Progress on $K_L \to \mu^+\mu^-$ from lattice QCD

En-Hung CHAO

Columbia University

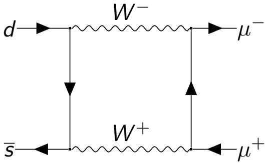
September 18, 2023 @ CKM 2023

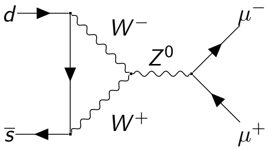
Based on on-going work with Norman Christ

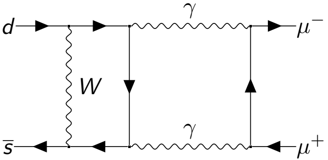
Outline

Introduction

- In the Standard Model, $K_{\mathrm{L}} \to \mu^+\mu^-$ comes in at one-loop level with exchange of two W-bosons or two W- and a Z-boson (short-distance contribution, SD).
- Precisely measured $\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.84(11) \times 10^{-9} \Rightarrow$ good test for the SM and potential interest for the physics beyond the SM. [BNL E871 Collab., PRL '00]
- Current theory limitation is the long-distance contribution (LD) involving two-photon exchange entering at $O(G_{\mathrm{F}}\alpha_{\mathrm{QED}}^2)$, parametrically comparable to the SD contribution: the real part of the amplitude is not well understood.





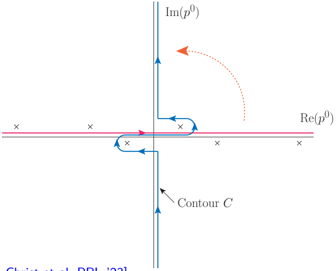


Formalism

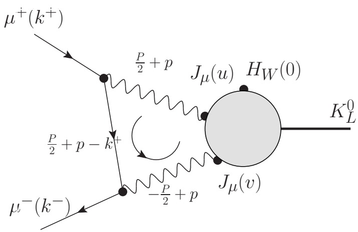
- Strategy: perturbatively expanded kernel function in G_{F} and α_{QED} + Euclidean hadronic correlation function computed on the lattice.

$$\mathcal{A}_{ss'}(k^+,k^-) = e^4 \int d^4p \int d^4u \int d^4v \, e^{-i\left(\frac{P}{2}+p\right)u} e^{-i\left(\frac{P}{2}-p\right)v} \frac{1}{(\frac{P}{2}-p)^2+m_\gamma^2-i\varepsilon} \cdot \frac{1}{(\frac{P}{2}+p)^2+m_\gamma^2-i\varepsilon}$$
$$\times \frac{\overline{u}_s(k^-)\gamma_\nu\{\gamma\cdot(\frac{P}{2}+p-k^+)+m_\mu\}\gamma_\mu v_{s'}(k^+)}{(\frac{P}{2}+p-k^+)^2+m_\mu^2-i\varepsilon} \cdot \left\langle 0 \middle| T\left\{J_\mu(u)J_\nu(v)\mathcal{H}_W(0)\right\} \middle| K_L \right\rangle .$$

- Analytic continuation of the kernel: ⇒ unphysical exponentially growing contribution from states lighter than the kaon at rest.
- Finite number of such states on a finite lattice ⇒ explicit, precise subtraction of such is possible.



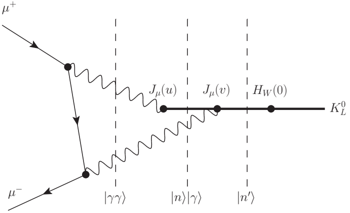
[cf. Christ et al, PRL '23]



Formalism

Time-ordering and Wick rotation

- Set an IR cutoff T and consider the possible intermediate states in the particular time-ordering $0 \le v_0 \le u_0$.



- The contribution from this time-ordering reads

$$\int_0^T du_0 \int_0^{u_0} dv_0 \int_{-\infty}^{\infty} dp_0\; e^{i\left(\frac{M_K}{2}+p_0\right)u_0} e^{i\left(\frac{M_K}{2}-p_0\right)v_0}$$
$$\times\, \tilde{\mathcal{L}}^{\mu\nu}(p) e^{-iE_n u_0} e^{-i(E_{n'}-E_n)v_0} \langle 0\,|J_\mu(0)|n\rangle\langle n|J_\nu(0)|n'\rangle\langle n'|\mathcal{H}_W(0)|\,K_L\rangle\,.$$

- Under Wick rotation $u_0 \leftarrow -iu_0$, it converges at $T \to \infty$ iff

$$\boxed{E_n' > M_K \quad (i) \qquad \text{and} \qquad E_n + |\vec{p}| \ge M_K \quad (ii)}$$

  Otherwise, unphysical exponential terms appear.

- Repeating the above analysis for all possible time-ordering and intermediate state, the two sources for the exponential terms are

  1. π^0 with zero spatial momentum, coming from $K_{\rm L}$ turned into π^0 by the weak Hamiltonian.
  2. $\pi\pi(\gamma)$ states with low kinetic energy, propagating between the electromagnetic currents.

Numerical implementation

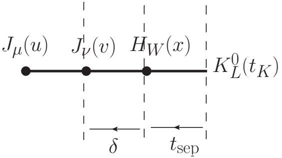
Parameter	Value
$L^3 \times T \times L_s$	$24^3 \times 64 \times 24$
m_π [MeV]	142
M_K [Mev]	515
a^{-1} [GeV]	1.023

- Lattice setup: Möbius Domain Wall fermion ensemble 24ID from the RBC/UKQCD collaboration.
- Master formula:

$$\mathcal{A}(t_{\rm sep}, \delta, x) \equiv \sum_{d \leq \delta} \sum_{u,v \in \Lambda} \delta_{v_0 - x_0, d}\; e^{M_K(v_0 - t_K)}\; K_{\mu\nu}(u - v) \left\langle J_\mu(u) J_\nu(v) \mathcal{H}_{\rm W}(x) K_{\rm L}(t_K) \right\rangle ,$$

$$\mathcal{H}_{\rm W}(x) = \frac{G_{\rm F}}{\sqrt{2}} V_{us}^* V_{ud} (C_1 Q_1 + C_2 Q_2) ,$$

$$Q_1 \equiv (\bar{s}_a \Gamma^L_\mu d_a)(\bar{u}_b \Gamma^L_\mu u_b) , \quad Q_2 \equiv (\bar{s}_a \Gamma^L_\mu d_b)(\bar{u}_b \Gamma^L_\mu u_a) .$$



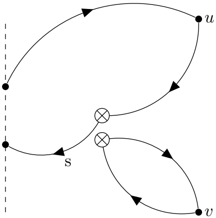
- Control of the contaminations from π^0 and low-energy $\pi\pi\gamma$ states:
  - The unphysical π^0 contribution can be measured and subtracted exactly

$$\frac{1}{2m_\pi} \sum_{\delta \geq 0} \sum_{u \in \Lambda} e^{(M_K - m_\pi)\delta} \left\langle 0 | J_\mu(u) J_\nu(v) | \pi^0 \right\rangle K_{\mu\nu}(u - v) \left\langle \pi^0 | \mathcal{H}_{\rm W}(v) | K_{\rm L} \right\rangle .$$

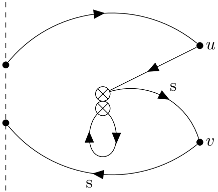
  - Control of the $\pi\pi\gamma$-intermediate state: use several kernels with different $|u - v| \leq R_{\rm max}$.

Contractions

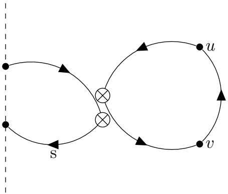
- Quark-connected Wick contractions for $\langle J_\mu(u) J_\nu(v) \mathcal{H}_W(x) K_L(t_K) \rangle$.
  Dashed line: $K_L(t_K)$, crosses: $\mathcal{H}_W(x)$, solid dots: $J_\mu(u)$ and $J_\nu(v)$



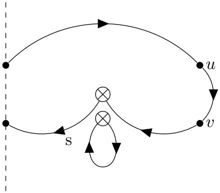
Type 1 diagram 1a



Type 2 diagram 1a



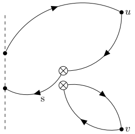
Type 3 diagram 1a

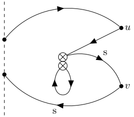


Type 4 diagram 1a

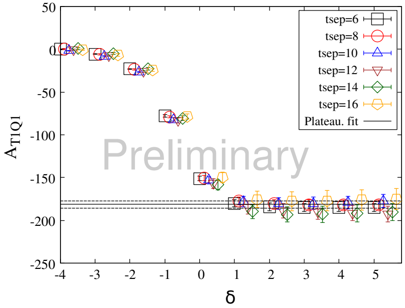
Preliminary results

Type 1 and 2

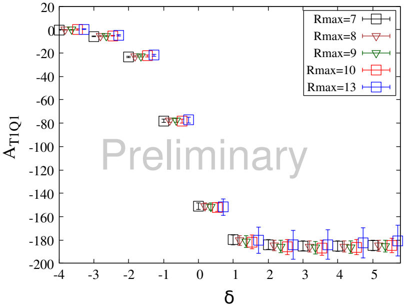




- Plateaux are formed at rather small value of δ.
- No unphysical contribution from the π^0-intermediate state expected.
- Consistency between results obtained with different $t_{\rm sep}$'s, allowing for an error-weighted average.
- Stable central values from different choices of $R_{\rm max}$, evidence of the absence of sizeable unphysical contribution from the $\pi\pi\gamma$ state.



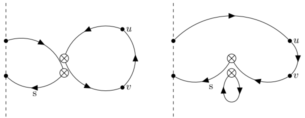
$\mathcal{A}_{\rm T1Q1}$ at fixed $R_{\rm max} = 7$



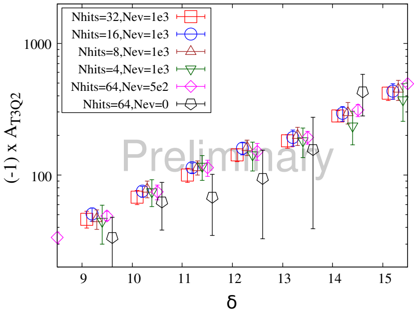
$\mathcal{A}_{\rm T1Q1}$ at fixed $t_{\rm sep} = 6$

Preliminary results

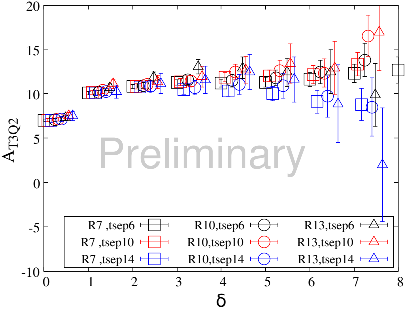
Type 3 and 4



- Stochastic all-to-all propagator with Z-Möbius low modes allowing computing with multiple $R_{\max}$.
- Expected exponentially-growing behavior due to the unphysical π^0 intermediate state.
- Plateau after subtracting the π^0 contamination. No strong sign of the $\pi\pi\gamma$ contamination by increasing $R_{\max}$.



$\mathcal{A}_{\mathrm{T3Q2}}$ in log scale with different hits and low modes.



$\mathcal{A}_{\mathrm{T3Q2}}$ after subtracting π^0

Conclusions and outlook

- A coordinate-space based lattice-QCD formalism for the $K_{\mathrm{L}} \to \mu^+\mu^-$ decay is proposed, enabling the determination of the phenomenologically inaccessible real part of the decay amplitude.
- Numerical strategies allowing to deal with different connected topologies have been developed, with possibility of keeping the $\pi\pi\gamma$ intermediate state under control.
- The so-far ignored disconnected part might not be negligible and can be much noisier due to the η intermediate state. More efficient sampling strategies will be needed.
- Possible finite-volume effects to worry about due to the $\pi\pi\gamma$ state.

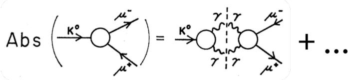
Back-up slides

Introduction

Various estimates

$$\boxed{\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.84(11) \times 10^{-9}}$$

- SD contribution computed with RG technique [Buchalla & Buras '94], known to NNLO with the charm quark effect included: $0.79(12) \times 10^{-9}$ [Gorbahn & Haisch '06]
- LD absorptive (imaginary) part from optical theorem
  - The 2γ cut dominates over other channels [Martin et al, PRD '70]



  - Estimate with the most recent $\Gamma(K_{\mathrm{L}} \to \gamma\gamma)$ saturates the experimental KL2mu decay rate: $\mathrm{Br}(K_{\mathrm{L}} \to \mu^+\mu^-) = 6.59(5) \times 10^{-9}$ [Ceccucci '17]

  $\Rightarrow$ *unitary bound* for the LD amplitude.
- Phenomenological attempts for the dispersive (real) part (+large-N_c)
  - Chiral perturbation with $\pi^0/\eta/\eta'$ pole [Dumm & Pich, PRL '98]
    $\Rightarrow$ GMO-suppressed, needs to go beyond $SU(3)_f$ and include mixings.
  - Lowest-meson dominance for the $K_{\mathrm{L}} \to \gamma^*\gamma^*$ transition form factor [Knecht et al, PRL '99]

Introduction

Lattice QCD

- Euclidean formulation of QCD regulated by the finite lattice spacing a (UV) and extent L (IR) with the $\mathrm{SU}(3)_{\mathrm{strong}}$ gauge field treated as a background.
- Positive (semi-)definite Boltzmann weight + gauge-invariant path-integral measure ⇒ suitable for Monte Carlo-based methods:

$$\langle \mathcal{O} \rangle \equiv \int \mathcal{D}[U] e^{-S[U]} \mathcal{O}[U] \approx \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathcal{O}[U_n] \,.$$

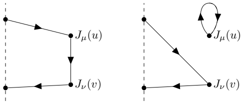
- Case with fermionic composite operators:

$$\langle \prod_n \psi_{i_n} \bar{\psi}_{j_n} \rangle = \frac{1}{Z} \int \mathcal{D}[U] \Big(e^{-S_G[U]} \prod_{f\in\mathrm{flavor}} \det[D_f] \Big) \times \Big\{ \Big(\prod_n \frac{\partial}{\partial \eta_{j_n}} \frac{\partial}{\partial \bar{\eta}_{i_n}} \Big) \Big[\prod_{f\in\mathrm{flavor}} \exp\Big(\sum_{f\in\mathrm{flavor}} \bar{\eta}_f D_f^{-1} \eta_f \Big) \Big] \Big\} \Big|_{\eta=\eta'=0} ,$$

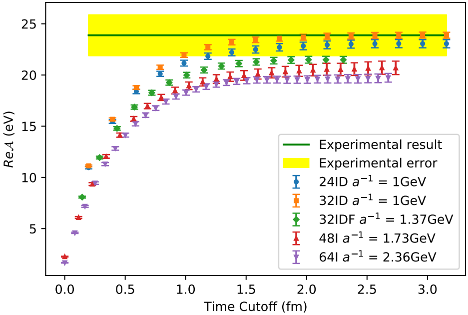
⇒ each quark line leads to a D_f^{-1} evaluated in the gauge background.

- Connection to the physical world:
  - Setting the scale (a) to a physical value.
  - Formalism Determination of the “pion mass" of each ensemble.
  - Extrapolation to the physical point: $(a, L, m_\pi) \to (0, \infty, m_\pi^{\mathrm{Phys.}})$

Intermezzo: $\pi^0 \to e^+e^-$ [Christ et al, PRL '23]



- After radiative corrections, $\mathrm{Br}(\pi^0 \to e^+e^-, \mathrm{exp}) = 6.86(27)_{\mathrm{stat.}}(23)_{\mathrm{syst.}} \times 10^{-8}$.
- Similar analytically continued kernel, without the weak Hamiltonian
- No intermediate state lighter than $\pi^0 \Rightarrow$ no unphysical exponential.
- Central value dominated by the quark-connected contribution (disc.$\sim$ 3% conn.) but comparable errors on both.
- Final result: Re/Im-ratio from the lattice and reconstruct the real part based on the $\pi^0 \to \gamma\gamma$ decay rate $\Rightarrow$ more precise



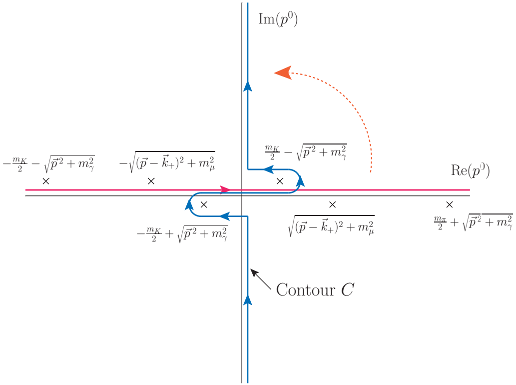
$$\mathrm{Re}\mathcal{A} = 20.2(0.4)_{\mathrm{stat}}(0.1)_{\mathrm{syst}}(0.2)_{\mathrm{expt}}\ \mathrm{eV}$$
$$\mathrm{Br}(\pi^0 \to e^+e^-) = 6.22(5)_{\mathrm{stat}}(2)_{\mathrm{syst}} \times 10^{-8}$$

Formalism

The QED kernel (1/2) [Christ et al '23, Zhao PhD thesis]

$$\mathcal{A} = \int_{u,v} \mathcal{L}_{\mu\nu}(u-v)\, \langle 0|\{\mathrm{T}\{J_\mu(u)J_\nu(v)\mathcal{H}_{\mathrm{W}}0\}|K_{\mathrm{L}}\rangle$$

- Wick rotation $u_0 \leftarrow -iu_0$
  $\Rightarrow$ integrating along $p_0 \leftarrow ip_0 \in i\mathbb{R}$ to keep the Fourier weight e^{ipu} unchanged.
- Unconventional contour avoiding the poles at $M_K/2 - |\vec{p}| \geq 0$



Formalism

The QED kernel (2/2)

- Numerical treatment:
  1. Due to CP, the Lorentz structure of the kernel is given by

$$\mathcal{L}_{\mu 0}(w) = \mathcal{L}_{0\nu}(w) = 0\,, \quad \mathcal{L}_{ij}(w) = \frac{\epsilon_{ijk} w^k}{|\vec{w}|^2} L(w^0, |\vec{w}|)$$

  2. Cauchy's theorem to get the pole on the p^0-plane, keeping the $i\varepsilon$.
  3. Principal value prescription for the $|\vec{p}|$-integral due to terms of type $\frac{1}{x - i\varepsilon}$.

- Finite in the $|\vec{w}| \to 0$ limit but exponentially growing with $w^0 \to \infty$
  $\Rightarrow$ suppressed for heavy intermediate states in the Eucl. hadronic correlator.

$$L^{\mathrm{re}}(w^0, |\vec{w}|) = 4 m_e \alpha^2 \left\{ \ln\left(\frac{1+\beta}{1-\beta}\right) \mathcal{P}\!\!\int_0^\infty \frac{\mathrm{d}|\vec{p}|}{M_\pi^2 \beta} \frac{e^{-|\vec{p}||w^0|}}{\left(\vec{p}^{\,2} - \frac{M_\pi}{2}\right)^2} F(|\vec{p}||\vec{w}|) H(|\vec{p}|, |w^0|) + \ldots \right\},$$

$$H(|M_\pi|, |w^0|) \equiv \left[\frac{M_\pi}{2} \sinh\left(\frac{M_\pi}{2}|w^0|\right) + |\vec{p}| \cosh\left(\frac{M_\pi}{2}|w^0|\right)\right],$$

$$F(x) \equiv \cos(x) - \frac{1}{x}\sin(x)\,.$$

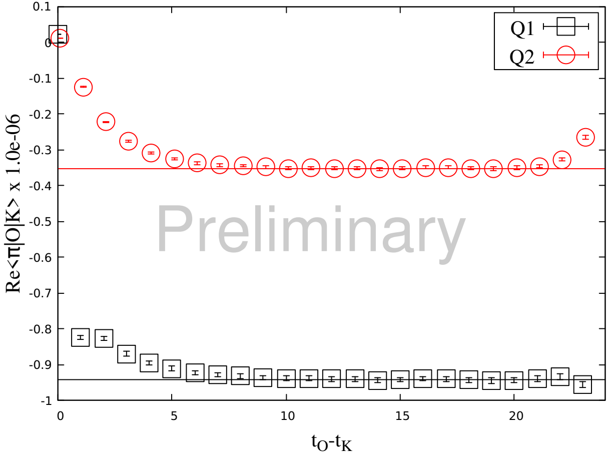
Numerical implementation

Some technical details

- Use of the (z-)Möbius accelerated Domain Wall Fermion solver: two-level solve where the loose inner solver solves the Dirac equation with a low-mode deflated z-Möbius operator.
- Coulomb-gauge-fixed wall sources to better overlap with the pseudoscalar meson ground states at large time-separation $t_{\rm sep}$.
- Randomly distributed reference points to sample the volume.

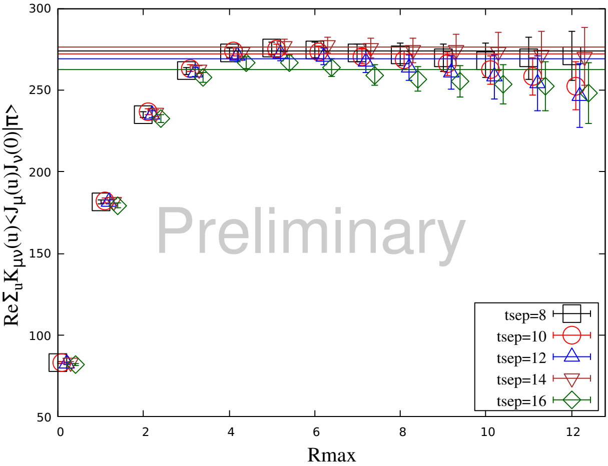
Study of the unphysical π^0 contribution

$$\frac{1}{2m_\pi}\sum_{\delta\geq 0}\sum_{u\in\Lambda} e^{(M_K-m_\pi)\delta}\left\langle 0|J_\mu(u)J_\nu(v)|\pi^0\right\rangle K_{\mu\nu}(u-v)\left\langle \pi^0|\mathcal{H}_{\rm W}(v)|K_{\rm L}\right\rangle .$$



Study of the unphysical π^0 contribution

$$\frac{1}{2m_\pi}\sum_{\delta\geq 0}\sum_{u\in\Lambda} e^{(M_K-m_\pi)\delta}\left\langle 0|J_\mu(u)J_\nu(v)|\pi^0\right\rangle K_{\mu\nu}(u-v)\left\langle \pi^0|\mathcal{H}_{\mathrm{W}}(v)|K_{\mathrm{L}}\right\rangle .$$



Preliminary results

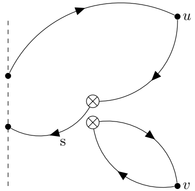
Type 1 and 2

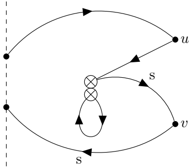
- No unphysical π^0 state expected.
- Construct blocks according to the t_{sep} dependence.
  E.g. Type 1 (a).

$$\mathcal{A}_1^{\mathrm{T1D1a}}(t_{\mathrm{sep}}, \delta, x) = \sum_{d \leq \delta} \sum_{\vec{z} \in \Lambda_0} \delta_{z_0 - x_0, d} e^{M_K(z - t_K)} \, \mathrm{Tr}_C \left[\hat{F}^1_{\nu\rho}(z, x, t_{\mathrm{sep}}) \right] \mathrm{Tr}_C \left[G_{\nu\rho}(z, x) \right] .$$

- Convolution with the kernel performed with Fast Fourier Transform

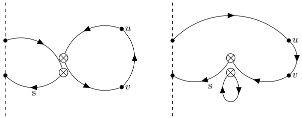
$$\hat{F}_{\nu\rho}(z, x, t_{\mathrm{sep}}) \equiv \sum_u K_{\mu\nu}(u - v) F_{\mu\rho}(u, t_{\mathrm{sep}}) = \mathcal{F}^{-1} \left[\tilde{K}_{\mu\nu}(-p) \tilde{F}_{\mu\rho}(p, t_{\mathrm{sep}}) \right] .$$





Preliminary results

Type 3 and 4



- All-to-all propagator estimator with Z-Möbius low modes h_i

$$\hat{L}(x,y) = \sum_{i=1}^{N_{\rm ev}} V^{45'} \left[\lambda_i^{-1} h_i h_i^\dagger\right]^{5'} U^{5'4} - \sum_{i=1}^{N_{\rm ev}} V^{45'} \left[\lambda_i^{-1} h_i h_i^\dagger\right]^{5'} U^{5'4} \sum_{j=1}^{N_{\rm hits}} \xi_j \xi_j^\dagger + V^{45} \left[D^{-1}\right]^{5} U^{54} \sum_{j=1}^{N_{\rm hits}} \xi_j \xi_j^\dagger \,,$$

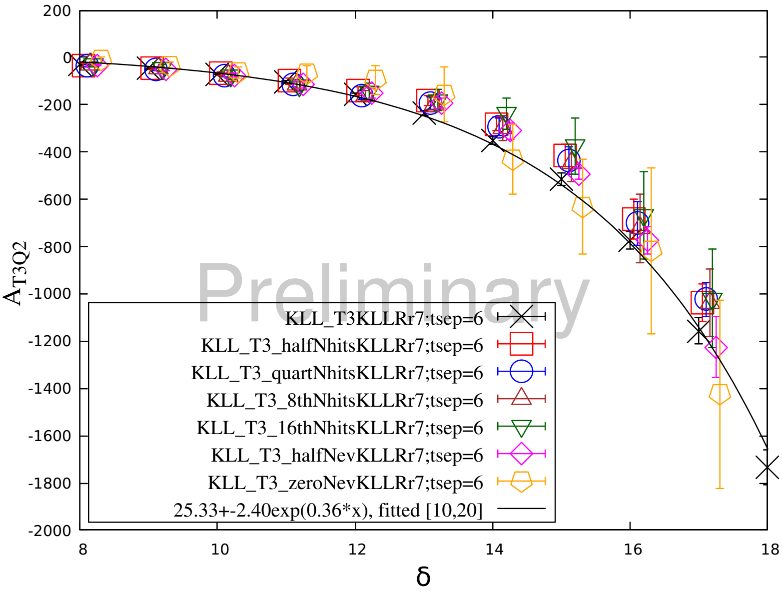
- Choice for the stochastic source ξ_i: $\mathbb{Z}_2$ time-diluted source for Type 3 and Gaussian volume source for Type 4.

- Reuse of the data for different kernels (only M and P are kernel dependent).
  E.g. building blocks for Type 3:

$$\begin{aligned} \hat{\mathcal{C}}_q^a(t_{\rm sep},\delta,v) = & \sum_{i=1}^{N_{\rm ev}} \left\langle M_i(v), N_{q,i}^a(t_{\rm sep},\delta,v)\right\rangle - \frac{1}{N_{\rm hits}} \sum_{i=1}^{N_{\rm ev}} \sum_{j=1}^{N_{\rm hits}} \left\langle w_i^l(z), \xi_j(z)\right\rangle \left\langle P_j(v), N_{q,i}^a(t_{\rm sep},\delta,v)\right\rangle \\ & + \frac{1}{N_{\rm hits}} \sum_{j=1}^{N_{\rm hits}} \left\langle P_j(v), Q_{q,j}^a(t_{\rm sep},\delta,v)\right\rangle \,. \end{aligned}$$

- Unphysical π^0 intermediate state contamination, expected by inspecting the quark flows, needs to be removed.

Unphysical π^0 from direct exp. fit



The disconnected diagram