle_{\alpha} =$$ 

$$\bar{u}_i^+ = \bar{v}_i^- = [i|_{\dot{\beta}} =$$ 

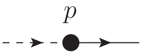
, $$u_j^- = v_j^+ = |j]^{\dot{\beta}} =$$ 

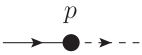
- Spinor inner products follow

$$\langle i|^{\alpha}|j\rangle_{\alpha} \equiv \langle ij\rangle = -\langle ji\rangle =$$ 

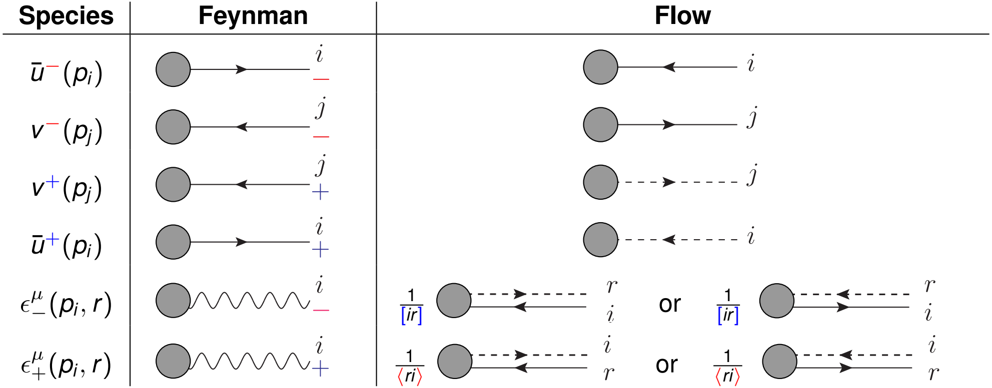
$$[i|_{\dot{\beta}}|j]^{\dot{\beta}} \equiv [ij] = -[ji] =$$ 

- Define slashed momentum as dot

$$p\!\!\!/ \equiv \sqrt{2}p^{\mu}\tau_{\mu}^{\dot{\alpha}\beta} =$$ 

, $$\bar{p\!\!\!/} \equiv \sqrt{2}p_{\mu}\bar{\tau}^{\mu}_{\alpha\dot{\beta}} =$$ 

The Massless QED Flow Rules: External Particles



Left-chiral $\equiv$ dotted lines right-chiral $\equiv$ solid lines

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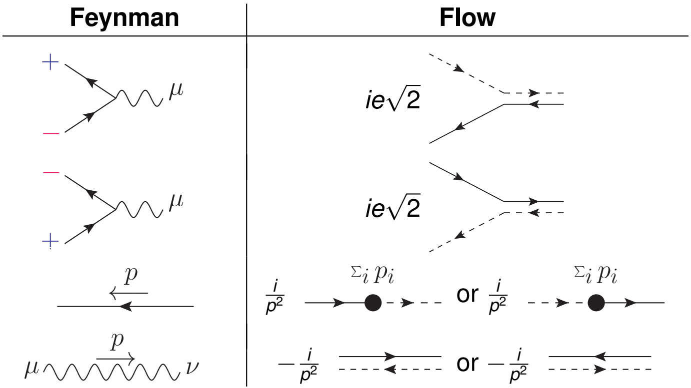
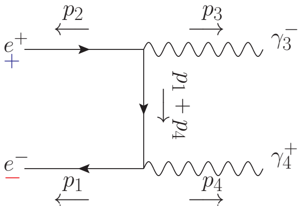
The QED Flow Rules: Vertices and Propagators



Left-chiral ≡ dotted lines   right-chiral ≡ solid lines

An Illuminating Example: $e^+e^- \to \gamma\gamma$



Spinor helicity:

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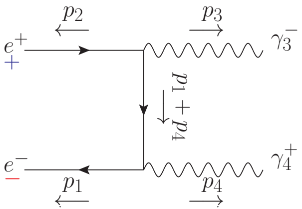
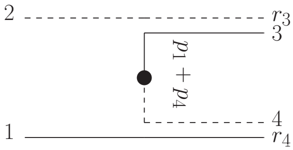


Chirality flow:

$$\sim \frac{1}{\langle r_3 3\rangle [4 r_4]}$$



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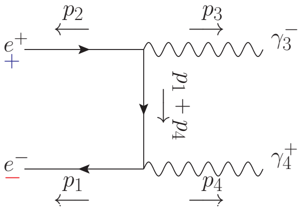

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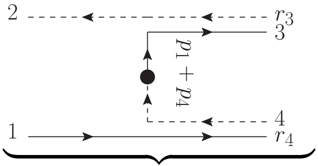
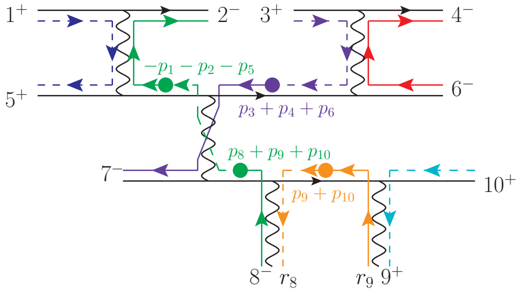
$$\sim \frac{1}{\langle r_3 3\rangle [4 r_4]}$$



$\langle 1 r_4\rangle [41] \langle 13\rangle [r_3 2]$

A complicated QED Example



Spinor-helicity analytic:

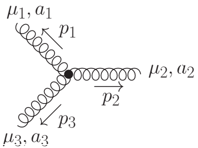
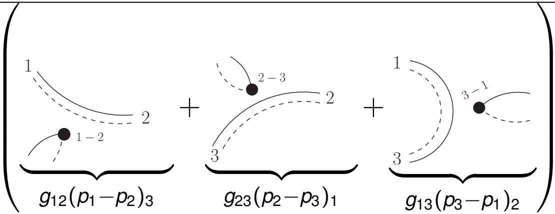
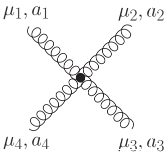
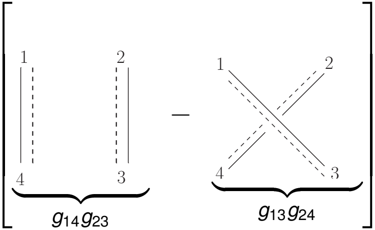
- 5 charge conjugation/Fierz + rearranging
- Not possible to fit on single slide!

$$= \underbrace{(\sqrt{2}ei)^8}_{\text{vertices}} \underbrace{\frac{(-i)^3}{s_{12}\, s_{34}\, s_{78910}}}_{\text{photon propagators}} \underbrace{\frac{(i)^4}{s_{125}\, s_{346}\, s_{8910}\, s_{910}}}_{\text{fermion propagators}} \underbrace{\frac{1}{[8r_8]\langle r_9 9\rangle}}_{\text{polarization vectors}} [15]\langle 64\rangle[10\,9]$$

$$\times \left(\langle r_9 9\rangle[9r_8] + \langle r_9 10\rangle[10 r_8]\right)\left(\underbrace{[33]}_{0}\langle 37\rangle + [34]\langle 47\rangle + [36]\langle 67\rangle\right)$$

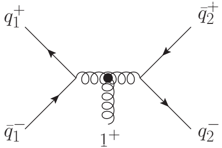
$$\times \left(-\langle 89\rangle[91]\langle 12\rangle - \langle 89\rangle[95]\langle 52\rangle - \langle 8\,10\rangle[10\;1]\langle 12\rangle - \langle 8\,10\rangle[10\;5]\langle 52\rangle\right)$$

The Non-abelian Massless QCD Flow Vertices

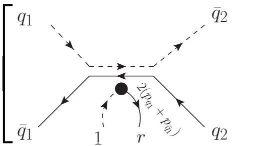
Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ 
	$ig_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 

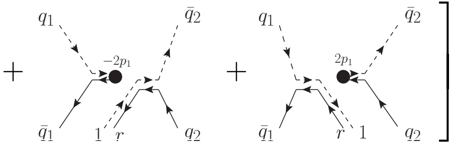
Arrow directions only consistently set within full diagram

QCD Example: $q_1 \bar{q}_1 \to q_2 \bar{q}_2 g$



$$= \frac{i g_s^3}{2 s_{q_1\bar{q}_1} s_{q_2\bar{q}_2} \langle r 1 \rangle}$$





$$\left[\cdots\right] \equiv \Big\{ 2[q_1\bar{q}_2]\langle q_2\bar{q}_1\rangle\big([1q_1]\langle q_1 r\rangle + [1\bar{q}_1]\langle 1 r\rangle\big)$$

$$- 2[q_1 1]\langle 1\bar{q}_1\rangle\langle q_2 r\rangle[1\bar{q}_2] + 2[q_1 1]\langle r\bar{q}_1\rangle\langle q_2 1\rangle[1 q_2]\Big\}$$

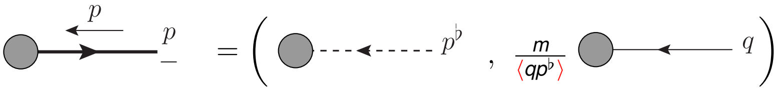
Massive Chirality Flow (hep-ph:2011.10075)

Decompose massive momentum into massless ones

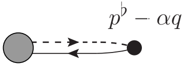
$$p^\mu = p^{\flat,\mu} + \alpha q^\mu\,, \quad (p^\flat)^2 = q^2 = 0\,, \quad \alpha = \frac{p^2}{2p^\flat\cdot q}$$

- Spinors contain both chiralities, e.g.

$\bar{v}^-(p) =$

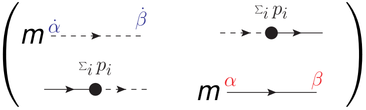


- Add new polarisation vector $\not{\epsilon}_0 = \frac{1}{m\sqrt{2}}$



- Need matrix structure in fermion propagators and vertices, e.g.

$p^\mu\gamma_\mu - m \sim$



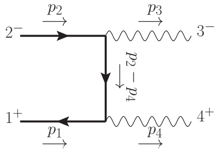
Massive Chirality Flow (hep-ph:2011.10075)

Main conclusion

Matrix structure unavoidable with massive fermions
Proceed as before to calculate without algebra

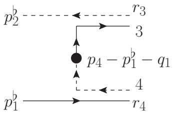
A Massive *Illuminating* Example

Consider the same diagram of $f_1^+ \bar{f}_2^- \rightarrow \gamma_3^+ \gamma_4^-$ as before but include mass m_f

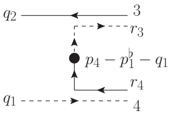


- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$

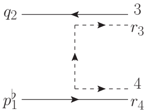
$$= \frac{-2ie^2}{(s_{23} - m_f^2)\langle r_3 3\rangle [4 r_4]} \Bigg\{$$



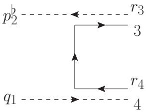
$$- \sqrt{\alpha_1 \alpha_2} e^{i(\varphi_2 - \varphi_1)}$$



$$+ m_f \Bigg(\sqrt{\alpha_2} e^{i\varphi_2}$$



$$- \sqrt{\alpha_1} e^{-i\varphi_2}$$



$$\Bigg) \Bigg\}$$

Summary

- So far: Numerical calculations use explicit multiplication rather than spin algebra analytically because quicker
- We have made the analytical spin algebra trivial
- Can we use this to make even faster numerics?

MadGraph and the Automation of Chirality Flow

Summary

- So far: Numerical calculations use explicit multiplication rather than spin algebra analytically because quicker
- We have made the analytical spin algebra trivial
- Can we use this to make even faster numerics?

Use MadGraph5_aMC@NLO (MG5aMC) for proof of concept automation

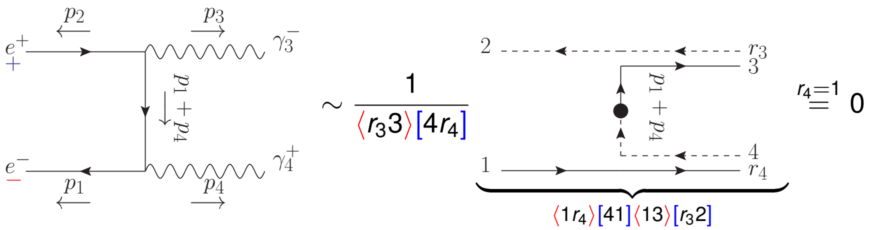
- Make minimal changes to massless QED in MG5aMC
- Pro: any difference in speed from our changes $\Rightarrow$ sound conclusions
- Con: MG5aMC not designed for chirality flow $\Rightarrow$ not optimal implementation

Sources of Expect Speed Gains

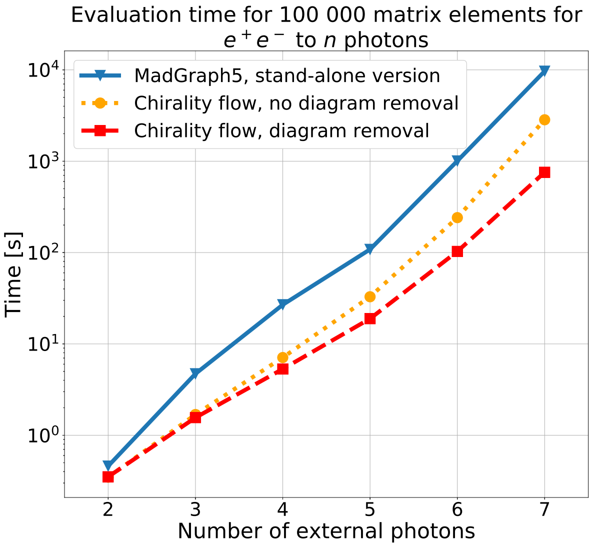
1. Simplified vertices and propagators
   - We minimise matrix multiplication
   - Each component of a calculation is simpler

Sources of Expect Speed Gains

1. Simplified vertices and propagators
   - We minimise matrix multiplication
   - Each component of a calculation is simpler
2. Gauge-based diagram removal
   - Polarisation vectors contain arbitrary gauge-reference spinor of momentum r
   - Spinor inner products antisymmetric $\Rightarrow \langle ii\rangle = [jj] = 0$
   - Chirality-flow makes optimal choice of r obvious $\Rightarrow$ remove diagrams!



Our Main Result (hep-ph:2203.13618)



Conclusions and Outlook

Conclusions:

- Chirality flow is the shortest route from Feynman diagram to complex number
- We have flow rules for full SM at tree level
- We automised it for massless QED, found significant gains in MadGraph

Outlook and other work in this area:

- Simon Plätzer and Malin Sjödahl used chirality flow as basis for resummation (hep-ph:2204.03258)
- Use method analytically to calculate loop amplitudes
  - Ongoing work by AL, Simon Plätzer, and Malin Sjödahl,
- Automate for rest of (tree-level) Standard Model and tweak algorithm to use all possible features of chirality flow
  - Two current master students working to achieve this

Reminder: Lorentz Group Representations

Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $\quad J_i \equiv$ rotations, $\quad K_i \equiv$ boosts

- Lorentz group generators $\simeq$ 2 copies of su(2) generators
  - $so(3,1)_{\mathbb{C}} \cong su(2) \oplus su(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i)\,, \quad [N_i^-, N_j^+] = 0\,,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^-\,, \qquad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations
  - $(0,0)$ scalar particles
  - $(\frac{1}{2}, 0)$ left-chiral and $(0, \frac{1}{2})$ right-chiral Weyl (2-component) spinors.
  - $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$, Dirac (4-component) spinors.
  - $(\frac{1}{2}, \frac{1}{2})$ vectors, e.g. gauge bosons

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Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors:

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp\rangle}, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

- r is a (massless) arbitrary reference momentum ($p \cdot r \neq 0$)
- Different r choices correspond to different gauges

$$\epsilon_+^\mu(p,r') - \epsilon_+^\mu(p,r) = -p^\mu \frac{\langle r'r\rangle}{\langle r'p\rangle\langle rp\rangle}$$

- Gauge invariant quantities must be r-invariant
  - Choose r as conveniently as possible (remember $\langle ij\rangle = -\langle ji\rangle$ s.t. $\langle ii\rangle = 0$) (4-gluon amplitude: can make 20/21 terms vanish)
  - Variance under $r \to r'$ good check of gauge invariance of (partial) amplitude

Spinor-Helicity: Vectors and Removing μ Indices

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Dirac matrices in chiral basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \quad \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

Remove $\tau/\bar{\tau}$ matrices in amplitude with

$$\underbrace{\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj]}_{\text{Fierz identity}}, \qquad \underbrace{\langle i|\bar{\tau}^\mu|j] = [j|\tau^\mu|i\rangle}_{\text{Charge Conjugation}}$$

Express (massless) p^μ in terms of spinors

$$p^\mu = \frac{[p|\tau^\mu|p\rangle}{\sqrt{2}} = \frac{\langle p|\bar{\tau}^\mu|p]}{\sqrt{2}}, \quad \sqrt{2}p^\mu\tau_\mu \equiv \not{p} = |p]\langle p|, \quad \sqrt{2}p^\mu\bar{\tau}_\mu \equiv \bar{\not{p}} = |p\rangle[p|$$

Spinor-Helicity: Gauge Bosons in Terms of Spinors

Lorentz algebra $so(3,1) \cong su(2) \oplus su(2)$
Consider massless particles: chirality $\sim$ helicity

Outgoing polarisation vectors ($r \equiv$ gauge choice, $r^2 = 0$, $r \cdot p \neq 0$):

$$\epsilon_+^\mu(p,r) = \frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp \rangle}, \qquad \epsilon_-^\mu(p,r) = \frac{[r|\tau^\mu|p\rangle}{[pr]}$$

$$p \cdot \epsilon_+(p,r) = \underbrace{\frac{\langle r|p^\mu \bar{\tau}_\mu|p]}{\langle rp \rangle} = 0}_{\text{Weyl eq. } p^\mu \bar{\tau}_\mu |p]=0} \qquad p \cdot \epsilon_-^\mu(p,r) = \underbrace{\frac{[r|p^\mu \tau_\mu|p\rangle}{[pr]} = 0}_{\text{Weyl eq. } p^\mu \tau_\mu |p\rangle=0}$$

$$\epsilon_+(p,r) \cdot (\epsilon_-)^*(p,r) = \underbrace{\frac{\langle r|\bar{\tau}^\mu|p]}{\langle rp \rangle} \frac{[r|\tau_\mu|p\rangle}{[pr]}}_{\epsilon_\pm = (\epsilon_\mp)^*} = \frac{\langle rp \rangle [rp]}{\langle rp \rangle [pr]} = \underbrace{-1}_{[pr] = -[rp]}$$

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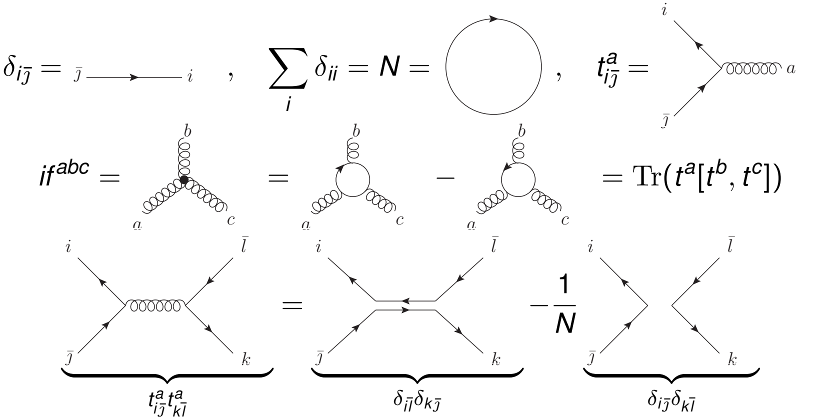
Colour Flow: a Quick Introduction

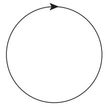
Standard method in $SU(N)$-colour calculations:

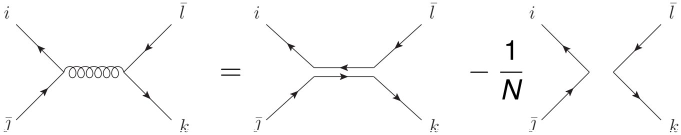
Write all objects in terms of $\delta_{i\bar{j}} \equiv$ flows of colour (for simplicity $\mathrm{T_R} = 1$)
Calculations done pictorially, not via indices

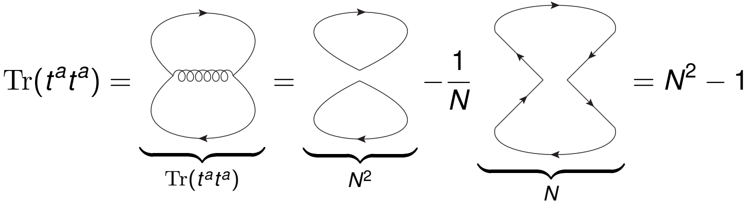


Colour Flow: a Quick Introduction

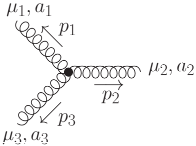
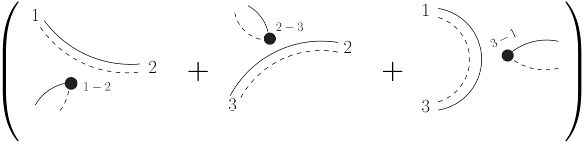
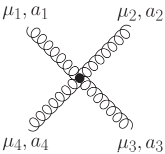
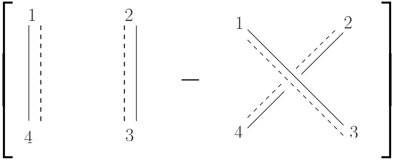
Standard method in $SU(N)$-colour calculations:

Calculations done pictorially, not via indices $\sum_i \delta_{ii} = N =$ 





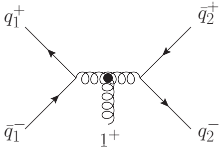
The Non-abelian Massless QCD Flow Vertices

Feynman	Flow
	$-\frac{g_s f^{abc}}{2}$ 
	$i g_s^2 \sum_{Z(2,3,4)} f^{a_1 a_2 b} f^{b a_4 a_3}$ 

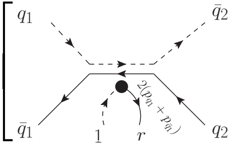
Arrow directions only consistently set within full diagram

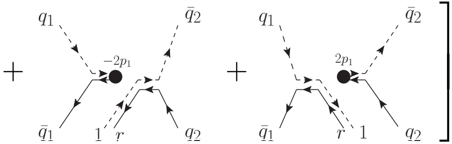
Double line $\equiv g_{\mu\nu}$, momentum dot $\equiv p_\mu$

QCD Example: $q_1 \bar{q}_1 \to q_2 \bar{q}_2 g$



$$= \frac{i g_s^3}{2 s_{q_1 \bar{q}_1} s_{q_2 \bar{q}_2} \langle r 1 \rangle}$$





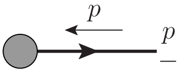
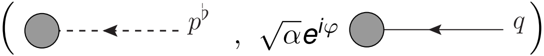
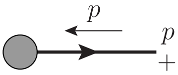
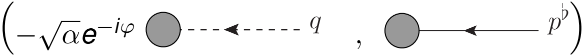
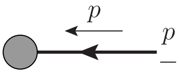
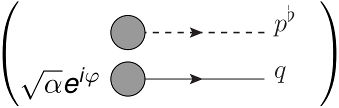
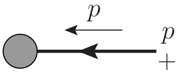
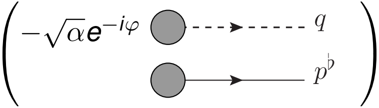
$$\left[\cdots\right] \equiv \Big\{ 2[q_1 \bar{q}_2]\langle q_2 \bar{q}_1\rangle \big([1 q_1]\langle q_1 r\rangle + [1 \bar{q}_1]\langle 1 r\rangle\big)$$

$$- 2[q_1 1]\langle 1 \bar{q}_1\rangle\langle q_2 r\rangle[1 \bar{q}_2] + 2[q_1 1]\langle r \bar{q}_1\rangle\langle q_2 1\rangle[1 q_2] \Big\}$$

Incoming Massive Spinors in Chirality Flow

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad e^{i\varphi}\sqrt{\alpha} = \frac{m}{\langle p^\flat q\rangle} \,, \qquad e^{-i\varphi}\sqrt{\alpha} = \frac{m}{[qp^\flat]}$$

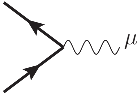
$$\text{Spin operator } -\frac{\Sigma^\mu s_\mu}{2} = \frac{\gamma^5 s^\mu \gamma_\mu}{2}, \qquad s^\mu = \frac{1}{m}(p^{\flat,\mu} - \alpha q^\mu)$$

Spinor	Feynman	Flow
$\bar{v}^-(p)$		
$\bar{v}^+(p)$		
$u^-(p)$		
$u^+(p)$		

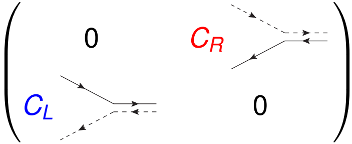
Some Fermion Flow Rules

$$p^\mu = p^{\flat,\mu} + \alpha q^\mu \,, \quad (p^\flat)^2 = q^2 = 0 \,, \quad \alpha = \frac{p^2}{2p\cdot q} \neq 0$$

Fermion-vector vertex

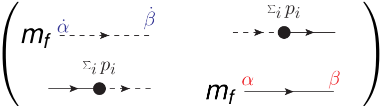


$$= ie(P_L C_L + P_R C_R)\gamma^\mu = ie\sqrt{2}$$



Fermion propagator

$$\frac{i}{p^2 - m_f^2}\begin{pmatrix} m_f \delta^{\dot\alpha}{}_{\dot\beta} & \sqrt{2}p^{\dot\alpha\beta} \\ \sqrt{2}\bar{p}_{\alpha\dot\beta} & m_f \delta_\alpha{}^\beta \end{pmatrix} = \frac{i}{p^2 - m_f^2}$$

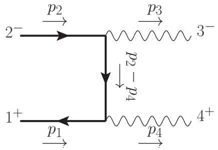


Left and right chiral couplings may differ

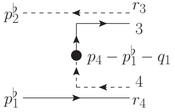
A Massive *Illuminating* Example

Consider the same diagram of $f_1^+ \bar{f}_2^- \rightarrow \gamma_3^+ \gamma_4^-$ as before but include mass m_f

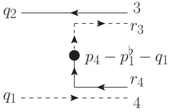
- Obtain 3 new terms
- Simplify with choices of q_1, q_2, r_3, r_4
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$



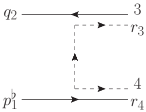
$$= \frac{-2ie^2}{(s_{23} - m_f^2)\langle r_3 3\rangle [4 r_4]} \Bigg\{$$



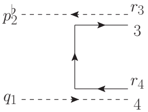
$$- \sqrt{\alpha_1 \alpha_2}\, e^{i(\varphi_2 - \varphi_1)}$$



$$+ m_f \Bigg(\sqrt{\alpha_2}\, e^{i\varphi_2}$$

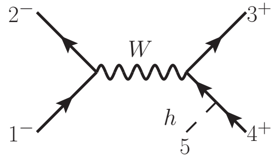


$$- \sqrt{\alpha_1}\, e^{-i\varphi_2}$$



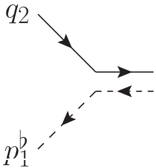
$$\Bigg) \Bigg\}$$

A Second Massive Example: $f_1 \bar{f}_2 \to W \to f_3 \bar{f}_4 h_5$

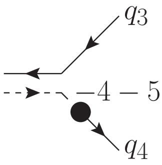


- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
- $e^{i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{\langle p_i^\flat q_i \rangle}$, $\quad e^{-i\varphi_i}\sqrt{\alpha_i} = \frac{m_i}{[q_i p_i^\flat]}$
- Scalar has no flow line

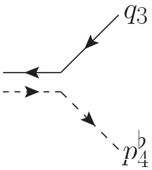
Step 1: Draw fermion lines: $\sim C_{L,12}\sqrt{\alpha_2}e^{i\varphi_2}$



$\times\, C_{L,34}\sqrt{\alpha_3}(-e^{i\varphi_3}) \Big[\sqrt{\alpha_4}(-e^{i\varphi_4})$



$+\, m_4$



$\Big]$

A Second Massive Example: $f_1\bar{f}_2 \to W \to f_3\bar{f}_4 h_5$

- W bosons simplifies ($C_R = 0$)
- Simplify with choices of $q_1, \cdots q_5$
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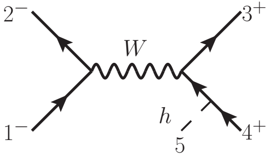
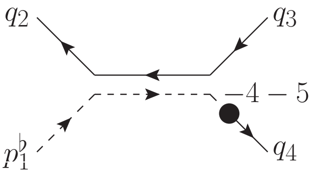
- Scalar has no flow line

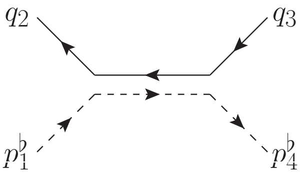


Step 2: Flip arrows and connect: $C_{L,12}C_{L,34}\sqrt{\alpha_2\alpha_3}e^{i(\varphi_2+\varphi_3)}$

$\times \Bigg[\sqrt{\alpha_4}e^{i\varphi_4}$



$- \; m_4$



$\Bigg]$

Lorentz Group Representations

Lorentz group elements: $e^{i(\theta_i J_i + \eta_i K_i)}$ $J_i \equiv$ rotations, $K_i \equiv$ boosts

- Lorentz group generators $\simeq$ 2 copies of su(2) generators
  - $so(3,1)_{\mathbb{C}} \cong su(2) \oplus su(2)$

Group algebra defined by commutator relations

$$[J_i, J_j] = i\epsilon_{ijk} J_k, \quad [J_i, K_j] = i\epsilon_{ijk} K_k, \quad [K_i, K_j] = -i\epsilon_{ijk} J_k$$

$$N_i^{\pm} = \frac{1}{2}(J_i \pm iK_i) , \quad [N_i^-, N_j^+] = 0 ,$$

$$[N_i^-, N_j^-] = i\epsilon_{ijk} N_k^- , \quad [N_i^+, N_j^+] = i\epsilon_{ijk} N_k^+$$

- Representations (i.e. realisations of $N_i^{\pm}$)
  - $(0,0)$ scalar particles
  - $(\frac{1}{2},0)$ left-chiral and $(0,\frac{1}{2})$ right-chiral Weyl (2-component) spinors.
  - $(\frac{1}{2},0) \oplus (0,\frac{1}{2})$, Dirac (4-component) spinors.
  - $(\frac{1}{2},\frac{1}{2})$ vectors, e.g. gauge bosons

How to Calculate? Spinor-Helicity

Give each particle a defined helicity $\Rightarrow$ amplitude now a number!

Spinors (in chiral basis):

$$u^+(p) = v^-(p) = \begin{pmatrix} 0 \\ |p\rangle \end{pmatrix} \qquad u^-(p) = v^+(p) = \begin{pmatrix} |p] \\ 0 \end{pmatrix}$$

$$\bar{u}^+(p) = \bar{v}^-(p) = \begin{pmatrix} [p| & 0 \end{pmatrix} \qquad \bar{u}^-(p) = \bar{v}^+(p) = \begin{pmatrix} 0 & \langle p| \end{pmatrix}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sqrt{2}\tau^\mu \\ \sqrt{2}\bar{\tau}^\mu & 0 \end{pmatrix} \qquad \sqrt{2}\tau^\mu = (1, \vec{\sigma}), \ \ \sqrt{2}\bar{\tau}^\mu = (1, -\vec{\sigma}),$$

- Amplitude written in terms of Lorentz-invariant spinor inner products

$$\langle ij\rangle = -\langle ji\rangle \equiv \langle i||j\rangle \text{ and } [ij] = -[ji] \equiv [i||j]$$

  - These are well known complex numbers, $\langle ij\rangle \sim [ij] \sim \sqrt{2p_i \cdot p_j}$
- Remove $\tau/\bar{\tau}$ matrices in amplitude with

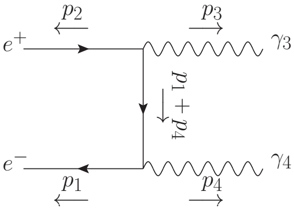
$$\langle i|\bar{\tau}^\mu|j][k|\tau_\mu|l\rangle = \langle il\rangle[kj] \,, \qquad \langle i|\bar{\tau}^\mu|j] = [j|\tau^\mu|i\rangle$$

How to Calculate a Process

Sum all Feynman diagrams, square, and integrate

Often spin structure is non-trivial

e.g.



$$\sim \underbrace{[\bar{u}(p_1)\gamma^\mu (p_1^\nu + p_4^\nu)\gamma_\nu \gamma^\rho v(p_2)]\,\epsilon_\rho(p_3)\epsilon_\mu(p_4)}_{\text{A mathematical expression we have simplify and square}}$$

Most common method: use helicity basis

Each diagram is a complex number, easy to square
Can use algebra to simplify first, or brute force matrix multiplication

Define Problem

Kinematic part of amplitude slowed by spin and vector structures

- Can we still improve on this?
  - Deriving spinor inner products $\langle ij \rangle$, $[kl]$ requires at least 2 steps
    - Re-write every object as spinors
    - Use Fierz identity $\bar{\tau}^{\mu}_{\alpha\dot{\beta}} \tau_{\mu}^{\dot{\alpha}\beta} = \delta_{\alpha}^{\ \beta} \delta^{\dot{\alpha}}_{\ \dot{\beta}}$
  - Not intuitive which inner products we obtain
- In SU(N) use graphical reps for calculations
  - E.g. using the colour-flow method
  - (Also birdtracks etc.)
- Spinor-helicity $\equiv su(2) \oplus su(2)$
  - Can we use graphical reps?

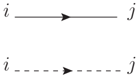
Creating Chirality Flow: Building Blocks

A flow is a directed line from one object to another
$su(2)$ objects have dotted indices and $su(2)$ objects undotted indices

- First step: Ansatz for spinor inner products (only possible Lorentz invariant)

$$\langle i|^{\alpha}|j\rangle_{\alpha} \equiv \langle ij\rangle = -\langle ji\rangle =$$

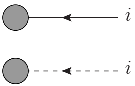
$$[i|_{\dot\beta}|j]^{\dot\beta} \equiv [ij] = -[ji] =$$

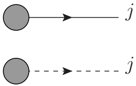


- Spinors and Kronecker deltas follow

$$\langle i|^{\alpha} = \quad , \qquad |j\rangle_{\alpha} =$$

$$[i|_{\dot\beta} = \quad , \qquad |j]^{\dot\beta} =$$





$$\delta_{\alpha}^{\ \beta} \equiv \mathbb{1}_{\alpha}^{\ \beta} = \quad , \qquad \delta^{\dot\beta}_{\ \dot\alpha} \equiv \mathbb{1}^{\dot\beta}_{\ \dot\alpha} =$$