Rings and strings for enhanced coherent branching algorithms

Jack Holguin

in collaboration with Jeff Forshaw and Simon Plätzer

Based on arXiv:2112.13124

Summary

$$\frac{\mathrm{d}(\cos\theta_{in})\mathrm{d}\phi_n}{1-\cos\theta_{in}}\Theta(\theta_{in}<\theta) \mapsto {}^{(n-1)}\mathrm{S}_i^{j,k}\, E_n^2\, \Theta(\theta_{in}<\theta)\, \mathrm{d}(\cos\theta_{in})\mathrm{d}\phi_n$$

$$^{(n-1)}\mathrm{S}_i^{j,k} = \frac{s_{ij}s_{nk}+s_{ik}s_{nj}-s_{jk}s_{ni}}{2s_{ni}s_{nj}s_{nk}}$$

for $s_{ij} = q_i \cdot q_j$. The emitted parton is labelled n, i is its parent, j, k are spectators (defined later), and θ the previous angular scale in the shower. This modification extends the angular-ordered framework to the three-jet limit.

Motivation

In previous work:

- “Parton branching at amplitude level” J. Forshaw, JH, S. Plätzer arXiv:1905.08686
- “Soft gluon evolution and non-global logarithms”
  R. Ángeles Martínez, M. De Angelis, J. Forshaw, S. Plätzer, M. Seymour arXiv:1802.08531

Collinear poles from soft gluons cause issues in the Monte Carlo implementation (CVolver).

General observation: collinear poles are always colour diagonal. They are “simple”, we must be able to exploit this.

Exploiting this is at the core of the coherent branching formalism.

Example: Collinear poles are colour diagonal

$$|M_n(q_1,\ldots,q_n)\rangle = \mathbf{J}_n(q_n)\,|M_{n-1}(q_1,\ldots,q_{n-1})\rangle + \mathcal{O}(\lambda^0)$$

$$\mathbf{J}_n^{(0)}(q_n) = \left(\frac{\alpha_\mathrm{s}}{\pi}\right)^{\frac{1}{2}} \sum_{i,\lambda_i} \mathbf{T}_i \frac{\varepsilon_{\lambda_i}(q_n)\cdot q_i}{q_n\cdot q_i}$$

$$\mathbf{A}_n = \mathbf{J}_n^{(0)}\,|M_{n-1}\rangle\,\langle M_{n-1}|\,\mathbf{J}_n^{(0)\,\dagger} = -\frac{\alpha_\mathrm{s}}{\pi}\sum_{i\neq j}\omega_{ij}(q_n)\mathbf{T}_i\,|M_{n-1}\rangle\,\langle M_{n-1}|\,\mathbf{T}_j^\dagger$$

$$\omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Example: Collinear poles are colour diagonal

$$\mathbf{A}_n = \mathbf{J}_n^{(0)} \left| M_{n-1} \right\rangle \left\langle M_{n-1} \right| \mathbf{J}_n^{(0)\,\dagger} = -\frac{\alpha_\mathrm{s}}{\pi} \sum_{i \neq j} \omega_{ij}(q_n) \mathbf{T}_i \left| M_{n-1} \right\rangle \left\langle M_{n-1} \right| \mathbf{T}_j^\dagger$$

$$[i \cdot j] = \mathbf{T}_i \left| M_{n-1} \right\rangle \left\langle M_{n-1} \right| \mathbf{T}_j^\dagger$$

$$\operatorname*{Pole}_{q_n || q_i} \left(\omega_{ij}(q_n) \frac{\mathrm{d}^{3-2\epsilon} \vec{q}_n}{2E_n} \right) = \frac{\mathrm{d}\theta_{ni}}{(\theta_{ni})^{1+2\epsilon}} \frac{E_i^\epsilon \mathrm{d}E_n}{(E_n)^{1+3\epsilon}} \mathrm{d}\phi$$

$$\textstyle\sum_j \mathbf{T}_j = 0$$

$$\operatorname*{Pole}_{q_n || q_i} \left(\mathbf{A}_n \frac{\mathrm{d}^{3-2\epsilon} \vec{q}_n}{2E_n} \right) = \frac{\alpha_\mathrm{s}}{\pi} [i \cdot i] \frac{\mathrm{d}\theta_{ni}}{(\theta_{ni})^{1+2\epsilon}} \frac{E_i^\epsilon \mathrm{d}E_n}{(E_n)^{1+3\epsilon}} \mathrm{d}\phi$$

Observation

We have shown the general observation at tree level: collinear poles are always colour diagonal.

However, this observation can be flipped.

Off-diagonal colour structures are always collinear finite.

It doesn't take much work to adapt the previous slides to show that each diagonal colour structure ($[i \cdot i]$) can be associated with a single colour pole.

Useful definitions

$$^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

A single collinear pole.

$$\underset{q_n||q_i}{\mathrm{Pole}}\left({}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}^{3-2\epsilon}\vec{q}_n}{2E_n}\right) = \underset{q_n||q_i}{\mathrm{Pole}}\left(\omega_{ij}(q_n)\frac{\mathrm{d}^{3-2\epsilon}\vec{q}_n}{2E_n}\right) = \underset{q_n||q_i}{\mathrm{Pole}}\left(\omega_{ik}(q_n)\frac{\mathrm{d}^{3-2\epsilon}\vec{q}_n}{2E_n}\right) = \frac{\mathrm{d}\theta_{ni}}{(\theta_{ni})^{1+2\epsilon}}\frac{E_i^{\epsilon}\mathrm{d}E_n}{(E_n)^{1+3\epsilon}}\mathrm{d}\phi$$

$$^{(n-1)}\mathrm{R}_{k,l}^{i,j} = \omega_{ij}(q_n) - \omega_{ik}(q_n) - \omega_{jl}(q_n) + \omega_{kl}(q_n)$$

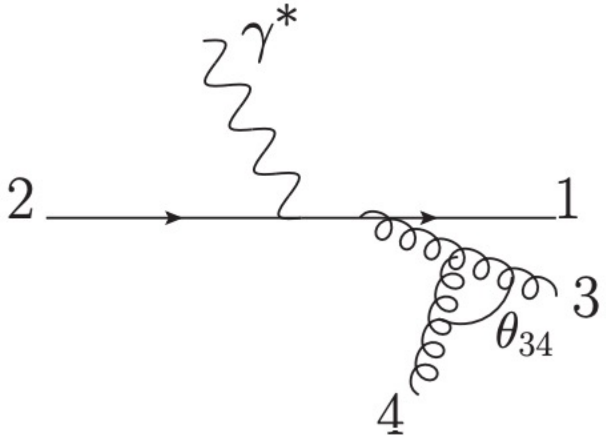
No collinear pole.

$$\mathbf{A}_n = \frac{2\alpha_\mathrm{s}}{\pi}\left({}^{(n-1)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(n-1)}\mathrm{S}_2^{1,3}[2\cdot 2] + \sum_{i\neq 1,2}{}^{(n-1)}\mathrm{S}_i^{1,2}[i\cdot i]\right) - \frac{\alpha_\mathrm{s}}{2\pi}\sum_{i\neq 1,2,3}{}^{(n-1)}\mathrm{R}_{2,3}^{i,1}([i\cdot 1-2]+[1-2\cdot i]) - \frac{\alpha_\mathrm{s}}{2\pi}\sum_{i\neq 1,2}\sum_{j\neq i,1,2}{}^{(n-1)}\mathrm{R}_{1,2}^{i,j}([i\cdot j]+[j\cdot i]).$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \quad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Coherence: traditional approach



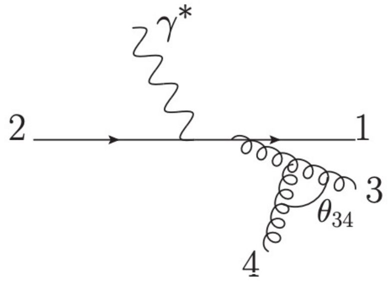
$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\textstyle\sum_j \mathbf{T}_j = 0 \quad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i \; q_n\cdot q_j}$$

Coherence: traditional approach



$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

$$2P_{ij} = w_{ij}^{(n)} + \frac{E_i}{E_n\, q_i\cdot q_n} - \frac{E_j}{E_n\, q_j\cdot q_n}$$

$$E_n^2 \int_0^{2\pi} \frac{\mathrm{d}\phi_q^{(i)}}{2\pi} P_{ij} = \frac{1}{1-\cos\theta_{in}}\Theta(\theta_{in} < \theta_{ij})$$

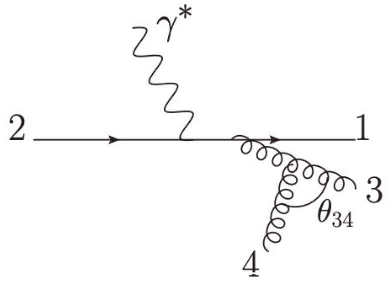
$$P_{ij}^{[i]} = -\frac{(\mathrm{d}\cos\theta_{in})\,\mathrm{d}\phi_q^{(i)}}{4\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi_q^{(i)}}{2\pi} P_{ij}$$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\textstyle\sum_j \mathbf{T}_j = 0 \qquad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Coherence: traditional approach



$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi} \approx \frac{1}{2}\left(P_{ij}^{[i]} + P_{ik}^{[i]}\right) + \frac{1}{2}\left(P_{ji}^{[j]} - P_{jk}^{[j]}\right) + \frac{1}{2}\left(P_{ki}^{[k]} - P_{kj}^{[k]}\right)$$

$$P_{ij}^{[i]} = -\frac{(\mathrm{d}\cos\theta_{in})\,\mathrm{d}\phi_q^{(i)}}{4\pi}\int_0^{2\pi}\frac{\mathrm{d}\phi_q^{(i)}}{2\pi}P_{ij}$$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}(\omega_{ij} + \omega_{ik} - \omega_{jk})$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \quad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Coherence: traditional approach

$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi} \approx \frac{1}{2}\left(P_{ij}^{[i]} + P_{ik}^{[i]}\right) + \frac{1}{2}\left(P_{ji}^{[j]} - P_{jk}^{[j]}\right) + \frac{1}{2}\left(P_{ki}^{[k]} - P_{kj}^{[k]}\right) $$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_i\to n_j} \approx P_{ij}^{[i]}, \qquad {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_i\to n_k} \approx P_{ik}^{[i]}, $$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_j\to n_k} \approx P_{ij}^{[i]} + \tilde{P}_{ji}^{[j]}(\theta_{jk}) \approx P_{ik}^{[i]} + \tilde{P}_{ki}^{[k]}(\theta_{jk}), $$

Up to terms of the order θ^0 and in a frame where $\theta_{\text{spectators}} = \pi$

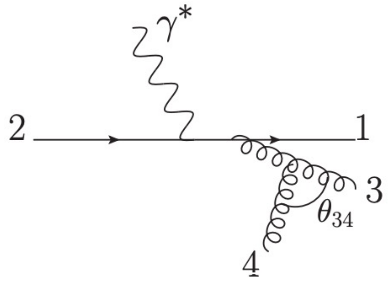
$$ \tilde{P}_{jk}^{[j]}(\theta_{ij}) = P_{jk}^{[j]}\Theta(\theta_{jn} > \theta_{ij}) $$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right) $$

$$ [i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger $$

$$ \textstyle\sum_j \mathbf{T}_j = 0 \qquad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j} $$

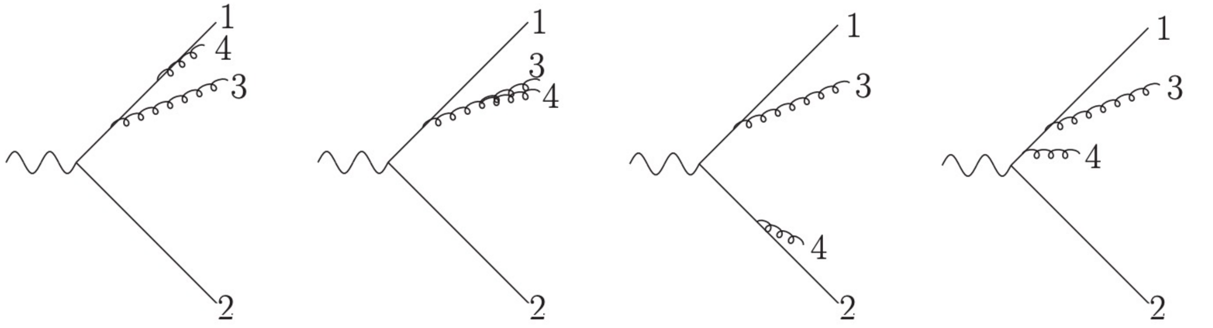
Coherence: traditional approach



$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

Up to terms of the order $(\theta_{13}^0\theta_{i4}^0)$ and in a frame where $\theta_{12} = \pi$

$$\mathbf{A}_4 \frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_1\to n_3} \approx \frac{2\alpha_s}{\pi}\left(P_{13}^{[1]}[1\cdot 1] + (P_{21}^{[2]} + \tilde{P}_{(1+3)2}^{[1+3]}(\theta_{13}))[2\cdot 2] + P_{31}^{[3]}[3\cdot 3]\right)$$



(a) $\propto P_{13}^{[1]}$ (b) $\propto P_{31}^{[3]}$ (c) $\propto P_{21}^{[2]}$ (d) $\propto \tilde{P}_{(1+3)2}^{[1+3]}(\theta_{13})$

$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \quad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Coherence: more generally?

But why azimuthally average if it constrains us to the $\theta_{12} = \pi$ frame?

It prevents us from applying the derivation outside the 2-jet limit where there is no azimuthal symmetry.

Outside the 2-jet limit in the $\theta_{12} = \pi$ frame terms with a θ_{13}^{-1} collinear pole depend on the azimuth.

Coherence: more generally?

But why azimuthally average if it constrains us to the $\theta_{12} = \pi$ frame?

What the averaging achieves is it handles the commutation of limits: i.e.

$$\lim_{q_n/E_n \to n_i} \lim_{n_i \to n_j} \omega_{ij}(q_n) \to 0, \qquad \lim_{n_i \to n_j} \lim_{q_n/E_n \to n_i} \omega_{ij}(q_n) \to \infty.$$

The theta function in $P_{ij}^{[i]}$ always screens us from complications due to conflicting limits. In our case the limits of concern are of the form

$$\lim_{q_n/E_n \to n_i} \lim_{n_i \to n_j} {}^{(n-1)}\mathrm{R}_{k,l}^{i,j} \to \lim_{q_n/E_n \to n_i} -2^{(n-1)}\mathrm{S}_i^{k,l} \to -\infty$$

$$\lim_{n_i \to n_j} \lim_{q_n/E_n \to n_i} {}^{(n-1)}\mathrm{R}_{k,l}^{i,j} \to \omega_{kl} + \mathcal{O}(\theta_{in}^0, \theta_{ij}^0)$$

Coherence: more generally?

The lack of commutativity of these angular limits suggest the presence of poles on the boundaries dividing the limits. It is these poles that average to give theta functions.

However, we do not need to average to handle the poles. What we are doing is computing Laurent series of the density matrix around the emission angles.

When computing Laurent series around a pole you divide the domain with other poles defining boundaries. Then compute the expansion in each region. The union of the expansion across the regions provides the complete expansion (taking care of overlaps).

Coherence: more generally

It turns out that for the simple string only one partition is really necessary:

$$^{(n-1)}\mathrm{S}_i^{j,k} = {}^{(n-1)}\mathrm{S}_i^{j,k}(\Theta(\theta_{in} < \theta_{ij}) + \Theta(\theta_{in} > \theta_{ij}))$$

which is sufficient for us to show that,

$$^{(n-1)}\mathrm{S}_i^{j,k}\big|_{n_j \to n_i} \approx {}^{(n-1)}\mathrm{S}_i^{j,k}\Theta(\theta_{in} < \theta_{ij})$$

since,

$$(\omega_{ij}(q_n) + \omega_{ik}(q_n) - \omega_{jk}(q_n))\Theta(\theta_{in} > \theta_{ij})\big|_{n_j \to n_i} \approx 0$$

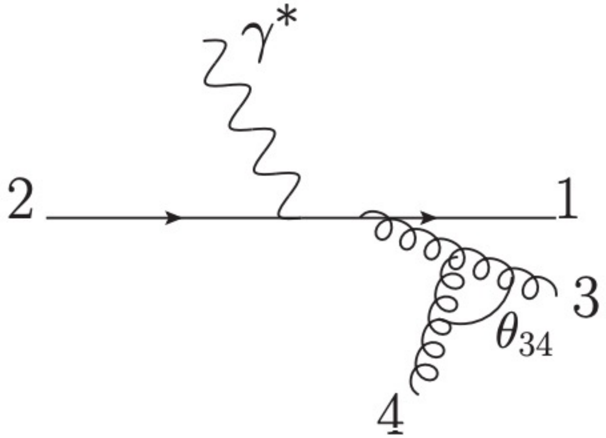
up to terms of the order θ^0.

$$^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}(\omega_{ij} + \omega_{ik} - \omega_{jk})$$

$$[i \cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \quad \omega_{ij}(q_n) = \frac{q_i \cdot q_j}{q_n \cdot q_i \; q_n \cdot q_j}$$

Coherence: more generally



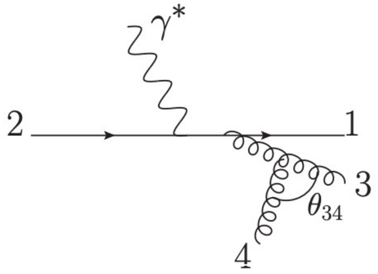
$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \quad \omega_{ij}(q_n) = \frac{q_i \cdot q_j}{q_n\cdot q_i \; q_n \cdot q_j}$$

Coherence: more generally



$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_i\to n_j} \approx P_{ij}^{[i]}, \qquad {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_i\to n_k} \approx P_{ik}^{[i]},$$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\frac{\mathrm{d}\Omega_n}{4\pi}\bigg|_{n_j\to n_k} \approx P_{ij}^{[i]} + \tilde{P}_{ji}^{[j]}(\theta_{jk}) \approx P_{ik}^{[i]} + \tilde{P}_{ki}^{[k]}(\theta_{jk}),$$

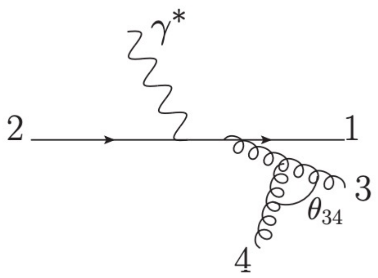
$$ {}^{(n-1)}\mathrm{S}_i^{j,k}\big|_{n_j\to n_i} \approx {}^{(n-1)}\mathrm{S}_i^{j,k}\Theta(\theta_{in} < \theta_{ij})$$

$$ {}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}(\omega_{ij} + \omega_{ik} - \omega_{jk})$$

$$[i\cdot j] = \mathbf{T}_i \left|M_{n-1}\right\rangle \left\langle M_{n-1}\right| \mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \qquad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Coherence: more generally



$$\mathbf{A}_4 = \frac{2\alpha_s}{\pi}\left({}^{(3)}\mathrm{S}_1^{2,3}[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}[3\cdot 3]\right)$$

$$\mathbf{A}_4\Big|_{n_1\to n_3} \approx \frac{2\alpha_s}{\pi}\Big[{}^{(3)}\mathrm{S}_1^{2,3}\Theta(\theta_{14}<\theta_{13})[1\cdot 1] + {}^{(3)}\mathrm{S}_2^{1,3}\Theta(\theta_{(1+3)4}>\theta_{13})[2\cdot 2] + {}^{(3)}\mathrm{S}_3^{1,2}\Theta(\theta_{34}<\theta_{13})[3\cdot 3]\Big]$$

Up to terms of the order $(\theta_{13}^0\theta_{i4}^0)$ and in any frame.

This and the averaged result precisely agree when $\theta_{12}=\pi$.

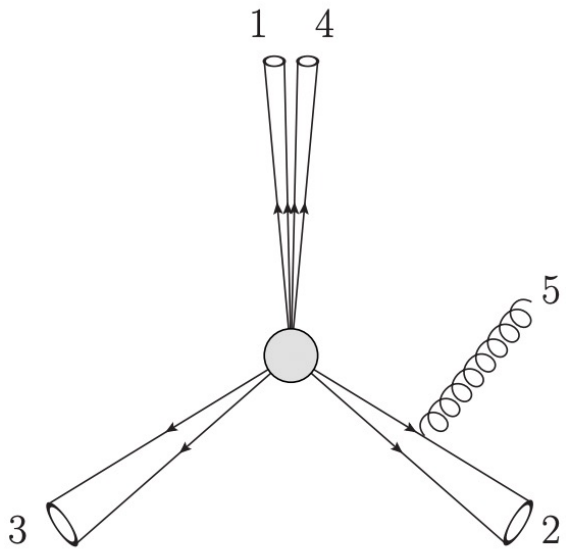
$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}(\omega_{ij}+\omega_{ik}-\omega_{jk})$$

$$[i\cdot j] = \mathbf{T}_i\,|M_{n-1}\rangle\langle M_{n-1}|\,\mathbf{T}_j^\dagger$$

$$\textstyle\sum_j \mathbf{T}_j = 0 \qquad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Coherence: more generally

$$\mathbf{A}_{4\,\mathrm{jet}+(g)} = \frac{2\alpha_s}{\pi}\Big[{}^{(4)}\mathrm{S}_1^{2,3}|1\cdot 1|_4 + {}^{(4)}\mathrm{S}_2^{1,3}|2\cdot 2|_4 + {}^{(4)}\mathrm{S}_3^{1,2}|3\cdot 3|_4 + {}^{(4)}\mathrm{S}_4^{1,2}|4\cdot 4|_4 \Big]$$
$$- \frac{\alpha_s}{2\pi}\Big[{}^{(4)}\mathrm{R}_{1,2}^{3,4}(|1\cdot 4|_4 + |3\cdot 4|_4 - |2\cdot 4|_4 + |4\cdot 1|_4 + |4\cdot 3|_4 - |4\cdot 2|_4)$$
$$+ {}^{(4)}\mathrm{R}_{1,2}^{4,3}(|2\cdot 4|_4 + |3\cdot 4|_4 - |1\cdot 4|_4 + |4\cdot 2|_4 + |4\cdot 3|_4 - |4\cdot 1|_4)\Big]$$



The same steps can be followed to find the 3-jet coherence limit of the matrix element above.

The derivation is a little more subtle, more regions must be identified, but the outcome is elegant (I think).

$$\mathbf{A}_{4\,\mathrm{jet}+(g)}\big|_{1||4} \approx \frac{2\alpha_s}{\pi}\Big[{}^{(4)}\mathrm{S}_1^{2,3}\Theta(\theta_{15} < \theta_{14})|1\cdot 1|_4 + {}^{(4)}\mathrm{S}_2^{1,3}|2\cdot 2|_4 + {}^{(4)}\mathrm{S}_3^{1,2}|3\cdot 3|_4 +$$
$${}^{(4)}\mathrm{S}_4^{1,2}|4\cdot 4|_4 + {}^{(4)}\mathrm{S}_1^{2,3}\Theta(\theta_{15} > \theta_{14})|1+4\cdot 1+4|_4\Big].$$

$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}(\omega_{ij} + \omega_{ik} - \omega_{jk})$$

$$[i\cdot j] = \mathbf{T}_i\,|M_{n-1}\rangle\,\langle M_{n-1}|\,\mathbf{T}_j^\dagger$$

$$\sum_j \mathbf{T}_j = 0 \qquad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Returning to slide 2

The complete outcome can be summarised

$$\frac{\mathrm{d}(\cos\theta_{in})\mathrm{d}\phi_n}{1-\cos\theta_{in}}\Theta(\theta_{in}<\theta) \mapsto \,^{(n-1)}\mathrm{S}_i^{j,k}\, E_n^2\,\Theta(\theta_{in}<\theta)\,\mathrm{d}(\cos\theta_{in})\mathrm{d}\phi_n$$

$$^{(n-1)}\mathrm{S}_i^{j,k} = \frac{s_{ij}s_{nk}+s_{ik}s_{nj}-s_{jk}s_{ni}}{2s_{ni}s_{nj}s_{nk}}$$

for $s_{ij} = q_i \cdot q_j$. The emitted parton is labelled n, i is its parent, j, k are spectators (defined later), and θ the previous angular scale in the shower. This modification extends the angular-ordered framework to the three-jet limit.

...up to terms of the order θ^0.

In the literature

Therefore the accompanying partons k_i are assumed to be soft and their distribution can be treated as independent. Assembling together the contributions from the three configurations $\mathcal{C}_\delta$ defined in (2.10), the distribution can be presented as

$$M_n^2 = \sum_{\delta=1}^{3} M_0^2(\mathcal{C}_\delta) \cdot \prod_i^n W_\delta(k_i)\,, \tag{2.21}$$

where $M_0(\mathcal{C}_\delta)$ is the Born $q\bar{q}g$ matrix element and W_δ is the distribution of the soft gluon radiation off the hard three-parton antenna in the momentum configuration $\mathcal{C}_\delta$. Here we discuss the real emission contribution to M_n; the virtual corrections will be accommodated later.

For the configuration $\delta = 3$, for example, the squared Born matrix element reads

$$M_0^2(\mathcal{C}_3) = \frac{C_F\alpha_s}{2\pi}\frac{x_1^2+x_2^2}{(1-x_1)(1-x_2)}\,, \qquad x_a \equiv \frac{2P_aQ}{Q^2}\,, \tag{2.22}$$

where P_3 is the gluon momentum and P_1, P_2 the quark-antiquark momenta (see (2.10)). For this configuration the single soft gluon radiation pattern is given, at the one-loop level, by

$$W_3(k) = C_F\, w_{12} + \frac{N_c}{2}(w_{13}+w_{23}-w_{12}) = \frac{N_c}{2}\left(w_{13}+w_{23}-\frac{1}{N_c^2}w_{12}\right). \tag{2.23}$$

$$(w_{13}+w_{23}-w_{12})$$

Here w_{ab} is the standard two-parton antenna of the ab-dipole, which, within the normalization convention prescribed by (2.20), is given by

$$w_{ab}(k) = \frac{\alpha_s}{\pi}\frac{(P_aP_b)}{2(P_ak)(kP_b)} = \frac{\alpha_s}{\pi k_{t,ab}^2}\,. \tag{2.24}$$

Here $k_{t,ab}$ is the invariant gluon transverse momentum with respect to the hyper-plane defined by the P_a, P_b momenta.

The first term $C_F w_{12}$ in (2.23) is the "Abelian" contribution describing soft gluon emission off the $q\bar{q}$ pair. The second term proportional to N_c is its "non-Abelian" counterpart that describes radiation off the hard gluon P_3. Similar expressions for two other kinematical configurations ($\delta = 1, 2$) is straightforward to write down by properly adjusting the parton indices,

$$W_1(k) = \frac{N_c}{2}\left(w_{13}+w_{12}-\frac{1}{N_c^2}w_{23}\right), \qquad W_2(k) = \frac{N_c}{2}\left(w_{23}+w_{12}-\frac{1}{N_c^2}w_{13}\right). \tag{2.25}$$

To reach SL accuracy it is necessary to treat multi-parton emission at the two-loop level. This involves allowing secondary gluon to split into two gluons or into a $q\bar{q}$ pair. In principle, perturbative analysis of a system consisting of three hard partons

hep-ph/0004027

$$\int \frac{\mathrm{d}\Omega_n^{3-2\varepsilon}}{4\pi}E_n^2\,{}^{(n-1)}\mathrm{R}_{k,l}^{i,j} = \frac{1}{-2\varepsilon}\frac{\pi^{\frac{1-2\varepsilon}{2}}}{\Gamma(\frac{1-2\varepsilon}{2})}\left(f_{ij}-f_{ik}-f_{jl}+f_{kl}\right), \\ = 2\ln\left(\frac{n_i\cdot n_j\, n_k\cdot n_l}{n_i\cdot n_k\, n_j\cdot n_l}\right) + \mathcal{O}(\epsilon) \equiv 2\ln(\rho_{ijkl}) + \mathcal{O}(\epsilon)$$

The jet anomalous dimensions $\gamma_{\mathcal{J}_k}$ can be computed using Eq. (3.8) in the fundamental representation. At one loop this yields

$$\gamma_{\mathcal{J}_k}(w_k,\alpha_s,\epsilon) = \frac{\alpha_s}{2\pi}C_F\left[-1+\ln\left(\frac{2(v_k\cdot n_k)^2}{n_k^2}\right)-\frac{1}{\epsilon}\right]+\mathcal{O}(\alpha_s^2)\,. \tag{B.5}$$

– 30 –

The subtracted terms are proportional to the unit matrix in colour space, so they affect only the diagonal elements of the anomalous dimension matrix. We end up with the following anomalous dimension for $\overline{\mathcal{S}}$, which is of course finite,

$$\Gamma^{\overline{\mathcal{S}}}(\rho_{ij},\alpha_s) = \frac{\alpha_s}{\pi}C_F\begin{pmatrix} 2+\frac{1}{2}\ln(\rho_{12}\,\rho_{34}) & 0 \\ 0 & 2+\frac{1}{2}\ln(\rho_{13}\,\rho_{24})\end{pmatrix} \\ + \frac{\alpha_s}{4\pi}\begin{pmatrix} \frac{1}{N_c}\ln\left(\frac{\rho_{14}\,\rho_{23}}{\rho_{13}\,\rho_{24}}\right) & \ln\left(\frac{\rho_{12}\,\rho_{34}}{\rho_{14}\,\rho_{23}}\right) \\ \ln\left(\frac{\rho_{13}\,\rho_{24}}{\rho_{14}\,\rho_{23}}\right) & \frac{1}{N_c}\ln\left(\frac{\rho_{14}\,\rho_{23}}{\rho_{12}\,\rho_{34}}\right)\end{pmatrix} - 2\pi\mathrm{i}\,\frac{\alpha_s}{\pi}\begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c}\end{pmatrix} + \mathcal{O}(\alpha_s^2)\,. \tag{B.6}$$

It is straightforward to check that the general expressions in Eq. (5.16) and Eq. (5.6) indeed reduce to Eq. (B.6) and Eq. (B.3), respectively, upon evaluating the colour factors in the chosen basis and substituting the one-loop values for γ_K, $\widehat{\delta}_{\overline{\mathcal{S}}}$ and $\widehat{\delta}_{\mathcal{S}}$.

$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij}+\omega_{ik}-\omega_{jk}\right)$$

$$\sum_j \mathbf{T}_j = 0 \qquad \omega_{ij}(q_n) = \frac{q_i\cdot q_j}{q_n\cdot q_i\; q_n\cdot q_j}$$

Momentum conservation

There is a question of momentum conservation…

$$^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

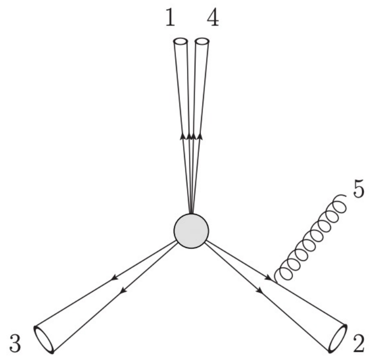
Two spectators for a given colour structure. Genuine 3->4 transition.

There isn't a known solution to conserving recoils for 3->4 transitions.

The problem is consistency between the 2-jet limit and 3-jet limit.

$$q_5 = \alpha q_2 + \beta q_1 + \gamma(q_3 + q_4) + k_\perp$$

$$q_5 = \alpha q_2 + \beta(q_1 + q_3 + q_4) + k_\perp$$



D.o.f. in $k_\perp$ changes

Momentum conservation

There is a question of momentum conservation…

$$^{(n-1)}\mathrm{S}_i^{j,k} = \frac{1}{2}\left(\omega_{ij} + \omega_{ik} - \omega_{jk}\right)$$

Two spectators for a given colour structure. Genuine 3->4 transition.

There isn't a known solution to conserving recoils for 3->4 transitions.

Solution could be to just pick one of the spectators with a 50% chance each time and use the large body of work on 2->3 recoils.

This is consistent with the 2-jet limit but doesn't "feel right". It would be sufficient for a Parton Shower implementation though.

Concluding

The complete outcome can be summarised

$$\frac{\mathrm{d}(\cos\theta_{in})\mathrm{d}\phi_n}{1-\cos\theta_{in}}\Theta(\theta_{in}<\theta) \mapsto {}^{(n-1)}\mathrm{S}_i^{j,k}\, E_n^2\, \Theta(\theta_{in}<\theta)\, \mathrm{d}(\cos\theta_{in})\mathrm{d}\phi_n$$

$${}^{(n-1)}\mathrm{S}_i^{j,k} = \frac{s_{ij}s_{nk}+s_{ik}s_{nj}-s_{jk}s_{ni}}{2s_{ni}s_{nj}s_{nk}}$$

for $s_{ij} = q_i \cdot q_j$. The emitted parton is labelled n, i is its parent, j, k are spectators (defined later), and θ the previous angular scale in the shower. This modification extends the angular-ordered framework to the three-jet limit.

…up to terms of the order θ^0.
Randomly pick j or k as a spectator for recoil in a 2->3 style.