Multi-emission Kernels for Parton Branching Algorithms[a]

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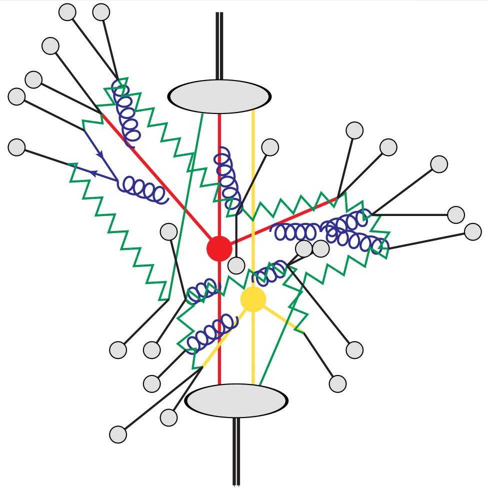
23 Sep 2022, Graz

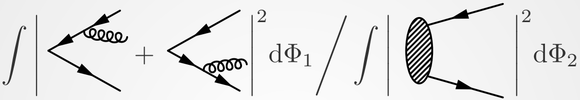
[a]in collaboration with Simon Plätzer and Emma Simpson Dore. arXiv:2112.14454



[Simon Plätzer]

$$\mathrm{d}\sigma \simeq \mathrm{d}\sigma_{\mathrm{hard}}(Q) \times \mathrm{PS}(Q \to \mu) \times \mathrm{Had}(\mu \to \Lambda) \times \ldots$$

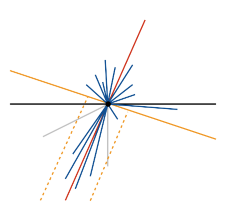
Parton shower status



- Despite pushes for higher orders in parton showers (*e.g.* [Prestel, Hoeche—Phys.Rev.D 96 (2017) 7, 074017], [Skands, Li—PLB 771 (2017) 59-66])
  Road to accuracy requires paradigm shift
  - **Recoil, ordering, colour, correlations**
    [Bewick, Seymour, Richardson—JHEP 04 (2020) 019], [Forshaw, Holguin, Plätzer–JHEP 09 (2020) 014], [Ruffa, Plätzer—JHEP 06 (2021) 007], [ML, Plätzer, Simpson—2112.14454], [also see PanScales]
- **Amplitude level** sets the complexity for resolving these
  [Nagy, Soper], [DeAngelis, Forshaw, Plätzer— PRL 126 (2021) 11, 112001 & JHEP 05 (2018) 044]

Non-global observables

- **Coherent branching via angular ordering** essential for including large-angle soft contributions
- No global measure of deviation from jet configuration: **Coherent branching fails**
- **Dipole shower**: correct LL@LC for non-global, but issues in NLL@LC and LL@NLC for global observables



Non-global observables

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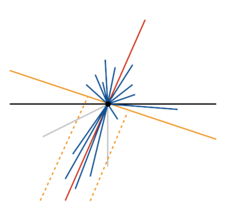


- Require dipole-type soft gluon evolution (to account for change in colour structure)
- Even with a dipole approach, $1/N_C$ effects possibly become comparable to subleading logs, and intrinsically $\sim 10\%$ effects

Bucket list

$\Rightarrow$ Study approximations in emission iterations rather than iterations of one emission approximation.
Or: amplitude vs. cross-section level

Goal: NLL@NLC accuracy for global and non-global observables

- Going beyond iterated $1 \to 2$ splittings in parton showers
- Combine with global recoil scheme
- Include color and spin correlations

} Systematic expansion to handle uncertainties $\Leftrightarrow$ higher logarithmic accuracy

- Refine ad hoc models of MC-programs, *e.g.* azimuthal correlations
- Define language for connecting fixed order to parton showers

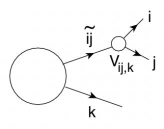
Comparison to CS dipoles

- Catani-Seymour dipole operators reproduce the partitioned soft and collinear behaviour for one emission:

$$\mathcal{D}_{ij,k}(p_1, ..., p_{m+1}) = -\frac{1}{2p_i \cdot p_j} \qquad (5.2)$$

$$\cdot\, {}_m\!< 1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 | \frac{\boldsymbol{T}_k \cdot \boldsymbol{T}_{ij}}{\boldsymbol{T}_{ij}^2} \boldsymbol{V}_{ij,k} | 1, .., \widetilde{ij}, .., \widetilde{k}, .., m+1 >_m \;.$$

$$< s | \boldsymbol{V}_{q_i g_j,k}(\tilde{z}_i; y_{ij,k}) | s' > \;=\; 8\pi\mu^{2\epsilon}\alpha_S\, C_F \left[\frac{2}{1-\tilde{z}_i(1-y_{ij,k})} - (1+\tilde{z}_i) - \epsilon(1-\tilde{z}_i) \right] \delta_{ss'}$$



[Catani, Seymour '97]

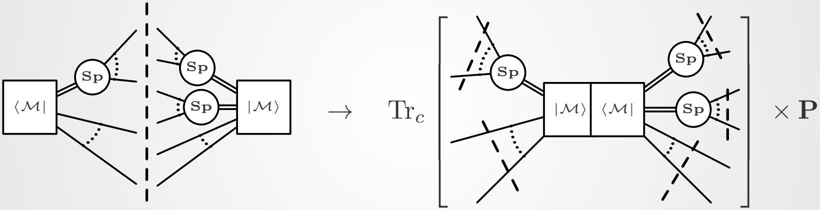
- Our idea: algorithmic generation of such splitting kernels for > 1 emission
- Generate partitioned soft behaviour via power counting instead of construction 'by hand'
- Potential for constructing subtraction terms

Splitting kernels

Splitting kernels from amplitudes

From the cross-section level to decomposed amplitudes:

$$\sigma = \sum_n \int \mathrm{Tr}\left[\, |\mathcal{M}(\mu)\rangle \langle \mathcal{M}(\mu)| \,\right] u(p_1, \ldots, p_n) \mathrm{d}\phi_n$$



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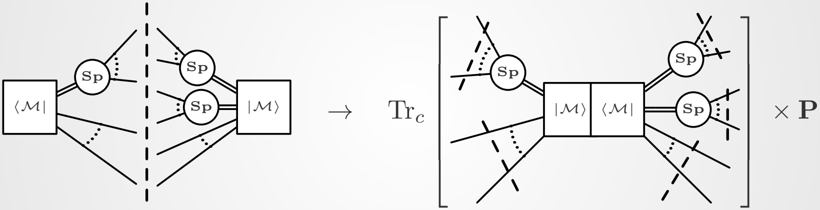
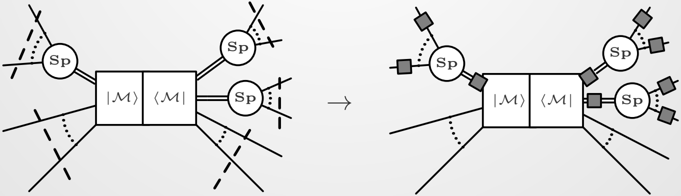
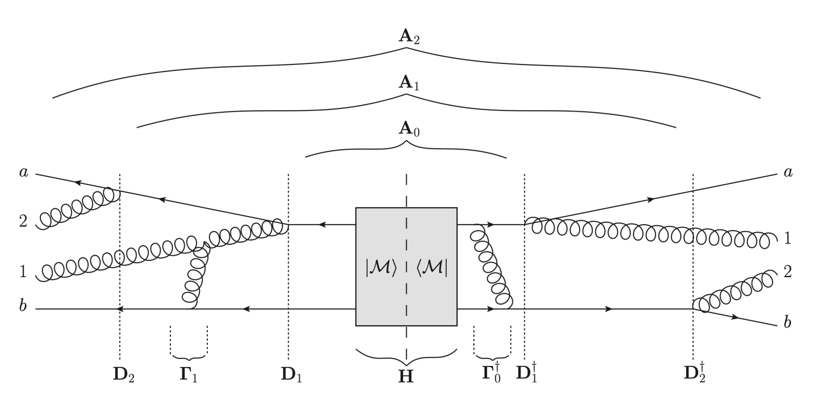




Splitting kernel iterations

Density operator language is useful for discussing emissions in iterative manner:



[Forshaw, Holguin, Plätzer–JHEP 09 (2020) 014]

Partitioning

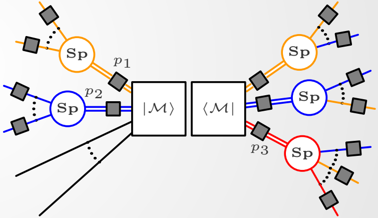
Disentangling different collinear sectors

- Use partition of one in terms of all possible collinear pairings

$$1 = \mathbb{P}_1^{(\mathcal{A})} + \mathbb{P}_2^{(\mathcal{A})} + \mathbb{P}_3^{(\mathcal{A})} + \dots$$

where $\mathbb{P}_i^{(\mathcal{A})}$ projects onto collinearity w.r.t. p_i for some amplitude $\mathcal{A}$

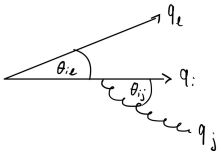
- **Disentangle overlapping collinear singularities**
- Keep smooth interpolation over whole phase space



Angular ordering and subtractions

- Radiation of a soft gluon leads to

$$\mathrm{d}\sigma_{n+1} = \mathrm{d}\sigma_n \times \frac{\mathrm{d}\omega}{\omega}\frac{\mathrm{d}\Omega}{2\pi}\frac{\alpha_S}{2\pi}\sum_{i,l} C_{il} W_{il}$$



where $\quad W_{il} = \dfrac{\omega^2 p_i \cdot p_l}{p_i \cdot p_j \, p_l \cdot p_j}$: 'Radiation function'

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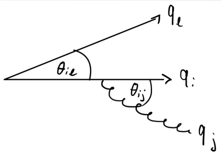

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  - Can decompose $W_{il} = W_{il}^{[i]} + W_{il}^{[l]}$

$$W_{il}^{[i]} = \frac{1}{2}\left(W_{il} - \frac{1}{1-\cos\theta_{jl}} + \frac{1}{1-\cos\theta_{ij}}\right)$$

  - Then azimuthal averaging confines emissions to cone

$$\int_0^{2\pi} \frac{\mathrm{d}\phi_{ij}}{2\pi} W_{il}^{[i]} = \begin{cases} \frac{1}{1-\cos\theta_{ij}} & \text{if } \theta_{ij} < \theta_{il}, \\ 0 & \text{otherwise.} \end{cases}$$

  - Textbook knowledge: subtraction partitioning implies angular ordering [Ellis, Stirling, Webber]

Subtraction partitioning

- As an alternative to fractional partitioning, define subtraction scheme:

$$\mathbb{P}_{(i\|j)}\left[\frac{1}{S_{ij}\,S_{jl}}\right] = \frac{1}{2}\left(\frac{1}{S_{ij}\,S_{jl}} - \Delta_{(j\|l)} + \Delta_{(i\|j)}\right),$$

$$\mathbb{P}_{(j\|l)}\left[\frac{1}{S_{ij}\,S_{jl}}\right] = \frac{1}{2}\left(\frac{1}{S_{ij}\,S_{jl}} - \Delta_{(i\|j)} + \Delta_{(j\|l)}\right),$$

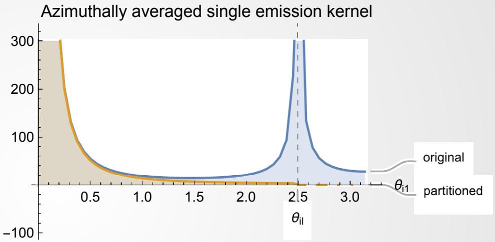
$$\Delta_{(i\|j)} = \frac{E_i}{E_j}\frac{1}{S_{il}S_{ij}}, \quad \Delta_{(j\|l)} = \frac{E_l}{E_j}\frac{1}{S_{il}S_{jl}}.$$

  by exploiting $S_{ij} \xrightarrow{(j\|l)} E_i E_j\, n_i \cdot n_l = \frac{E_j}{E_l} S_{il}$
- $\mathbb{P}_{(i\|j)}\left[\ldots\right]$ non-singular in $(j \parallel l)$-limit while original singular behaviour is reproduced in $(i \parallel j)$-limit
- **Algorithmic generalisation to multi emissions under control**

Subtraction partitioning behaviour

Recent work: subtraction partitioning $\Longrightarrow$ angular ordering for 2E?

$$\mathcal{A} \propto \frac{1}{S_{ij} S_{jl}}:$$



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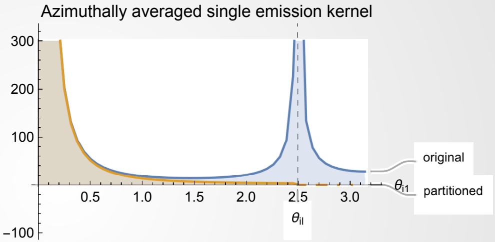
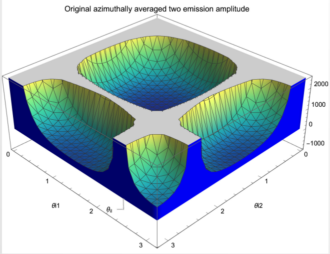
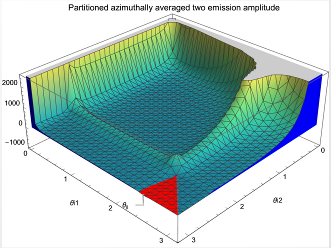
$$\mathcal{A} \propto \frac{1}{S_{ij}S_{jl}}:$$

$$\mathcal{A} \propto \frac{1}{S_{ij}S_{jk}S_{jl}S_{kl}}:$$





partitioning $\xrightarrow{\hspace{2cm}}$ $(i\|j)(k\|l)$



Fractional partitioning for two emissions

Alternatively: cancel out 'unwanted' collinear singularities by partitioning factors

- Read
  $(i \parallel j \parallel k) : S_{ijk} = (q_i + q_j + q_k)^2 \to 0$
- Collect non-singular factors in triple collinear and coll-coll pairings

configuration	$\mathcal{A} \propto \frac{1}{S_{ij} S_{kl} S_{ijk} S_{jkl}}$
$i \parallel j \parallel k$	$S_{kl} S_{jkl}$
$i \parallel j \parallel l$	$S_{kl} S_{ijk} S_{jkl}$
$i \parallel k \parallel l$	$S_{ij} S_{ijk} S_{jkl}$
$j \parallel k \parallel l$	$S_{ij} S_{ijk}$
$(i \parallel j), (k \parallel l)$	$S_{ijk} S_{jkl}$
$(i \parallel k), (j \parallel l)$	$\times$
$(i \parallel l), (j \parallel k)$	$\times$

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$(i \parallel j), (k \parallel l)$	$S_{ijk} S_{jkl}$
$(i \parallel k), (j \parallel l)$	×
$(i \parallel l), (j \parallel k)$	×

$\Rightarrow$ Construct partitioning factors of the form

$$\mathbb{P}^{(\mathcal{A})}_{(ijk)} = \frac{S_{kl} S_{jkl}}{S_{kl} S_{jkl} + S_{ij} S_{ijk} + S_{ijk} S_{jkl} + (S_{kl} + S_{ij}) S_{ijk} S_{jkl}}$$

- $\mathbb{P}^{(\mathcal{A})}_{(ijk)} \times \mathcal{A}$ **extracts the** $(i \parallel j \parallel k)$**- singular behaviour**
- $\mathbb{P}^{(\mathcal{A})}_{(ijk)}$ **is non-singular in any collinear configuration**

Power Counting

Power counting

- Discuss soft and collinear scaling of internal lines in general way
- Sudakov-like decomposition of momenta:

$$q_I^\mu = \sum_{k \in I} r_{ik} = z_I \, {p_i}^\mu + \frac{S_I + p_{\perp,I}^2}{2 z_I \; p_i \cdot n} \, n^\mu + {k_{\perp,I}}^\mu \; ,$$

- Decompose fermion and gluon lines (factors of $\sqrt{z_I}$ absorbed in vertices for fermions):

 $= \not{p}_i,$

 $= d^{\mu\nu}(p_i),$

 $= \frac{S_I + p_{\perp,I}^2}{2 z_I^2 \, p_i \cdot n} \not{n},$

 $= \frac{S_I + p_{\perp,I}^2}{(z_I \, p_i \cdot n)^2} \, n^\mu n^\nu,$

 $= \frac{\not{k}_{\perp,I}}{z_I},$

 $= \frac{k_{\perp,I}^\mu n^\nu + n^\mu k_{\perp,I}^\nu}{z_I \, p_i \cdot n}.$

- Leads to power counting rules with potential connection to SCET

Soft and collinear scaling

- Algorithmically determine soft or collinear scaling of an emission amplitude via scaling of internal lines (and propagators)

Scaling of hard lines:

	h	h+c	h+s	h+c+s
	λ	λ	λ	λ (bal.)
	0	λ	λ	λ (unbal.)
	λ^2	λ^2	λ	λ (bal.)
	0	λ^2	λ	λ (unbal.)

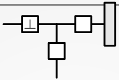
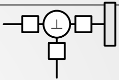
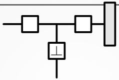
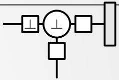
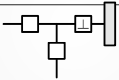
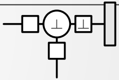
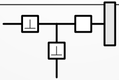
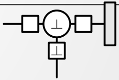
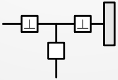
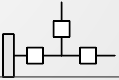
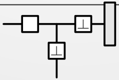
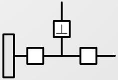
Scaling of emissions:

	s	c	s+c
	1	λ	λ
	1	λ^2	λ

- Note differences between mappings, *e.g.* with and without balanced $k_\perp$-components

One emission amplitudes

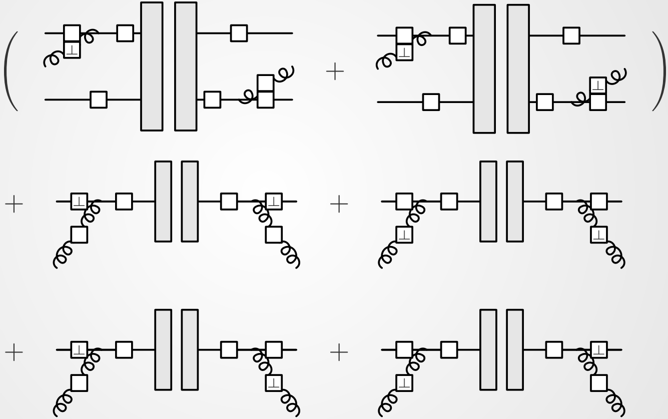
- Determine list of all relevant sub-amplitudes via power counting rules
- Combine these in density operator ($\simeq$ squared amp) to find full splitting kernel

	C	S		C	S
	λ	λ		λ	λ
	λ	1		λ^2	λ^2
	λ	λ		λ^2	λ^2
	λ^2	λ		λ^2	λ
	λ^2	λ^2		1	1
	λ^2	λ		λ	1

One emission example

Full one emission (ij)-splitting kernel (balanced mapping) consists of

$$\mathbb{U}_{(ij)} = \mathbb{P}_{(ij)} \Bigg(\quad + \quad \Bigg)$$



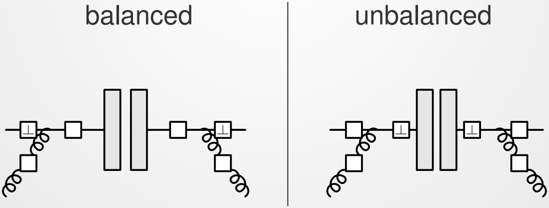
- **Exhibits factorisation to hard amplitude**
- **Smooth interpolation between soft and collinear limits**
- **Algorithmically generalizable for more emissions**

Balanced vs. unbalanced mapping

- Can test different implementations of momentum mappings, *e.g.* the balancing of transverse components

$$k_{\perp,I}{}^{\mu} = \sum_{i \in I} k_{\perp,I}^{(i),\mu} \,,$$

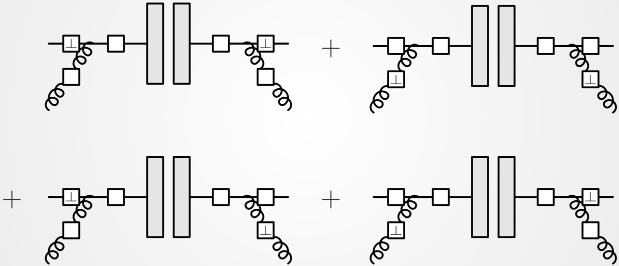
- Yields different sets of diagrammatic contributions
- Nevertheless, the same collinear and soft behaviour is reproduced for one emission



- Still: can compare mappings and check for inconsistencies for > 1 emission

Check: One emission splitting function

- Reproduce **Splitting function** P_{qg} as a crosscheck



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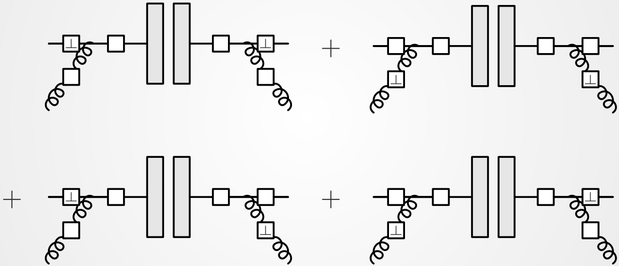
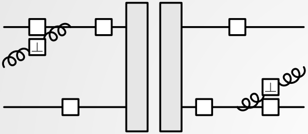


$$\rightarrow \frac{4\pi\alpha_s \mathbf{T}_i^2}{S_{ij}} \left[(d-2)\alpha_i + 4\frac{(1-\alpha_i)^2}{\alpha_i} + 4(1-\alpha_i) \right] \not{p}_i + \mathcal{O}(\lambda^{-1}).$$

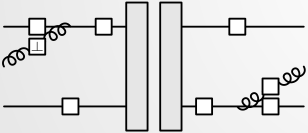
Soft-Collinear Interplay

- Soft singular part of splitting function cancelled by:



$$\propto \frac{4\pi\alpha_s \mathbf{T}_i\cdot\mathbf{T}_k}{S_{jk}} \frac{4(1-\alpha_i)^2}{\alpha_i} \frac{p_k\cdot n}{p_i\cdot n}[\not{p}_i][\not{p}_k]$$

- Eikonal part remains:

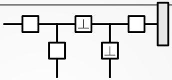
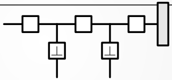
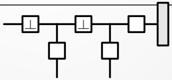
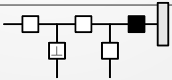


$$\propto \frac{4\pi\alpha_s \mathbf{T}_i\cdot\mathbf{T}_k}{S_{ij}S_{jk}} 4(1-\alpha_i)\sqrt{\frac{\tilde{\beta}_i}{\alpha_i}} p_k\cdot n_\perp [\not{p}_i][\not{p}_k]$$

- Smooth interpolation between soft and collinear limits in $\mathbb{U}_{(ij)}$
- Current work: investigate this interplay for two emissions

Two emissions: splitting amplitudes

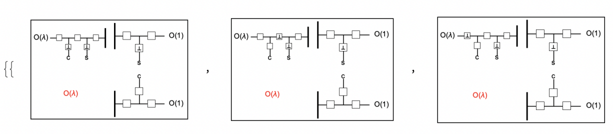
- Same procedure applies to two emissions
- Some amplitudes can not be achieved by single emission iteration
- Signals for violation of exact factorisation (drop out for two emissions though)

	C_1C_2	C_1S_2	S_1C_2	S_1S_2
	λ^2	λ	λ^2	λ
	λ^2	λ	λ	1
	λ^2	λ^2	λ^2	λ^2
	λ^3	λ^2	λ	λ
$\vdots$				

Two emissions: combined contributions

- Determine amplitude scaling algorithmically:

```
combinedAmpsB2[{c, s, c, s}, 1]
```



- Combine with partitioned propagator scaling to find all leading contributions for full kernel

	CC	CS	SC	SS
$A^{(1)}$	$1/\lambda^2$	$1/\lambda^3$	$1/\lambda$	$1/\lambda^4$
$A^{(2)}$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda^4$
$A^{(3)}$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda$	$1/\lambda^3$
$A^{(4)}$	$1/\lambda^3$	$1/\lambda^2$	$1/\lambda$	$1/\lambda^4$
$A^{(5)}$	$1/\lambda^3$	$1/\lambda$	$1/\lambda$	$1/\lambda^4$
$B^{(1)}$	$1/\lambda^3$	$1/\lambda^4$	$1/\lambda^2$	$1/\lambda^4$
$B^{(2)}$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^4$
$B^{(3)}$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^2$	$1/\lambda^4$
$B^{(4)}$	$1/\lambda^3$	$1/\lambda^4$	$1/\lambda^2$	$1/\lambda^4$
$B^{(5)}$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^4$
$B^{(6)}$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^2$	$1/\lambda^4$
$X^{(1)}$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda^4$	$1/\lambda^4$
$X^{(2)}$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda^2$
$E^{(1)}$	$1/\lambda^4$	$1/\lambda^4$	$1/\lambda^2$	$1/\lambda^4$
$E^{(2)}$	$1/\lambda^4$	$1/\lambda^3$	$1/\lambda^3$	$1/\lambda^4$
$E^{(3)}$	$1/\lambda^4$	$1/\lambda^3$	$1/\lambda^2$	$1/\lambda^4$
$E^{(4)}$	$1/\lambda^4$	$1/\lambda^3$	$1/\lambda^2$	$1/\lambda^4$
$E^{(5)}$	$1/\lambda^4$	$1/\lambda^2$	$1/\lambda^2$	$1/\lambda^4$

Conclusions

Goal: **universal algorithm for handling accuracy in multiple emissions** (for applications in parton showers and beyond)

- Density-operator formalism to study iterative behaviour of emissions
- Partitioning algorithms to separate overlapping singularities
- Momentum mapping for exposing collinear and soft factorization
- Global recoil via Lorentz transformation
- Set of power counting rules to single out leading amplitudes
- Can handle and compare different momentum mappings
- Two-emission kernels/power counting under control

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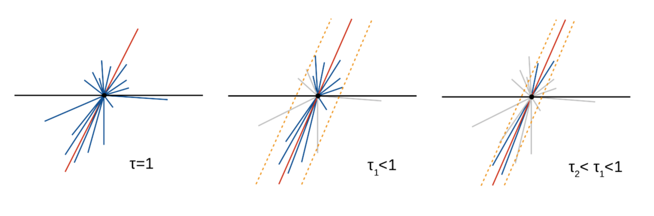
Thank you!

Backup slides

Coherent branching

- **Coherent emission of soft large angle gluons** from systems of collinear partons
- **Angular ordering** essential for including large-angle soft contributions



[Simon Plätzer]

- Resummation of global jet observables such as thrust τ
- NLL accurate @Next-to-Leading-Colour (NLC) if inclusive over secondary soft gluon emission

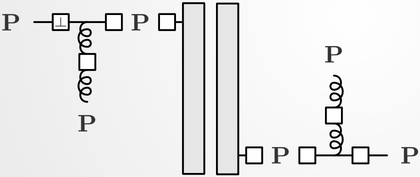
Applications

- Use projectors and helicity sums to represent emission amplitudes as (complex) weights for numerical evaluation

$$\mathbf{P}(q) \equiv \begin{cases} P^{\rho\sigma}(p) = d^{\rho\sigma}(p), & \text{(gluon)}, \\ \not{P}(p) = \frac{\not{n}}{2n\cdot p}, & \text{(quark)}, \end{cases}$$

$$d^{\mu\nu}(p) = \epsilon_+^{\mu}(p,n)\epsilon_-^{\nu}(p,n) + (\mu \leftrightarrow \nu),$$

$$\not{n} = \sum_\lambda u_\lambda(n)\bar{u}_\lambda(n),$$



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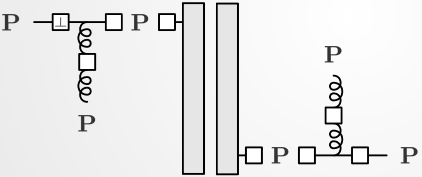
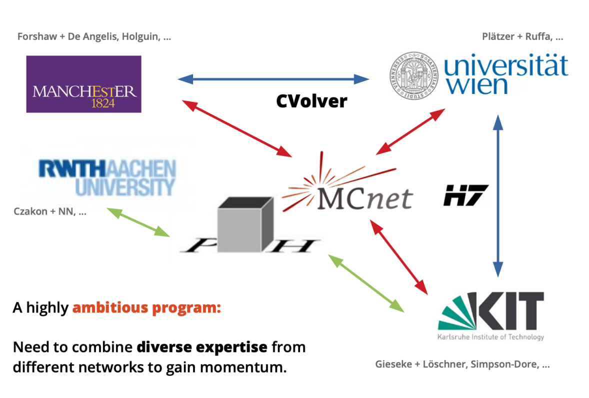


$$\to \sum_{\lambda_i,\bar{\lambda}_i} \frac{u_{\lambda_1}}{\sqrt{2n\cdot p_i}} \left[\frac{\bar{u}_{\lambda_1}}{\sqrt{2n\cdot p_i}} \not{k}_\perp \not{\epsilon}_{\lambda_3} \not{p}_i \frac{u_{\lambda_2}}{\sqrt{2n\cdot p_i}} \right] \frac{\bar{u}_{\lambda_2}}{\sqrt{2n\cdot p_i}} \epsilon^\sigma_{\lambda_3}$$

$$\times \frac{u_{\bar{\lambda}_1}}{\sqrt{2n\cdot p_i}} \left[\frac{\bar{u}_{\bar{\lambda}_1}}{\sqrt{2n\cdot p_k}} \not{p}_k \frac{u_{\bar{\lambda}_2}}{\sqrt{2n\cdot p_k}} \, p_k \cdot \epsilon_{\bar{\lambda}_3} \right] \frac{\bar{u}_{\bar{\lambda}_2}}{\sqrt{2n\cdot p_k}} \epsilon_{\bar{\sigma},\bar{\lambda}_3}.$$

Team

Karlsruhe/Manchester/Vienna network with support from **SFB** drives significant parts of the development, also relating to aspects such as **color reconnection** [*e.g.* Gieseke, Kirchgaesser, Plätzer-JHEP 11 (2018) 149]



[Plätzer—Annual CRC Meeting 2019]

Algorithm for subtraction partitioning

- General form of partitioned propagator P for config σ

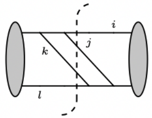
$$\mathbb{P}_{\sigma}[P] = \frac{1}{m}\left(P + (m-1)\Delta_{\sigma;\tau_1,\ldots,\tau_{m-1}}[P] - \sum_{i=1}^{m-1} \Delta_{\tau_i;\tau_1,\ldots,\tau_{i-1},\sigma,\tau_{i+1},\ldots,\tau_{m-1}}[P] \right) ,$$

- with Subtraction terms

$$\Delta_{\tau_1;\tau_2,\ldots,\tau_m}[P] = \underbrace{\mathbb{F}_{\tau_1}[P]}_{\text{non-singular bits}} \left(\underbrace{\mathbb{S}_{\tau_1}[P]}_{\text{singular bits}} - \overline{\sum_{\mathcal{S}/\tau_1}} \Delta_{\tau_{i_1};\tau_{i_2},\ldots,\tau_{i_{m-1}}} [\mathbb{S}_{\tau_1}[P]] \right) ,$$

- When partitioning *e.g.* to $\sigma = (i \parallel j \parallel k)$, subtract off all (sub-)divergences of other singular configs τ_i for propagator factor P.
- Combinatorial factor m: number of singular configs for P

Two emission example

$A^{(1)}_{ijkl} =$ 

- Partitioned version of $A^{(1)} \propto 1/S_{ij}S_{ijk}S_{kl}S_{jkl}$

$$\mathcal{P}(A^{(1)}) = \frac{1}{3}\left(\frac{1}{S_{ij}S_{ijk}S_{kl}S_{jkl}} + 2\Delta_{(ijk)}[\mathcal{P}(A^{(1)})] - \Delta_{(jkl)}[\mathcal{P}(A^{(1)})] - \Delta_{(ij)(kl)}[\mathcal{P}(A^{(1)})]\right),$$
$$+ \frac{1}{3}\left(\frac{1}{S_{ij}S_{ijk}S_{kl}S_{jkl}} - \Delta_{(ijk)}[\mathcal{P}(A^{(1)})] + 2\Delta_{(jkl)}[\mathcal{P}(A^{(1)})] - \Delta_{(ij)(kl)}[\mathcal{P}(A^{(1)})]\right),$$
$$+ \frac{1}{3}\left(\frac{1}{S_{ij}S_{ijk}S_{kl}S_{jkl}} - \Delta_{(ijk)}[\mathcal{P}(A^{(1)})] - \Delta_{(jkl)}[\mathcal{P}(A^{(1)})] + 2\Delta_{(ij)(kl)}[\mathcal{P}(A^{(1)})]\right),$$

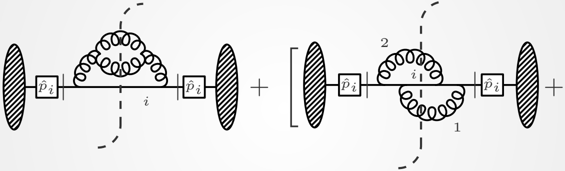
where *e.g.*

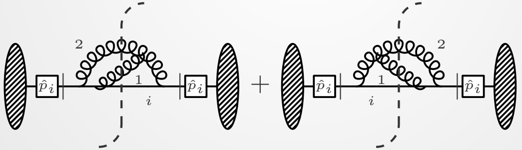
$$\Delta_{(jkl)}[\mathcal{P}(A^{(1)})] = \frac{E_l^2}{E_j(E_j+E_k)}\frac{1}{S_{il}^2}\left(\frac{1}{S_{kl}S_{jkl}} - \frac{E_iE_l}{E_j(E_l+E_k)}\frac{1}{S_{il}S_{kl}}\right),$$

Check: Two Emissions

- **Reproduced from general two-emission kernel** which includes soft-limit too (here: in lightcone-gauge)

$$\frac{\mu^{2\varepsilon}}{\hat{\alpha}^2 S_{i12}^2} \Bigg\{$$



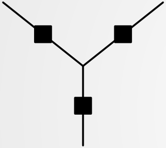
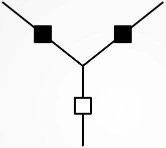
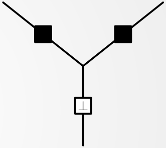


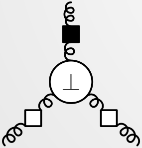
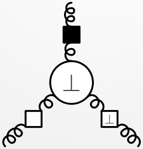
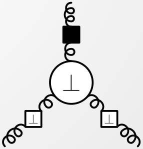
$$+ (1 \leftrightarrow 2) \Bigg] \Bigg\}_{C_A C_F}$$

$$= \left(\frac{8\pi\alpha_S}{\hat{\alpha} S_{i12}} \mu^{\varepsilon} \right)^2 C_A C_F \, \langle \hat{P}_{ggq}^{\text{(non-Ab)}} \rangle \hat{\not{p}}_i + \mathcal{O}\left(\beta_{il}^{-3/2} \right).$$

Vertex rules

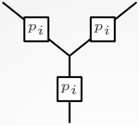
- Can find vertex rules such as:

 $= 0,$  $= 0,$  $= 0.$

 $= 0,$  $= 0,$  $= 0.$

Insights from Power Counting Rules

- Powerful vertex rule for lines belonging to same collinear sector:



$$= 0,$$

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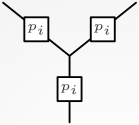
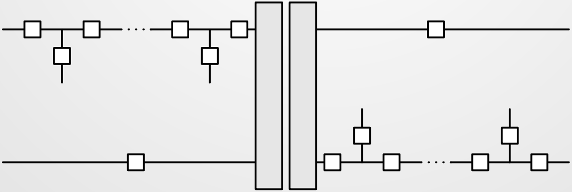


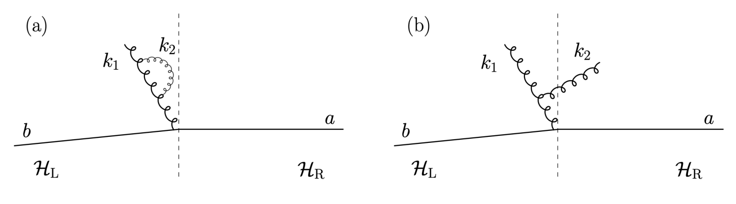
$$= 0,$$

- Shows (known fact) that interference diagrams do not contribute in splitting function in a physical gauge
- Reason: denominator goes as $1/\lambda^{2k} S^k_{(\mathrm{col})}$ for k coll. emissions
- Can only contribute in splitting function ($\propto 1/\lambda^{2k} S^k_{(\mathrm{col})}$) if numerator goes as $\mathcal{O}(1)$, but the only possible contribution $\equiv 0$



$$= 0.$$

Global and non-global observables



[Dasgupta, Salam (2001)]

- Example: heavy and light jet mass (global) vs. hemisphere jet mass (non-global)
- Cancellations between large angle-soft and virtual contributions (from k_2) not guaranteed
  $\Rightarrow$ **NLL enhancement from leftover $\alpha_S^2 L^2$ terms**

Partitioning

Amplitudes carry different singular S-invariants

$$\mathcal{A}(S_1, S_2) = \frac{\mathcal{N}(S_1, S_2)}{S_1 S_2},$$

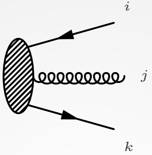
Decomposition using partitioning factors:

$$\mathbb{P}_{(1)}^{(\mathcal{A})} = \frac{S_2}{S_1 + S_2}, \quad \mathbb{P}_{(2)}^{(\mathcal{A})} = \frac{S_1}{S_1 + S_2},$$

we can decompose $\mathcal{A}$ into

$$\mathcal{A} = \left[\mathbb{P}_{(1)}^{(\mathcal{A})} + \mathbb{P}_{(2)}^{(\mathcal{A})}\right] \mathcal{A} = \frac{\mathcal{N}(S_1, S_2)}{S_1(S_1 + S_2)} + \frac{\mathcal{N}(S_1, S_2)}{S_2(S_1 + S_2)}.$$

Parton Shower



$$\int \left| \quad \cdots \quad \right|^2 \mathrm{d}\Phi_3$$

- Soft and collinear regions are of special interest:

$$S_{ij} \equiv (q_i + q_j)^2 = 2\, q_i \cdot q_j = 2 q_i^0 q_j^0 \left[1 - \cos\theta_{ij}\right], \quad \text{for } q_{i/j}^2 = 0$$

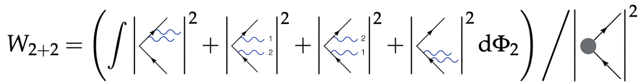
- Amplitude goes as $\propto 1/S_{ij}$
  $\Rightarrow$ becomes singular/enhanced when $S_{ij} \to 0$
- Large logarithms due to phase space integrations of the kind

$$\frac{\mathrm{d}q_j^0}{q_j^0}, \quad \frac{\mathrm{d}\theta_{ij}}{\theta_{ij}} \quad \to \alpha_S \log^2 \frac{Q}{Q_0} \sim 1$$

for some scale $Q \in \{\theta, p_\perp, \dots\}$ and cut-off Q_0

Parton shower: collinear limit

- Single emission approach is then usually iterated in a probabilistic manner



$$= 2^2 \int_{t_0}^{t} \mathrm{d}t' \int_{t_0}^{t'} \mathrm{d}t''\, W(t') W(t'') = \frac{2^2}{2!} \left(\int_{t_0}^{t} \mathrm{d}t\, W(t) \right)^2 .$$

[Stefan Gieseke]

- Sum over any number of emissions: result exponentiates

$$\sigma_{>2}(t_0) = \sigma_2(t_0) \sum_{k=1}^{\infty} \frac{2^k}{k!} \left(\int_{t_0}^{t} \mathrm{d}t\, W(t) \right)^k$$

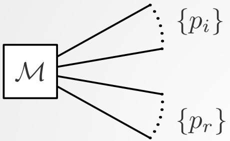
- Sudakov Form Factor ($\simeq$ no emission probability in range $t \to t_0$)

$$\Delta(t_0) = \exp\left[- \int_{t_0}^{t} \mathrm{d}t\, W(t) \right], \quad W(t) = \int_{z_-}^{z_+} \frac{\alpha_S(z,t)}{2\pi} \frac{\hat{P}(z,t)}{t} \mathrm{d}z.$$

Momentum mapping

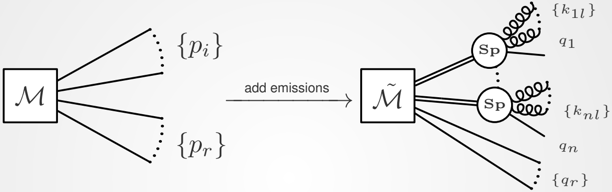
Momentum mapping

Adding emissions



- Start with **on-shell** (OS) momenta p_i (to be **emitters**) and p_r (to be **recoilers**) with overall momentum transfer $Q \equiv \sum_i p_i + \sum_r p_r$

Momentum mapping
Adding emissions



- Start with **on-shell** (OS) momenta p_i (to be **emitters**) and p_r (to be **recoilers**) with overall momentum transfer $Q \equiv \sum_i p_i + \sum_r p_r$
- Add emissions to the process with:
  1. Momentum conservation: $\sum_i q_i + \sum_{i,l} k_{il} + \sum_r q_r = Q$
  2. On-shellness of all partons
  3. Parametrization of soft & collinear behaviour for any # of emissions

Momentum mapping

$$q_r = \frac{\Lambda}{\alpha_L} p_r$$

$$k_{il} = \frac{\Lambda}{\alpha_L}\left[\alpha_{il}\, p_i + \tilde{\beta}_{il}\, n_i + \sqrt{\alpha_{il}\tilde{\beta}_{il}}\; n_{il}^{\perp}\right], \quad A_i \equiv \sum_l \alpha_{il}, \quad \tilde{\beta}_{il} = (1 - A_i)\beta_{il}$$

$$q_i = \frac{\Lambda}{\alpha_L}\left[(1 - A_i)p_i + \Big(y_i - \sum_l \tilde{\beta}_{il}\Big) n_i - \sum_l \sqrt{\alpha_{il}\tilde{\beta}_{il}}\; n_{il}^{\perp}\right]$$

- Decomposition w/ light-like momentum n_i and $n_{il}^{\perp} \cdot p_i = n_{il}^{\perp} \cdot n_i = 0$
- Need $\alpha_L^2 = (Q+N)^2/Q^2$ for momentum conservation

$$Q = \sum_r q_r + \sum_i q_i + \sum_{i,l} k_{il} = \frac{\Lambda}{\alpha_L}\Big[\underbrace{\sum_r p_r + \sum_i}_{Q}\big(p_i + \underbrace{y_i n_i}_{N}\big)\Big]$$

- Lorentz transformation $\Lambda, \alpha_L \Rightarrow$ non-trivial **global recoil**

Momentum mapping II

- Using Λ and α_L, recoil effects are removed from considerations about factorization, due to Lorentz invariance and known mass dimension of the amplitudes:

$$|\mathcal{M}(q_1, ..., q_n)\rangle = \frac{1}{\alpha_L^{2n-4}} |\mathcal{M}(\hat{q}_1, ..., \hat{q}_n)\rangle \ .$$

- Soft and collinear power counting possible via scaling of α_{il} and β_{il}, *i.e.* $(p_i, n_i, n_{il}^{\perp})$-components

	$(\alpha_{il}, y_i, \beta_{il})$
(forward) collinear	$(1, \lambda^2, \lambda^2)$
soft	$(\lambda, \lambda, \lambda)$.

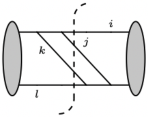
- Facilitates study of an amplitude's singular behaviour for implementation in splitting kernels
- This mapping is just one possible instance. Can *e.g.* use different balancing of transverse components.

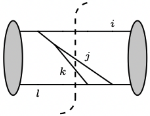
Two emissions: topologies

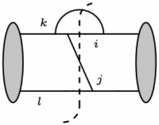
- Decompose squared amplitude in terms of set of topologies

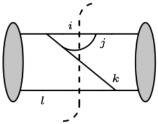
$$|\mathcal{M}_{n+2}|^2 = \sum_i \sum_\alpha \left(E_{ijk}^{(\alpha)} + (j \leftrightarrow k) \right) + \sum_i \sum_{l \neq i} \sum_\alpha \left(A_{ijkl}^{(\alpha)} + B_{ijkl}^{(\alpha)} + X_{ijkl}^{(\alpha)} + (j \leftrightarrow k) \right) + \dots$$

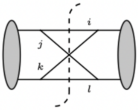
- Examples:

$A_{ijkl}^{(1)} =$ 

$A_{ijkl}^{(3)} =$ 

$B_{ijkl}^{(2)} =$ 

$B_{ijkl}^{(3)} =$ 

$X_{ijkl}^{(1)} = \frac{1}{2}$ 

$E_{ijk}^{(2)} =$