Multi-emission Kernels for Parton Branching Algorithms\textsuperscript{a}

Maximilian Löschner

Institute for Theoretical Physics (Project B1d, PI: S. Gieseke)
Erwin Schrödinger Institut (research stays in Vienna)

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\textsuperscript{a}in collaboration with Simon Plätzer and Emma Simpson Dore. \texttt{arXiv:2112.14454}
\[ \frac{d\sigma}{dQ} \simeq \frac{d\sigma_{\text{hard}}}{dQ} \times \text{PS}(Q \to \mu) \times \text{Had}(\mu \to \Lambda) \times \ldots \]
Parton shower status

\[ \int \left[ \begin{array}{c} \text{\begin{tikzpicture}[baseline=0pt, thick, scale=0.5] \draw[->, line width=1pt] (0,0) -- (1,1); \end{tikzpicture}} \end{array} \right] ^2 d\Phi_1 \int \left[ \begin{array}{c} \text{\begin{tikzpicture}[baseline=0pt, thick, scale=0.5] \draw[->, line width=1pt] (0,0) -- (1,1); \end{tikzpicture}} \end{array} \right] ^2 d\Phi_2 \]

- Despite pushes for higher orders in parton showers (e.g. [Prestel, Hoeche—Phys.Rev.D 96 (2017) 7, 074017], [Skands, Li—PLB 771 (2017) 59-66])
- Road to accuracy requires paradigm shift
  - **Recoil, ordering, colour, correlations**
    - [Bewick, Seymour, Richardson—JHEP 04 (2020) 019], [Forshaw, Holguin, Plätzer—JHEP 09 (2020) 014], [Ruffa, Plätzer—JHEP 06 (2021) 007], [ML, Plätzer, Simpson—2112.14454], [also see PanScales]
  - **Amplitude level** sets the complexity for resolving these
    - [Nagy, Soper], [DeAngelis, Forshaw, Plätzer—PRL 126 (2021) 11, 112001 & JHEP 05 (2018) 044]
Non-global observables

- Coherent branching via angular ordering essential for including large-angle soft contributions
- No global measure of deviation from jet configuration: Coherent branching fails
- Dipole shower: correct LL@LC for non-global, but issues in NLL@LC and LL@NLC for global observables
Non-global observables

- **Coherent branching via angular ordering** essential for including large-angle soft contributions
- No global measure of deviation from jet configuration: **Coherent branching fails**
- **Dipole shower**: correct LL@LC for non-global, but issues in NLL@LC and LL@NLC for global observables

- Require dipole-type soft gluon evolution (to account for change in colour structure)
- Even with a dipole approach, $1/N_C$ effects possibly become comparable to subleading logs, and intrinsically $\sim 10\%$ effects
Study approximations in emission iterations rather than iterations of one emission approximation.
Or: amplitude vs. cross-section level

Goal: NLL@NLC accuracy for global and non-global observables

- Going beyond iterated $1 \rightarrow 2$ splittings in parton showers
- Combine with global recoil scheme
- Include color and spin correlations
- Refine ad hoc models of MC-programs, e.g. azimuthal correlations
- Define language for connecting fixed order to parton showers

Systematic expansion to handle uncertainties ⇔ higher logarithmic accuracy
Comparison to CS dipoles

- Catani-Seymour dipole operators reproduce the partitioned soft and collinear behaviour for one emission:

\[
D_{ij,k}(p_1, ..., p_{m+1}) = -\frac{1}{2p_i \cdot p_j} m < 1, ..., \tilde{i}, j, ..., \tilde{k}, ..., m + 1 \left| \frac{T_k \cdot T_{ij}}{T_{ij}^2} \right| V_{ij,k} \left| 1, ..., \tilde{i}, j, ..., \tilde{k}, ..., m + 1 >_m \right.
\]

\[
< s | V_{\alpha\beta \gamma \delta}(z_i; y_{ij,k}) | s' > = 8\pi \mu^2 \alpha_s C_F \left[ \frac{2}{1 - z_i(1 - y_{ij,k})} - (1 + z_i) - \epsilon(1 - z_i) \right] \delta_{ss'}
\]

- Our idea: algorithmic generation of such splitting kernels for \( > 1 \) emission

- Generate partitioned soft behaviour via power counting instead of construction ‘by hand’

- Potential for constructing subtraction terms
Splitting kernels
Splitting kernels from amplitudes

From the cross-section level to decomposed amplitudes:

\[
\sigma = \sum_n \int \text{Tr} \left[ |M(\mu)\rangle \langle M(\mu)| \right] u(p_1, \ldots, p_n) d\phi_n
\]
Splitting kernels from amplitudes

From the cross-section level to decomposed amplitudes:

\[ \sigma = \sum \int \text{Tr} \left[ |M(\mu)\rangle \langle M(\mu)| \right] u(p_1, \ldots, p_n) d\phi_n \]
Splitting kernel iterations

Density operator language is useful for discussing emissions in iterative manner:

[Forshaw, Holguin, Plätzer–JHEP 09 (2020) 014]
Partitioning
Disentangling different collinear sectors

- Use partition of one in terms of all possible collinear pairings

\[ 1 = P_1^{(A)} + P_2^{(A)} + P_3^{(A)} + \ldots \]

where \( P_i^{(A)} \) projects onto collinearity w.r.t. \( p_i \) for some amplitude \( A \)

- Disentangle overlapping collinear singularities

- Keep smooth interpolation over whole phase space
Angular ordering and subtractions

Radiation of a soft gluon leads to

$$d\sigma_{n+1} = d\sigma_n \times \frac{d\omega \, d\Omega \, \alpha_s}{\omega \, 2\pi \, 2\pi} \sum_{i,l} C_{il} W_{il}$$

where

$$W_{il} = \frac{\omega^2 p_i \cdot p_l}{p_i \cdot p_j \, p_l \cdot p_j} : \text{‘Radiation function’}$$
Angular ordering and subtractions

Radiation of a soft gluon leads to

\[
d\sigma_{n+1} = d\sigma_n \times \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{\alpha_s}{2\pi} \sum_{i,l} C_{il} W_{il}
\]

where

\[
W_{il} = \frac{\omega^2 p_i \cdot p_l}{p_i \cdot p_j p_l \cdot p_j} : \quad \text{‘Radiation function’}
\]

Can decompose

\[
W_{il} = W_{il}^{[i]} + W_{il}^{[l]}
\]

\[
W_{il}^{[i]} = \frac{1}{2} \left( W_{il} - \frac{1}{1 - \cos \theta_{jl}} + \frac{1}{1 - \cos \theta_{ij}} \right)
\]

Then azimuthal averaging confines emissions to cone

\[
\int_0^{2\pi} \frac{d\phi_{ij}}{2\pi} W_{il}^{[i]} = \begin{cases} 
\frac{1}{1 - \cos \theta_{ij}} & \text{if } \theta_{ij} < \theta_{il}, \\
0 & \text{otherwise}.
\end{cases}
\]

Textbook knowledge: subtraction partitioning implies angular ordering [Ellis, Stirling, Webber]
Subtraction partitioning

As an alternative to fractional partitioning, define subtraction scheme:

\[
\frac{1}{S_{ij}} = \frac{S_{il} - \Delta_{(j||l)} + \Delta_{(i||j)}}{2},
\]

\[
\frac{1}{S_{jl}} = \frac{S_{il} - \Delta_{(i||j)} + \Delta_{(j||l)}}{2},
\]

\[
\Delta_{(i||j)} = \frac{E_i}{E_j S_{il} S_{ij}}, \quad \Delta_{(j||l)} = \frac{E_l}{E_j S_{il} S_{jl}}.
\]

by exploiting

\[
S_{ij} \xrightarrow{(j||l)} E_i E_j n_i \cdot n_l = \frac{E_j}{E_i} S_{il}
\]

\[ \mathbb{P}_{(i||j)} \text{ non-singular in } (j \parallel l)-\text{limit while original singular behaviour is reproduced in } (i \parallel j)-\text{limit} \]

**Algorithmic generalisation to multi emissions under control**
Subtraction partitioning behaviour

Recent work: subtraction partitioning $\Rightarrow$ angular ordering for 2E?

$$A \propto \frac{1}{S_{ij} S_{jl}} :$$

Azimuthally averaged single emission kernel

Maximilian Löschner | ITP @ KIT
Subtraction partitioning behaviour

Recent work: subtraction partitioning $\implies$ angular ordering for 2E?

$$A \propto \frac{1}{S_{ij} S_{jl}} :$$

$$A \propto \frac{1}{S_{ij} S_{jk} S_{jl} S_{kl}} :$$

Azimuthally averaged single emission kernel

Original azimuthally averaged two emission amplitude

Partitioned azimuthally averaged two emission amplitude

partitioning $(i \parallel j) (k \parallel l)$
Fractional partitioning for two emissions

**Alternatively:** cancel out ‘unwanted’ collinear singularities by partitioning factors

- **Read**
  \[(i \parallel j \parallel k) : S_{ijk} = (q_i + q_j + q_k)^2 \to 0\]

- **Collect non-singular factors** in triple collinear and coll-coll pairings

<table>
<thead>
<tr>
<th>configuration</th>
<th>( \mathcal{A} \propto \frac{1}{S_{ijkl}S_{ijkl}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i \parallel j \parallel k)</td>
<td>(S_{kl}S_{jkl})</td>
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<tr>
<td>(i \parallel j \parallel l)</td>
<td>(S_{kl}S_{ijkl}S_{jkl})</td>
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<td>(i \parallel k \parallel l)</td>
<td>(S_{ij}S_{ijk}S_{jkl})</td>
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<td>(j \parallel k \parallel l)</td>
<td>(S_{ij}S_{ijk})</td>
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<td>((i \parallel j), (k \parallel l))</td>
<td>(S_{ijk}S_{jkl})</td>
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<td>((i \parallel k), (j \parallel l))</td>
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<tr>
<td>((i \parallel l), (j \parallel k))</td>
<td>(\times)</td>
</tr>
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Fractional partitioning for two emissions

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  ![Configuration Table]

<table>
<thead>
<tr>
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<th>[A \propto \frac{1}{S_{ij}S_{kl}S_{ijk}S_{jkl}}]</th>
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<tbody>
<tr>
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<td>[\times]</td>
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</table>

**⇒ Construct partitioning factors of the form**

\[
P^{(A)}_{(ijk)} = \frac{S_{kl}S_{jkl}}{S_{kl}S_{jkl} + S_{ij}S_{ijk} + S_{ijk}S_{jkl} + (S_{kl} + S_{ij})S_{ijk}S_{jkl}}
\]

- **\[P^{(A)}_{(ijk)} \times A\]** extracts the \[(i \parallel j \parallel k)\]-singular behaviour

- **\[P^{(A)}_{(ijk)}\]** is non-singular in any collinear configuration
Power Counting
Power counting

- Discuss soft and collinear scaling of internal lines in general way
- Sudakov-like decomposition of momenta:

\[
q_I^\mu = \sum_{k \in I} r_{ik} = z_I p_i^\mu + \frac{S_I + p_{\perp,I}^2}{2 z_I p_i \cdot n} n^\mu + k_{\perp,I}^\mu,
\]

- Decompose fermion and gluon lines (factors of $\sqrt{z_I}$ absorbed in vertices for fermions):

\[
\begin{align*}
\text{fermion line} & = \mathcal{P}_i, \\
\text{gluon line} & = d^{\mu\nu}(p_i), \\
\end{align*}
\]

- Leads to power counting rules with potential connection to SCET
Soft and collinear scaling

- Algorithmically determine soft or collinear scaling of an emission amplitude via scaling of internal lines (and propagators)

Scaling of hard lines:

<table>
<thead>
<tr>
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<th>h+c</th>
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<td>λ</td>
<td>λ (unbal.)</td>
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Scaling of emissions:

<table>
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<th></th>
<th>s</th>
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<tr>
<td></td>
<td>1</td>
<td>λ²</td>
<td>λ</td>
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</table>

- Note differences between mappings, e.g. with and without balanced $k_\perp$-components
One emission amplitudes

- Determine list of all relevant sub-amplitudes via power counting rules
- Combine these in density operator ($\simeq$ squared amp) to find full splitting kernel
One emission example

Full one emission \((ij)\)-splitting kernel (balanced mapping) consists of

\[
U_{ij} = P_{ij} \left( \right)
\]

- Exhibits factorisation to hard amplitude
- Smooth interpolation between soft and collinear limits
- Algorithmically generalizable for more emissions
Balanced vs. unbalanced mapping

- Can test different implementations of momentum mappings, e.g. the balancing of transverse components
  \[ k_{\perp,I}^\mu = \sum_{i \in I} k_{\perp,I}^{(i),\mu}, \]

- Yields different sets of diagrammatic contributions

- Nevertheless, the same collinear and soft behaviour is reproduced for one emission

- Still: can compare mappings and check for inconsistencies for > 1 emission
Check: One emission splitting function

▶ Reproduce **Splitting function** $P_{gq}$ as a crosscheck
Check: One emission splitting function

- Reproduce **Splitting function** $P_{qq}$ as a crosscheck

\[ \rightarrow \frac{4\pi\alpha_s T_i^2}{S_{ij}} \left[ (d - 2)\alpha_i + 4\frac{(1 - \alpha_i)^2}{\alpha_i} + 4(1 - \alpha_i) \right] \phi_i + \mathcal{O}(\lambda^{-1}). \]
Soft-Collinear Interplay

- Soft singular part of splitting function cancelled by:

\[
\propto 4\pi \alpha_s \frac{T_i \cdot T_k}{S_{jk}} \frac{4(1 - \alpha_i)^2}{\alpha_i} \frac{p_k \cdot n}{p_i \cdot n} [\phi_i][\phi_k]
\]

- Eikonal part remains:

\[
\propto 4\pi \alpha_s \frac{T_i \cdot T_k}{S_{ij} S_{jk}} 4(1 - \alpha_i) \sqrt{\frac{\beta_i}{\alpha_i}} \frac{p_k \cdot n_\perp}{p_i \cdot n} [\phi_i][\phi_k]
\]

- Smooth interpolation between soft and collinear limits in \( \mathbb{U}_{(ij)} \)
- Current work: investigate this interplay for two emissions
Two emissions: splitting amplitudes

- Same procedure applies to two emissions
- Some amplitudes cannot be achieved by single emission iteration
- Signals for violation of exact factorisation (drop out for two emissions though)

<table>
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<tr>
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<th>$C_1 C_2$</th>
<th>$C_1 S_2$</th>
<th>$S_1 C_2$</th>
<th>$S_1 S_2$</th>
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<td>$\lambda^3$</td>
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Two emissions: combined contributions

- Determine amplitude scaling algorithmically:

  \[
  \text{combinedAmpsB2}\{(c, s, c, s), 1\}
  \]

- Combine with partitioned propagator scaling to find all leading contributions for full kernel
Conclusions

Goal: **universal algorithm for handling accuracy in multiple emissions** (for applications in parton showers and beyond)

- Density-operator formalism to study iterative behaviour of emissions
- Partitioning algorithms to separate overlapping singularities
- Momentum mapping for exposing collinear and soft factorization
- Global recoil via Lorentz transformation
- Set of power counting rules to single out leading amplitudes
- Can handle and compare different momentum mappings
- Two-emission kernels/power counting under control
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Thank you!
Backup slides
Coherent branching

- **Coherent emission of soft large angle gluons** from systems of collinear partons
- **Angular ordering** essential for including large-angle soft contributions

Resummation of global jet observables such as thrust $\tau$

- **NLL accurate** @Next-to-Leading-Colour (NLC) if inclusive over secondary soft gluon emission
Applications

- Use projectors and helicity sums to represent emission amplitudes as (complex) weights for numerical evaluation

\[
P(q) \equiv \begin{cases} 
P^{\rho\sigma}(p) = d^{\rho\sigma}(p), & \text{(gluon),} \\
\bar{p}(p) = \frac{\eta}{2n \cdot p}, & \text{(quark),}
\end{cases}
\]

\[
d^{\mu\nu}(p) = \epsilon^\mu_+(p, n)\epsilon^\nu_-(p, n) + (\mu \leftrightarrow \nu), \\
\eta = \sum_\lambda u_\lambda(n)\bar{u}_\lambda(n),
\]
Applications

- Use projectors and helicity sums to represent emission amplitudes as (complex) weights for numerical evaluation

\[ P(q) \equiv \begin{cases} P^{\rho\sigma}(p) = d^{\rho\sigma}(p), & \text{(gluon)}, \\ \mathcal{P}(p) = \frac{\mathcal{P}}{2n \cdot p}, & \text{(quark)}, \end{cases} \]

\[ d^{\mu\nu}(p) = \epsilon^\mu_+(p, n) \epsilon^\nu_-(p, n) + (\mu \leftrightarrow \nu), \]

\[ \mathcal{P} = \sum_\lambda u_\lambda(n) \bar{u}_\lambda(n), \]

\[ \rightarrow \sum_{\lambda_i, \bar{\lambda}_i} \frac{u_\lambda_1}{\sqrt{2n \cdot p_i}} \left[ \frac{\bar{u}_\lambda_1}{\sqrt{2n \cdot p_i}} \frac{k_\perp \not\epsilon_{\lambda_3} \phi_i}{\sqrt{2n \cdot p_i}} \frac{u_\lambda_2}{\sqrt{2n \cdot p_i}} \right] \frac{\bar{u}_\lambda_2}{\sqrt{2n \cdot p_i}} \epsilon_{\lambda_3}^\sigma \]

\[ \times \frac{u_{\bar{\lambda}_1}}{\sqrt{2n \cdot p_i}} \left[ \frac{\bar{u}_{\bar{\lambda}_1}}{\sqrt{2n \cdot p_k}} \phi_k \frac{u_{\bar{\lambda}_2}}{\sqrt{2n \cdot p_k}} p_k \cdot \epsilon_{\bar{\lambda}_3} \right] \frac{\bar{u}_{\bar{\lambda}_2}}{\sqrt{2n \cdot p_k}} \epsilon_{\sigma, \bar{\lambda}_3}. \]
Team

Karlsruhe/Manchester/Vienna network with support from SFB drives significant parts of the development, also relating to aspects such as color reconnection [e.g. Gieseke, Kirchgaesser, Plätzer-JHEP 11 (2018) 149]

A highly ambitious program:

Need to combine diverse expertise from different networks to gain momentum.

[Plätzer—Annual CRC Meeting 2019]
Algorithm for subtraction partitioning

▶ General form of partitioned propagator $P$ for config $\sigma$

$$\mathbb{P}_\sigma[P] = \frac{1}{m} \left( P + (m - 1) \Delta_{\sigma;\tau_1,\ldots,\tau_{m-1}}[P] - \sum_{i=1}^{m-1} \Delta_{\tau_i;\tau_1,\ldots,\tau_{i-1},\sigma,\tau_{i+1},\ldots,\tau_{m-1}}[P] \right),$$

▶ with Subtraction terms

$$\Delta_{\tau_1,\tau_2,\ldots,\tau_m}[P] = \left[ \begin{array}{c} \text{F}_{\tau_1}[P] \\ \text{S}_{\tau_1}[P] \end{array} \right] - \sum_{S/\tau_1} \left[ \begin{array}{c} \text{F}_{\tau_1}[P] \\ \text{S}_{\tau_1}[P] \end{array} \right],$$

▶ When partitioning e.g. to $\sigma = (i \parallel j \parallel k)$, subtract off all (sub-)divergences of other singular configs $\tau_i$ for propagator factor $P$.

▶ Combinatorial factor $m$: number of singular configs for $P$
Two emission example

Partitioned version of $A^{(1)} \propto 1/ S_{ij} S_{ijk} S_{kl} S_{jkl}$

\[
\mathcal{P}(A^{(1)}) = \frac{1}{3} \left( \frac{1}{S_{ij} S_{ijk} S_{kl} S_{jkl}} + 2\Delta_{(ijk)}[\mathcal{P}(A^{(1)})] - \Delta_{(jkl)}[\mathcal{P}(A^{(1)})] - \Delta_{(ij)(kl)}[\mathcal{P}(A^{(1)})] \right), \\
+ \frac{1}{3} \left( \frac{1}{S_{ij} S_{ijk} S_{kl} S_{jkl}} - \Delta_{(ijk)}[\mathcal{P}(A^{(1)})] + 2\Delta_{(jkl)}[\mathcal{P}(A^{(1)})] - \Delta_{(ij)(kl)}[\mathcal{P}(A^{(1)})] \right), \\
+ \frac{1}{3} \left( \frac{1}{S_{ij} S_{ijk} S_{kl} S_{jkl}} - \Delta_{(ijk)}[\mathcal{P}(A^{(1)})] - \Delta_{(jkl)}[\mathcal{P}(A^{(1)})] + 2\Delta_{(ij)(kl)}[\mathcal{P}(A^{(1)})] \right),
\]

where e.g.

\[
\Delta_{(jkl)}[\mathcal{P}(A^{(1)})] = \frac{E_l^2}{E_j(E_j + E_k)} \frac{1}{S_{il} S_{kl} S_{jkl}} \left( \frac{1}{S_{kl} S_{jkl}} - \frac{E_i E_l}{E_j (E_j + E_k)} \frac{1}{S_{il} S_{kl}} \right),
\]
Check: Two Emissions

- **Reproduced from general two-emission kernel** which includes soft-limit too (here: in lightcone-gauge)

\[
\frac{\mu^{2\varepsilon}}{\hat{\alpha}^2 S_{i12}^2} \left\{ \begin{array}{c}
\hat{p}_i & \hat{p}_i \\
\end{array} \right\} + \left\{ \begin{array}{c}
\hat{p}_i & \hat{p}_i \\
\end{array} \right\} + \left\{ \begin{array}{c}
\hat{p}_i & \hat{p}_i \\
\end{array} \right\} + (1 \leftrightarrow 2) \right\} C_A C_F
\]

\[
= \left( \frac{8\pi \alpha_S}{\hat{\alpha} S_{i12}^2} \mu^{\varepsilon} \right)^2 C_A C_F \langle \hat{P}_{(non-Ab)}^{ggq} \rangle \hat{p}_i + O \left( \beta_{il}^{-3/2} \right).
\]
Vertex rules

- Can find vertex rules such as:

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{vertex_rule1}
\end{array} & = 0, & \begin{array}{c}
\includegraphics[width=0.3\textwidth]{vertex_rule2}
\end{array} = 0, & \begin{array}{c}
\includegraphics[width=0.3\textwidth]{vertex_rule3}
\end{array} = 0.
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{ccc}
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{vertex_rule4}
\end{array} & = 0, & \begin{array}{c}
\includegraphics[width=0.3\textwidth]{vertex_rule5}
\end{array} = 0, & \begin{array}{c}
\includegraphics[width=0.3\textwidth]{vertex_rule6}
\end{array} = 0.
\end{array}
\end{align*}
\]
Insights from Power Counting Rules

Powerful vertex rule for lines belonging to same collinear sector:

\[ p_i p_i = 0, \]

Shows (known fact) that interference diagrams do not contribute in splitting function in a physical gauge.

Reason: denominator goes as \( \frac{1}{\lambda^2 k_{S_{k_{\text{coll}}}}} \) for \( k_{\text{coll}} \) emissions.

Can only contribute in splitting function (\( \propto \frac{1}{\lambda^2 k_{S_{k_{\text{col}}}}} \)) if numerator goes as \( O(1) \), but the only possible contribution \( \equiv 0 \).
Insights from Power Counting Rules

- Powerful vertex rule for lines belonging to same collinear sector:
  \[ p_i \cdot p_i = 0, \]

- Shows (known fact) that interference diagrams do not contribute in splitting function in a physical gauge

- **Reason:** denominator goes as \( 1/\lambda^{2k} S^{k}_{(col)} \) for \( k \) coll. emissions

- Can only contribute in splitting function \( (\propto 1/\lambda^{2k} S^{k}_{(col)}) \) if numerator goes as \( \mathcal{O}(1) \), but the only possible contribution \( \equiv 0 \)

\[ = 0. \]
Global and non-global observables

Example: heavy and light jet mass (global) vs. hemisphere jet mass (non-global)

- Cancellations between large angle-soft and virtual contributions (from $k_2$) not guaranteed
  - $\Rightarrow$ NLL enhancement from leftover $\alpha_s^2 L^2$ terms
Partitioning

Amplitudes carry different singular $S$-invariants

$$\mathcal{A}(S_1, S_2) = \frac{\mathcal{N}(S_1, S_2)}{S_1 S_2},$$

Decomposition using partitioning factors:

$$\mathbb{P}(\mathcal{A})_{(1)} = \frac{S_2}{S_1 + S_2}, \quad \mathbb{P}(\mathcal{A})_{(2)} = \frac{S_1}{S_1 + S_2},$$

we can decompose $\mathcal{A}$ into

$$\mathcal{A} = \left[ \mathbb{P}(\mathcal{A})_{(1)} + \mathbb{P}(\mathcal{A})_{(2)} \right] \mathcal{A} = \frac{\mathcal{N}(S_1, S_2)}{S_1(S_1 + S_2)} + \frac{\mathcal{N}(S_1, S_2)}{S_2(S_1 + S_2)}.$$
Soft and collinear regions are of special interest:

\[ S_{ij} \equiv (q_i + q_j)^2 = 2 q_i \cdot q_j = 2 q_i^0 q_j^0 [1 - \cos \theta_{ij}] , \quad \text{for } q_{i/j}^2 = 0 \]

- Amplitude goes as \( \propto 1/S_{ij} \)
  \( \Rightarrow \) becomes singular/enhanced when \( S_{ij} \rightarrow 0 \)
- Large logarithms due to phase space integrations of the kind

\[
\begin{align*}
\frac{dq_j^0}{q_j^0}, & \quad \frac{d\theta_{ij}}{\theta_{ij}} \rightarrow \alpha_s \log^2 \frac{Q}{Q_0} \sim 1 \\
\end{align*}
\]

for some scale \( Q \in \{ \theta, p_{\perp}, \ldots \} \) and cut-off \( Q_0 \)
Parton shower: collinear limit

Single emission approach is then usually iterated in a probabilistic manner

\[ W_{2+2} = \left( \int |\langle \Phi_2 |^2 + |\langle \tilde{\Phi}_2 |^2 + |\langle \Phi_1 |^2 \right| \frac{d\Phi_2}{\sqrt{2}} \right)^2 \]

\[ = 2^2 \int_{t_0}^{t} dt' \int_{t_0}^{t} dt'' W(t') W(t'') = \frac{2^2}{2!} \left( \int_{t_0}^{t} dt W(t) \right)^2. \]

Sum over any number of emissions: result exponentiates

\[ \sigma_{>2}(t_0) = \sigma_2(t_0) \sum_{k=1}^{\infty} \frac{2^k}{k!} \left( \int_{t_0}^{t} dt W(t) \right)^k. \]

Sudakov Form Factor \((\simeq \text{no emission probability in range } t \to t_0)\)

\[ \Delta(t_0) = \exp \left[ - \int_{t_0}^{t} dt W(t) \right], \quad W(t) = \int_{z_-}^{z_+} \frac{\alpha_S(z,t) \hat{P}(z,t)}{2\pi} \frac{t}{z} dz. \]
Momentum mapping
Start with **on-shell** (OS) momenta $p_i$ (to be **emitters**) and $p_r$ (to be **recoilers**) with overall momentum transfer $Q \equiv \sum_i p_i + \sum_r p_r$.
Momentum mapping

Adding emissions

Start with **on-shell** (OS) momenta $p_i$ (to be **emitters**) and $p_r$ (to be **recoilers**) with overall momentum transfer $Q \equiv \sum_i p_i + \sum_r p_r$

Add emissions to the process with:

1. Momentum conservation: $\sum_i q_i + \sum_{i,l} k_{il} + \sum_r q_r = Q$
2. On-shellness of all partons
3. Parametrization of soft & collinear behaviour for any # of emissions
Momentum mapping

\[ q_r = \frac{\Lambda}{\alpha_L} p_r \]

\[ k_{il} = \frac{\Lambda}{\alpha_L} \left[ \alpha_{il} p_i + \tilde{\beta}_{il} n_i + \sqrt{\alpha_{il} \tilde{\beta}_{il} n_{il}^\perp} \right], \quad A_i \equiv \sum_l \alpha_{il}, \quad \tilde{\beta}_{il} = (1 - A_i) \beta_{il} \]

\[ q_i = \frac{\Lambda}{\alpha_L} \left[ (1 - A_i) p_i + (y_i - \sum_l \tilde{\beta}_{il}) n_i - \sum_l \sqrt{\alpha_{il} \tilde{\beta}_{il} n_{il}^\perp} \right] \]

- Decomposition w/ light-like momentum \( n_i \) and \( n_{il}^\perp \cdot p_i = n_{il}^\perp \cdot n_i = 0 \)
- Need \( \alpha_L^2 = (Q + N)^2 / Q^2 \) for momentum conservation

\[ Q = \sum_r q_r + \sum_i q_i + \sum_{i,l} k_{il} = \frac{\Lambda}{\alpha_L} \left[ \sum_r p_r + \sum_i \left( p_i + y_i n_i \right) \right] \]

- Lorentz transformation \( \Lambda, \alpha_L \Rightarrow \) non-trivial \textbf{global recoil}
Using $\Lambda$ and $\alpha_L$, recoil effects are removed from considerations about factorization, due to Lorentz invariance and known mass dimension of the amplitudes:

$$|M(q_1, \ldots, q_n)\rangle = \frac{1}{\alpha_L^{2n-4}} |M(\hat{q}_1, \ldots, \hat{q}_n)\rangle.$$ 

Soft and collinear power counting possible via scaling of $\alpha_{il}$ and $\beta_{il}$, i.e. $(p_i, n_i, n_{il}^\perp)$-components

<table>
<thead>
<tr>
<th>(forward) collinear</th>
<th>$(\alpha_{il}, y_i, \beta_{il})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>soft</td>
<td>$(1, \lambda^2, \lambda^2)$</td>
</tr>
<tr>
<td></td>
<td>$(\lambda, \lambda, \lambda)$</td>
</tr>
</tbody>
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Facilitates study of an amplitude’s singular behaviour for implementation in splitting kernels

This mapping is just one possible instance. Can e.g. use different balancing of transverse components.
Two emissions: topologies

- Decompose squared amplitude in terms of set of topologies

\[ |\mathcal{M}_{n+2}|^2 = \sum_i \sum_\alpha \left( E_{ijk}^{(\alpha)} + (j \leftrightarrow k) \right) + \sum_i \sum_{l \neq i} \sum_\alpha \left( A_{ijkl}^{(\alpha)} + B_{ijkl}^{(\alpha)} + X_{ijkl}^{(\alpha)} + (j \leftrightarrow k) \right) + \ldots \]

- Examples: