Quantum Information with Top Quarks

Y. Afik, JRMdN, EPJ Plus 136, 907 (2021)
Y. Afik, JRMdN, Quantum 6, 820 (2022)
Y. Afik, JRMdN, PRL 130, 221801 (2023)

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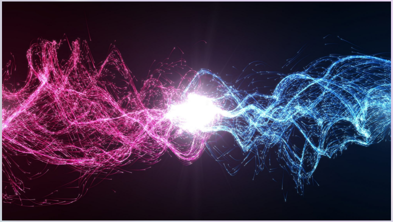
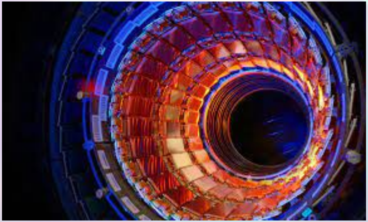
Motivation

- Standard Model is a Relativistic Quantum Field Theory = Special Relativity + Quantum Mechanics.
- Quantum Mechanics can be tested via Standard Model.
- Implementation of canonical techniques of Quantum Information $\rightarrow$ Quantum Information Theory at High-Energy Colliders.
- Highest-energy study at the frontier of the known Physics!
- Interest: Genuinely relativistic environment, exotic interactions and symmetries, fundamental nature...



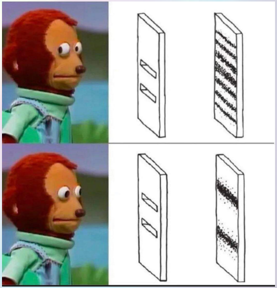


Part I: Quantum Information Theory

Quantum vs. Classical

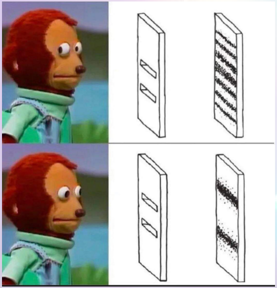
- Quantum Mechanics: Particles are in superposition of states → Probabilistic description of measurements.
- Classical Mechanics can also describe random outputs using classical probability distributions (noise, experimental variations...).
- Something genuinely quantum? Yes: Wave nature of quantum mechanics!

- Quantum Correlations=Correlations not accounted by classical theories.
  1. Quantum Discord
  2. Entanglement
  3. Steering
  4. Bell nonlocality



Quantum State

- Quantum descriptions:
  - **Pure state** $\rightarrow$ Wave function $\rightarrow$ Coherent mixture of quantum states $\rightarrow |\Psi\rangle = \sum_n \alpha_n \cdot |\phi_n\rangle$, α_n are amplitudes
  - **Mixed state** $\rightarrow$ Density matrix $\rightarrow$ Incoherent mixture of quantum states $\rightarrow \rho = \sum_n p_n \cdot |\phi_n\rangle \langle\phi_n|$, p_n are probabilities
- Density matrix: Most general quantum state.
- Classical descriptions accounted by density matrices.



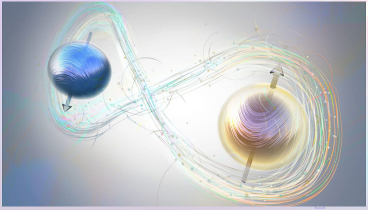
Qubits

- Qubit: Two-level quantum system $|\uparrow\rangle, |\downarrow\rangle \rightarrow$ Most simple quantum system.
- General density matrix (2×2) for 1 qubit $\rightarrow$ 3 parameters B_i:

$$\rho = \frac{1 + \sum_i B_i \sigma^i}{2}$$

- Two qubits $\rightarrow$ Most simple example of quantum correlations.
- General density matrix (4×4) for 2 qubits $\rightarrow$ 15 parameters $B_i^{\pm}$, C_{ij}

$$\rho = \frac{1 + \sum_i \left(B_i^+ \sigma^i \otimes 1 + B_i^- 1 \otimes \sigma^i \right) + \sum_{i,j} C_{ij} \sigma^i \otimes \sigma^j}{4}$$



Quantum Discord

- Classically, two equivalent expressions for mutual information of bipartite system A and B (Alice and Bob):

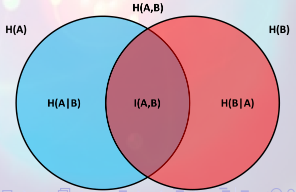
$$
\begin{aligned}
I(A,B) &= H(A) + H(B) - H(A,B) = H(A) - H(A|B) \\
H(A,B) &= -\sum_{x,y} p(x,y) \log_2 p(x,y) \\
H(A|B) &= \sum_{y} p(y) H(A|B=y)
\end{aligned}
$$

- Quantum mechanics can introduce a "discord" between both expressions:

$$\mathcal{D}(A,B) \equiv H(B) - H(A,B) + H(A|B) \neq 0$$

- Most basic form of quantum correlations!
- Quantum Discord is asymmetric $\mathcal{D}(A,B) \neq \mathcal{D}(B,A)$

Ollivier, Zurek PRL 88, 017901 (2001)



Quantum Discord: Two qubits

- How do we translate classical into quantum?

Quantum Discord: Two qubits

- How do we translate classical into quantum?
- Shannon entropy → Von Neumann entropy ($p_n \geq 0$, ρ eigenvalues)

$$H(A,B) \quad \to \quad H(\rho) = -\sum_n p_n \log_2 p_n$$

$$H(A) \quad \to \quad H(\rho_A),\ H(B) \to H(\rho_B),\ \rho_{A,B} = \mathrm{Tr}_{B,A}\rho$$

- Conditional probability → Conditional state $\rho_{A|B}$ = One-qubit state after Bob's spin measurement along $\hat{\mathbf{n}}$:

$$H(A|B) = p_{\hat{\mathbf{n}}} H(\rho_{\hat{\mathbf{n}}}) + p_{-\hat{\mathbf{n}}} H(\rho_{-\hat{\mathbf{n}}})$$

$$\rho_{\hat{\mathbf{n}}} = \frac{\Pi^B_{\hat{\mathbf{n}}}\rho\Pi^B_{\hat{\mathbf{n}}}}{p_{\hat{\mathbf{n}}}} = \frac{1+\mathbf{B}^+_{\hat{\mathbf{n}}}\cdot\sigma}{2},\ \mathbf{B}^+_{\hat{\mathbf{n}}} = \frac{\mathbf{B}^+ + \mathbf{C}\cdot\hat{\mathbf{n}}}{1+\hat{\mathbf{n}}\cdot\mathbf{B}^-},\ p_{\hat{\mathbf{n}}} = \frac{1+\hat{\mathbf{n}}\cdot\mathbf{B}^-}{2}$$

- Genuine quantumness → Minimization over all spin directions:

$$\mathcal{D}(A,B) = H(\rho_B) - H(\rho) + \min_{\hat{\mathbf{n}}} p_{\hat{\mathbf{n}}} H(\rho_{\hat{\mathbf{n}}}) + p_{-\hat{\mathbf{n}}} H(\rho_{-\hat{\mathbf{n}}}) \neq 0$$

Entanglement

- Entanglement: Most genuine feature of Quantum Mechanics. Key resource for quantum technologies.
- Separability: $\rho = \sum_n p_n \rho^a_n \otimes \rho^b_n$, $\sum_n p_n = 1$, $p_n \geq 0$
- Classically correlated state in $\mathcal{H} \rightarrow$ Separable.
- Non-separability=**Entanglement** $\rightarrow$ Non-classical state.

R. F. Werner, PRA 40, 4277 (1989)

Entanglement: Two qubits

- Two qubits: Separability=Positive P-representation $P(\mathbf{n}_A, \mathbf{n}_B) \geq 0$:

$$\rho = \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \, P(\mathbf{n}_A, \mathbf{n}_B) \left|\mathbf{n}_A \mathbf{n}_B\right\rangle \left\langle \mathbf{n}_A \mathbf{n}_B\right|, \quad \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \, P(\mathbf{n}_A, \mathbf{n}_B) = 1$$

- Classical spins pointing at directions $\mathbf{n}_A, \mathbf{n}_B$!
- Separability=Purely classical correlations

$$C_{ij} = \langle \sigma^i \otimes \sigma^j \rangle = \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B \; P(\mathbf{n}_A, \mathbf{n}_B) n_A^i n_B^j$$

- Entanglement=NO probability distribution $\rightarrow$ Genuine non-classical!

Steering: Two qubits

- Measurements of Bob can "steer" quantum state of Alice.
- Steering: Original conception of Schrödinger of EPR paradox → Only well-defined in 2007! (Wiseman, Jones, Doherty, PRL 98, 140402 (2007))
- Alice post-measurement state described by local-hidden states:

$$\tilde{\rho}_{\hat{\mathbf{n}}} = \Pi^B_{\hat{\mathbf{n}}} \rho \Pi^B_{\hat{\mathbf{n}}} = \int d\lambda \; p(1|\hat{\mathbf{n}}\lambda) p(\lambda) \rho_B(\lambda)$$

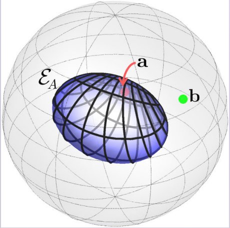
- If not, quantum state is **steerable**.

Steering: Two qubits

- Alice post-measurement state: same as for quantum discord.

$$\rho_{\hat{\mathbf{n}}} = \frac{\tilde{\rho}_{\hat{\mathbf{n}}}}{\mathrm{Tr}\tilde{\rho}_{\hat{\mathbf{n}}}} = \frac{1 + \mathbf{B}^{+}_{\hat{\mathbf{n}}} \cdot \sigma}{2}, \quad \mathbf{B}^{+}_{\hat{\mathbf{n}}} = \frac{\mathbf{B}^{+} + \mathbf{C} \cdot \hat{\mathbf{n}}}{1 + \hat{\mathbf{n}} \cdot \mathbf{B}^{-}}$$

- Set of conditional polarizations $\mathbf{B}^{+}_{\hat{\mathbf{n}}}$ describes an ellipsoid.
- Steering ellipsoid: Fundamental QI object, containing all information about the system.
- Similar for Bob $\rightarrow$ Steering: also asymmetric between Alice and Bob.



Jevtic, Pusey, Jennings, Rudolph
PRL 113, 020402 (2014)

Bell inequality: Two qubits

- Local realism: Joint Alice and Bob measurements M_A, M_B accounted by local hidden-variable model

$$p(a, b|M_A M_B) = \int d\lambda \; p(a|M_A\lambda)p(b|M_B\lambda)p(\lambda)$$

- Local realism holds if Bell inequality is satisfied. Two qubits → **CHSH inequality** ($\mathbf{a}_i, \mathbf{b}_i$ spin axes of measurements M_A, M_B)

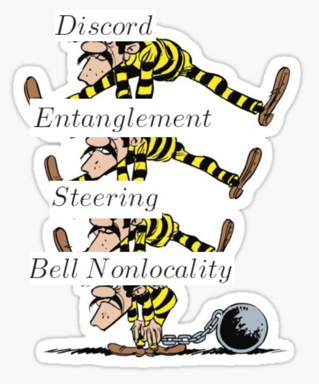
$$|\mathbf{a}_1^{\mathrm{T}}\mathbf{C}\left(\mathbf{b}_1 - \mathbf{b}_2\right) + \mathbf{a}_2^{\mathrm{T}}\mathbf{C}\left(\mathbf{b}_1 + \mathbf{b}_2\right)| \leq 2$$

- Stronger condition than entanglement → "Spooky action at distance"

Hierarchy of Quantum Correlations

- Steering and Discord can be asymmetric between Alice and Bob.
- Bell Nonlocality and Entanglement are always symmetric.
- Quantum Hierarchy:

$$\textit{Bell Nonlocality} \subset \textit{Steering} \subset \textit{Entanglement} \subset \textit{Discord}$$



Quantum Tomography: Two qubits

- **Quantum Tomography**: Reconstruction of quantum state from measurement of a set of observables.
- Quantum tomography $\rightarrow$ Measurement of ALL quantum correlations.
- Most general density matrices for $1, 2$ qubits:

$$\rho = \frac{1 + \sum_i B_i \sigma^i}{2}, \ \rho = \frac{1 + \sum_i \left(B_i^+ \sigma^i + B_i^- \bar{\sigma}^i\right) + \sum_{i,j} C_{ij} \sigma^i \bar{\sigma}^j}{4}$$

- One-qubit quantum tomography=Measurement of 3 parameters, polarization vector $\mathbf{B}$: $B_i = \langle \sigma^i \rangle$
- Two-qubit quantum tomography=Measurement of 15 parameters, polarization vectors $\mathbf{B}^{\pm}$ and correlation matrix $\mathbf{C}$:

$$B_i^+ = \langle \sigma^i \rangle, \ B_i^- = \langle \bar{\sigma}^i \rangle, \ C_{ij} = \langle \sigma^i \bar{\sigma}^j \rangle$$

Part II: Top Quark Physics

Who Top Quarks?

- Top quark is the most massive fundamental particle known to exist ($m_t c^2 \approx 173$ GeV).
- First discovered by the D0 and CDF collaborations at the Tevatron in 1995.
- Top quarks produced in top-antitop ($t\bar{t}$) pairs through QCD or Electroweak processes.

Why Top Quarks?

- Large Width $\Gamma_t \sim 1$ GeV $\rightarrow$ Very short lifetime $\tau = 1/\Gamma_t \sim 10^{-25}\mathrm{s}$
- Tops decay before
  - Hadronisation $\sim 10^{-23}\mathrm{s}$.
  - Spin-decorrelation $\sim 10^{-21}\mathrm{s}$.
- $\rightarrow$ NO DECOHERENCE OR RANDOMIZATION!
- Rotational invariance in $t\bar{t}$ rest frames $\rightarrow$ $t\bar{t}$ spins measured from decay products.
- Measurements by D0 and CDF (Tevatron), ATLAS and CMS (LHC) $\rightarrow$ Well-established technique!

Top pair kinematics

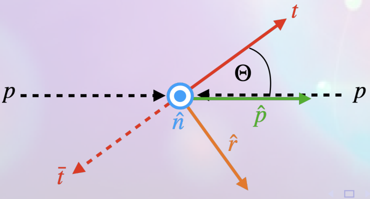
- $t\bar{t}$ pair kinematically described by invariant mass $M_{t\bar{t}}$ and top direction $\hat{k}$ in c.m. frame

$$\begin{aligned} k_t^\mu &= (k_t^0, \mathbf{k}), k_{\bar{t}}^\mu = (k_t^0, -\mathbf{k}) \\ M_{t\bar{t}}^2 &\equiv s_{t\bar{t}} \equiv (k_t + k_{\bar{t}})^2 \end{aligned}$$

- Invariant mass is simply related to top c. m. velocity β

$$M_{t\bar{t}} = \frac{2m_t}{\sqrt{1-\beta^2}} \to \beta = 0 \to M_{t\bar{t}} = 2m_t$$

- Threshold production: $M_{t\bar{t}} = 2m_t \approx 346$ GeV

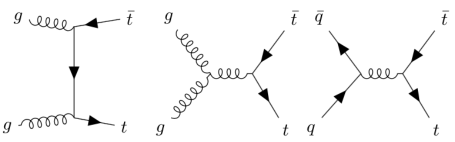


LO QCD Elementary Process

- Illustrative example: QCD analytical LO calculation.
  - Analytical results.
  - NLO corrections are small.
  - Building blocks of actual high-energy processes.
- Most elementary QCD processes:

$$\begin{aligned} q+\bar{q} &\rightarrow t+\bar{t}, \; q=u,d\ldots \\ g+g &\rightarrow t+\bar{t} \end{aligned}$$

- Each initial state $I = q\bar{q}, gg$ gives rise to quantum state $\rho^I(M_{t\bar{t}}, \hat{k})$



LO QCD Realistic

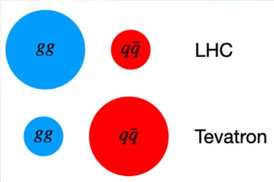
- No free quarks or gluons $\to$ Hadrons: Bound states of quarks and gluons (partons).
- LHC, Tevatron: pp, $p\bar{p}$ collisions at high c.m. energies $\sqrt{s}$.

$$
\begin{aligned}
p + p &\to \ldots \to t + \bar{t} \qquad \text{LHC} \\
p + \bar{p} &\to \ldots \to t + \bar{t} \qquad \text{Tevatron}
\end{aligned}
$$

- Quantum state depends now on c.m. energy $\sqrt{s}$:

$$
\rho(M_{t\bar{t}}, \hat{k}) = \sum_{I=q\bar{q},gg} w_I(M_{t\bar{t}}, \sqrt{s})\rho^I(M_{t\bar{t}}, \hat{k})
$$

- Total QCD process: *Incoherent* sum of elementary QCD processes with probability w_I.
- QCD Input: $w_I(M_{t\bar{t}}, \sqrt{s}), \rho^I(M_{t\bar{t}}, \hat{k}) \to$ QI Output: Textbook problem of *convex sum* of quantum states!



Part III: Quantum Tops

$t\bar{t}$ Quantum Correlations

- Quantum state $\rho(M_{t\bar{t}}, \hat{k})$: Function of scattering angle Θ and $M_{t\bar{t}}$.
- Two main regions of quantumness:
  - High-p_T for both $q\bar{q}$ and gg (spin triplet)
  - Threshold for gg (spin singlet).

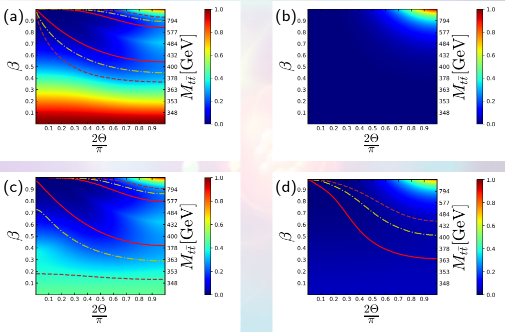
- Colorbar: Discord.
- Solid, dashed-dotted, dashed: Boundaries of Entanglement, Steering, Bell Nonlocality $\rightarrow$ Hierarchy!

a) $gg \rightarrow t\bar{t}$

b) $q\bar{q} \rightarrow t\bar{t}$

c) Run 2 LHC $\sqrt{s} = 13$ TeV

d) Tevatron $\sqrt{s} = 1.96$ TeV



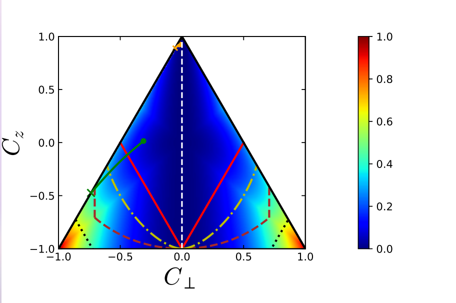
Total Quantum State

- Realistic measurement: Average over many different processes.
- Total quantum state: Events in window $[2m_t, M_{t\bar{t}}]$

$$\rho(M_{t\bar{t}}) \equiv \frac{1}{\sigma(M_{t\bar{t}})}\int_{2m_t}^{M_{t\bar{t}}} \mathrm{d}M \int \mathrm{d}\Omega\, \frac{\mathrm{d}\sigma}{\mathrm{d}M\mathrm{d}\Omega}\rho(M,\hat{k})$$

- Intuitively: Total quantum state=Sum of $t\bar{t}$ quantum states weighted with the differential cross-section.

- Rotational invariance around beam axis $\rightarrow$ Correlation matrix diagonal in beam basis $C_{ij} = C_i\delta_{ij}$, $C_x = C_y = C_\perp \rightarrow$ 2D dependence on $C_\perp, C_z$.
- Green: LHC. Orange: Tevatron.
- Cross: $\beta = 0$; Circle: $\beta = 1$.



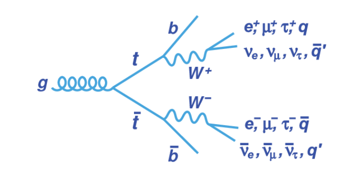
Part IV: Experimental Analysis

Top pair Quantum Tomography

- $\rho(M_{t\bar{t}}) \to$ Two qubit quantum state $\to$ Quantum tomography $=$ Measurement of spin polarizations and spin correlations.
- Spin polarizations $\mathbf{B}^{\pm}$ and spin correlation matrix $\mathbf{C}$ extracted from cross-section $\sigma_{\ell\bar{\ell}}$ of dileptonic decay

$$\frac{1}{\sigma_{\ell\bar{\ell}}}\frac{d\sigma_{\ell\bar{\ell}}}{d\Omega_+ d\Omega_-} = \frac{1}{(4\pi)^2}\left[1 + \mathbf{B}^+ \cdot \hat{\ell}_+ - \mathbf{B}^- \cdot \hat{\ell}_- - \hat{\ell}_+ \cdot \mathbf{C} \cdot \hat{\ell}_-\right]$$

- $\hat{\ell}_{\pm}$: lepton directions in each top (antitop) rest frames.

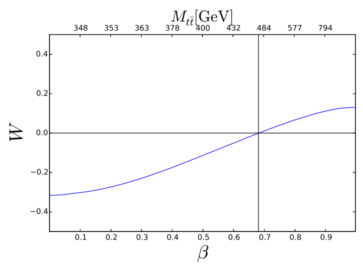


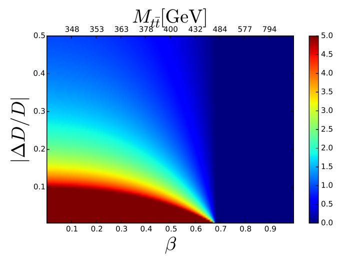
Entanglement in $t\bar{t}$ production at LHC $\sqrt{s} = 13$ TeV

- Entanglement witness:
  $W = D + 1/3 < 0,\ D \equiv \mathrm{tr}\ \mathbf{C}/3 \rightarrow$
  Entanglement only close to threshold.
- D directly measurable from decay cross-sections:

$$\frac{1}{\sigma}\frac{d\sigma}{d\cos\varphi} = \frac{1}{2}(1 - D\cos\varphi)$$

- Entanglement detection from one single magnitude! $\rightarrow$ No need for Quantum Tomography!
- High-statistical significance!
- Entanglement also at high-p_T: Fabbrichesi, Floreanini, Panizzo, PRL 127, 161801 (2021), Severi, Boschi, Maltoni, Sioli, EPJC 82, 285 (2022)





Discord and Steering

- Normalized dileptonic cross-section → Angular probability distribution:

$$p(\hat{\ell}_+, \hat{\ell}_-) = \frac{1}{\sigma_{\ell\bar{\ell}}} \frac{\mathrm{d}\sigma_{\ell\bar{\ell}}}{\mathrm{d}\Omega_+ \mathrm{d}\Omega_-} = \frac{1 + \mathbf{B}^+ \cdot \hat{\ell}_+ - \mathbf{B}^- \cdot \hat{\ell}_- - \hat{\ell}_+ \cdot \mathbf{C} \cdot \hat{\ell}_-}{(4\pi)^2}$$

- Direct one-qubit tomography of $\rho_{A,B}, \rho_{\hat{\mathbf{n}}}$ from Bloch vectors $\mathbf{B}^{\pm}, \mathbf{B}^{\pm}_{\hat{\mathbf{n}}}$:

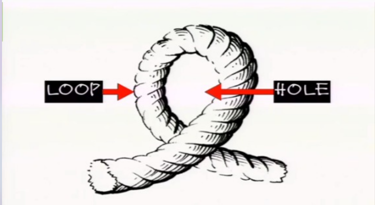
$$p(\hat{\ell}_\pm) = \int \mathrm{d}\Omega_\mp \; p(\hat{\ell}_+, \hat{\ell}_-) = \frac{1 \pm \mathbf{B}^\pm \cdot \hat{\ell}_\pm}{4\pi}$$

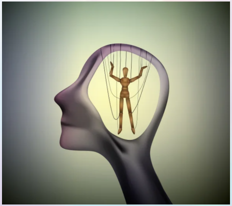
$$p(\hat{\ell}_\pm | \hat{\ell}_\mp = \mp\hat{\mathbf{n}}) = \frac{p(\hat{\ell}_\pm, \hat{\ell}_\mp = \mp\hat{\mathbf{n}})}{p(\hat{\ell}_\mp = \mp\hat{\mathbf{n}})} = \frac{1 \pm \mathbf{B}^+_{\hat{\mathbf{n}}} \cdot \hat{\ell}_\pm}{4\pi}$$

- Actual discord → Evaluated from minimization over $\hat{\mathbf{n}}$.
- Measurement of $\mathbf{B}^{\pm}_{\hat{\mathbf{n}}}$ → Reconstruction of $t, \bar{t}$ steering ellipsoids.
- Highly-challenging measurements in conventional setups → Natural implementation in colliders!

Bell Test Loopholes in a Collider Experiment

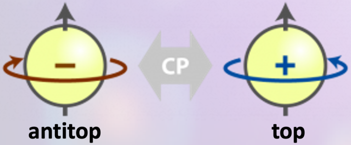
- Loopholes: Experimental tests of Bell's inequality may not fulfill all hypotheses of Bell's theorem.
- Collider experiment:
  1. Free-will loophole: Spin measurement directions should be free, independent from hidden-variables. $\rightarrow$ Not even single-detection events from Alice and Bob!
  2. Detection loophole: Only a subset of events selected for measurement $\rightarrow$ Bias!
- Quite natural: Colliders were not designed to test Bell's Inequality!





New Physics Witnesses

- Approximate *CP*-invariance of Standard Model $\rightarrow \mathbf{C} = \mathbf{C}^{\mathrm{T}}$, $\mathbf{B}^{+} = \mathbf{B}^{-}$ $\rightarrow$ Symmetric discord and steering!
- Therefore: Discord and/or Steering asymmetry $\rightarrow$ New Physics!
- New physics witnesses: Symmetry protected observables by SM, only non-zero for New Physics:
  - $\Delta\mathcal{D}_{t\bar{t}} \equiv \mathcal{D}_{t} - \mathcal{D}_{\bar{t}}$
  - Asymmetries in ellipsoid centers and/or semiaxes.
- No SM contribution to New Physics witnesses!



Conclusions and outlook

- Quantum Information theory $\longleftrightarrow$ High-Energy Physics. Interdisciplinary, huge potential and great interest!
- QI perspective:
  1. Highest-energy observation of entanglement ever!
  2. Genuinely relativistic, exotic symmetries and interactions, fundamental nature $\rightarrow$ Frontier of known Physics!
  3. Highly-demanding measurements naturally implemented at LHC.
- HEP perspective:
  1. Quantum Tomography: Novel experimental tool.
  2. QI techniques can inspire new approaches for searching New Physics:
     - Aoude, Madge, Maltoni, Mantani, PRD (2022).
     - Severi, Vryonidou, JHEP (2023).
     - Fabbrichesi, Floreanini, Gabrielli, EPJC (2023).
- Extension to e^+e^- colliders: Spin of initial state can be controlled! $\rightarrow$ Manipulation of qubits? Quantum gates?
- Adaptation to τ leptons, qutrits $W^\pm, Z^0$.

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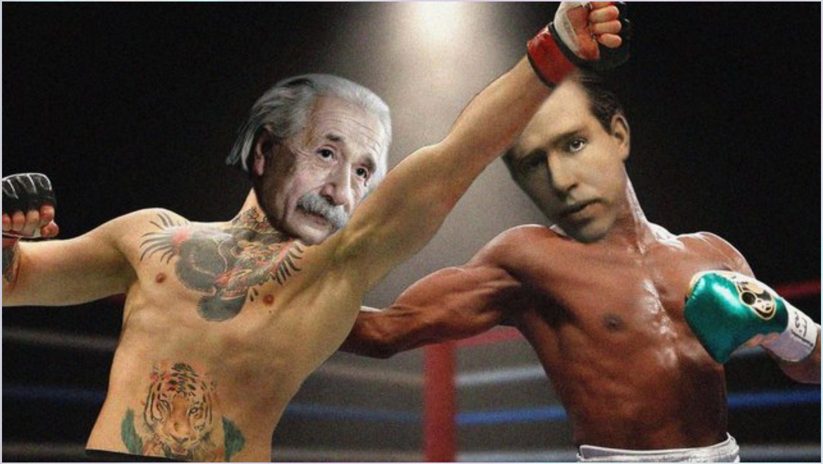
- The first measurements of entanglement between a pair of top-quarks are ongoing within ATLAS and CMS.
- **The results are expected to be public soon - stay tuned!**

Thank You



Backup

Quantum Discord: Two qubits

- Classical two-qubit state:

$$\begin{aligned}\rho &= p_{++}\,|\hat{\mathbf{n}}\,\hat{\mathbf{n}}'\rangle\langle\hat{\mathbf{n}}\,\hat{\mathbf{n}}'| + p_{+-}\,|\hat{\mathbf{n}}\,-\hat{\mathbf{n}}'\rangle\langle\hat{\mathbf{n}}\,-\hat{\mathbf{n}}'| \\ &+ p_{-+}\,|-\hat{\mathbf{n}}\,\hat{\mathbf{n}}'\rangle\langle-\hat{\mathbf{n}}\,\hat{\mathbf{n}}'| + p_{--}\,|-\hat{\mathbf{n}}\,-\hat{\mathbf{n}}'\rangle\langle-\hat{\mathbf{n}}\,-\hat{\mathbf{n}}'|\end{aligned}$$

- Incoherent statistical mixture of $|\pm\hat{\mathbf{n}}\pm\hat{\mathbf{n}}'\rangle$ states $\rightarrow$ Quantum mechanics if it was just some probabilistic theory
- Classical states $\iff$ Zero discord when measuring B along $\hat{\mathbf{n}}'$ direction!

$$\mathcal{D}(A,B) = H(\rho_B) - H(\rho) + p_{\hat{\mathbf{n}}'}H(\rho_{\hat{\mathbf{n}}'}) + p_{-\hat{\mathbf{n}}'}H(\rho_{-\hat{\mathbf{n}}'}) = 0$$

- Classical states included in separable states:

$$\rho = \int \mathrm{d}\Omega_A \mathrm{d}\Omega_B\; P(\mathbf{n}_A,\mathbf{n}_B)\,|\mathbf{n}_A\mathbf{n}_B\rangle\langle\mathbf{n}_A\mathbf{n}_B|$$

- Entanglement $\rightarrow$ Quantum discord