Luka Leskovec

Lattice outlook on $B \to \rho \ell \bar{\nu}$ and $B \to K^\star \ell \ell$

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CKM 2023
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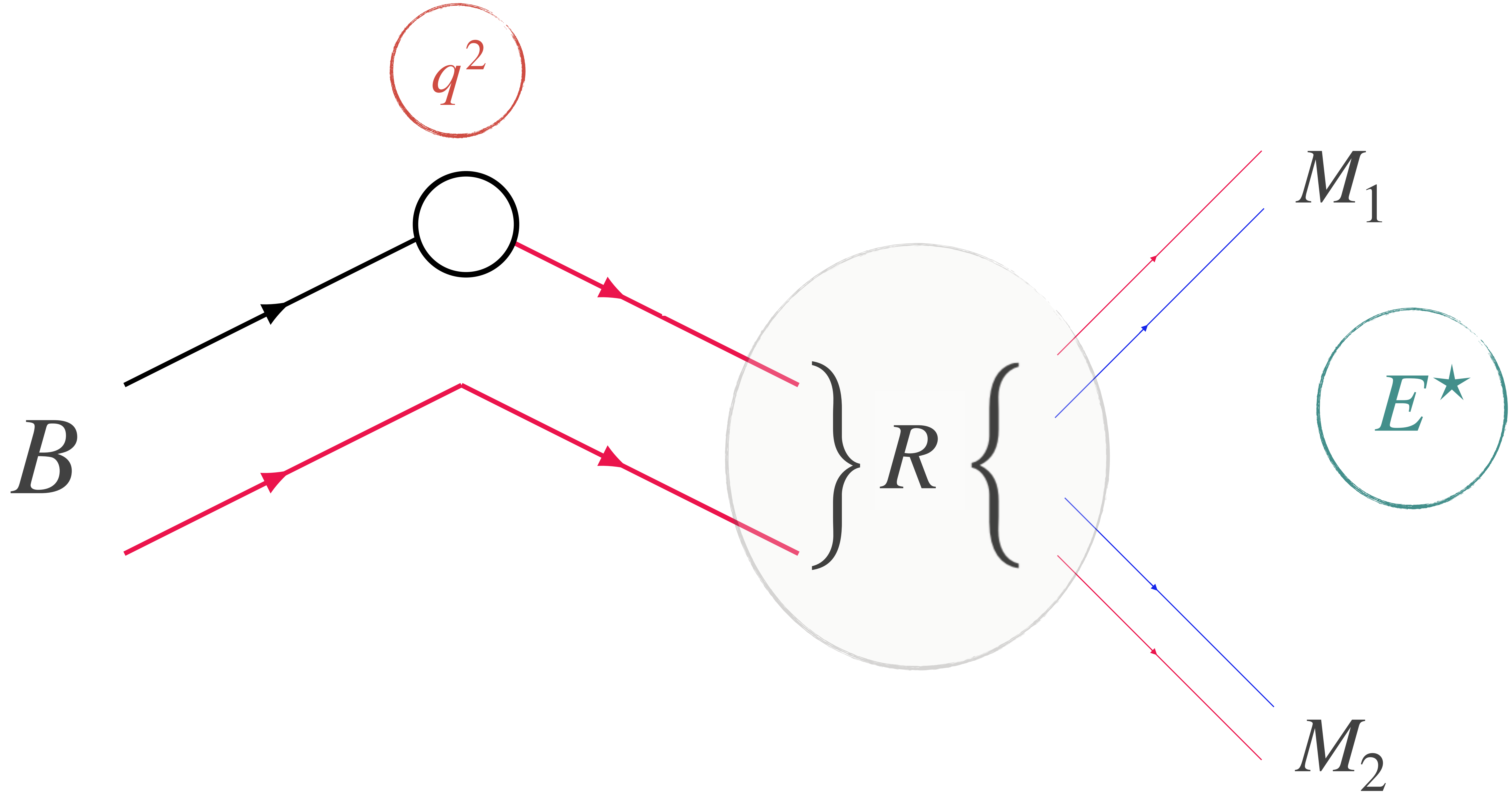
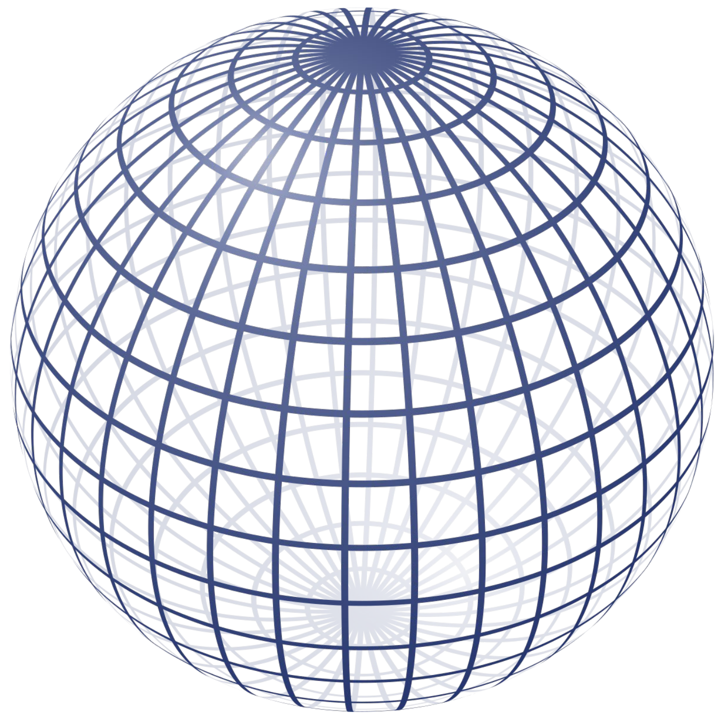
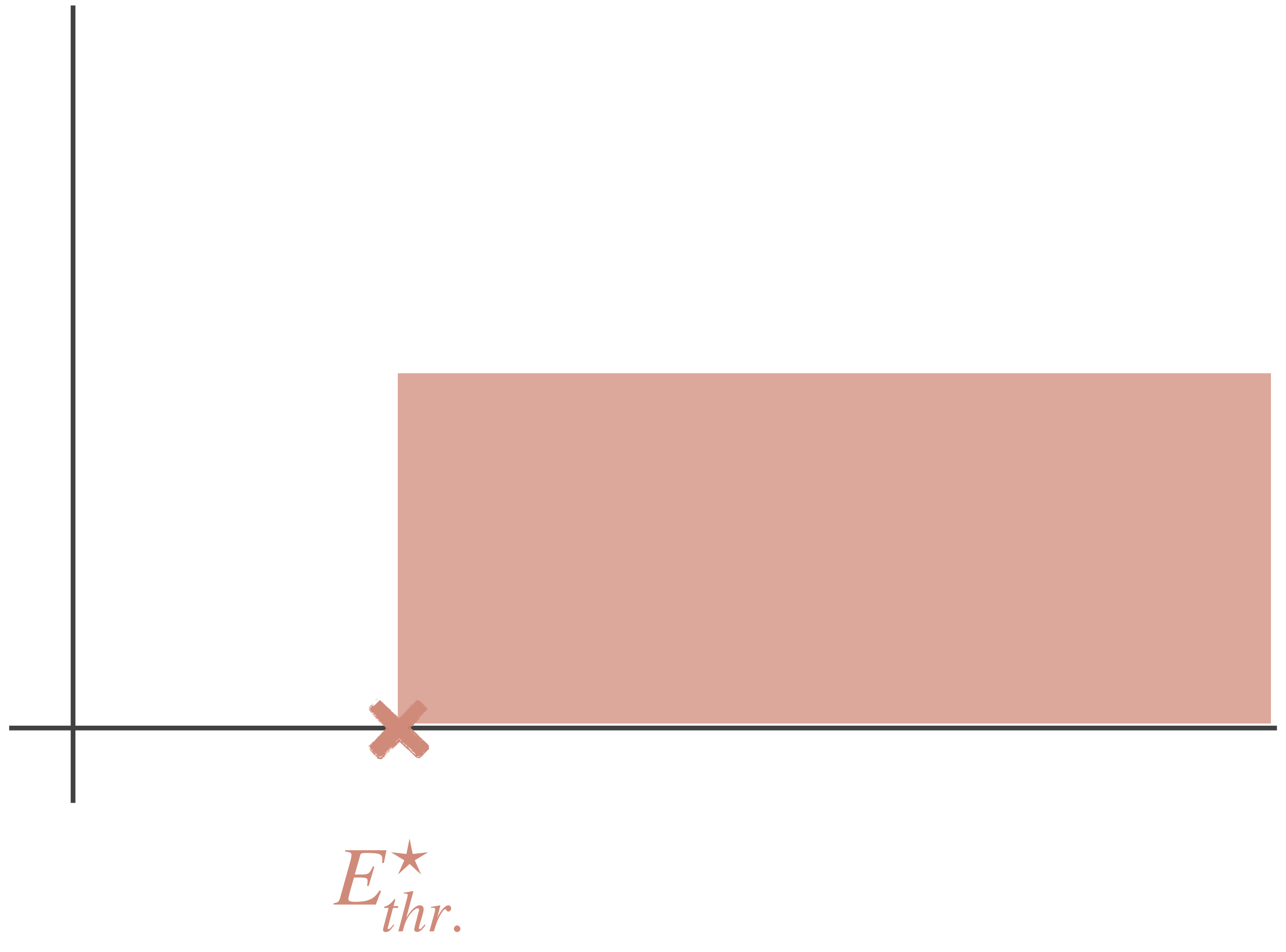

q^2
B
R
M_1
M_2
$E^\star$

resonances: ρ and $K^\star(892)$

infinite volume:



- $O(3)$ symmetry
- scattering amplitude T
- partial wave expansion
- infinite irreps (J^P)



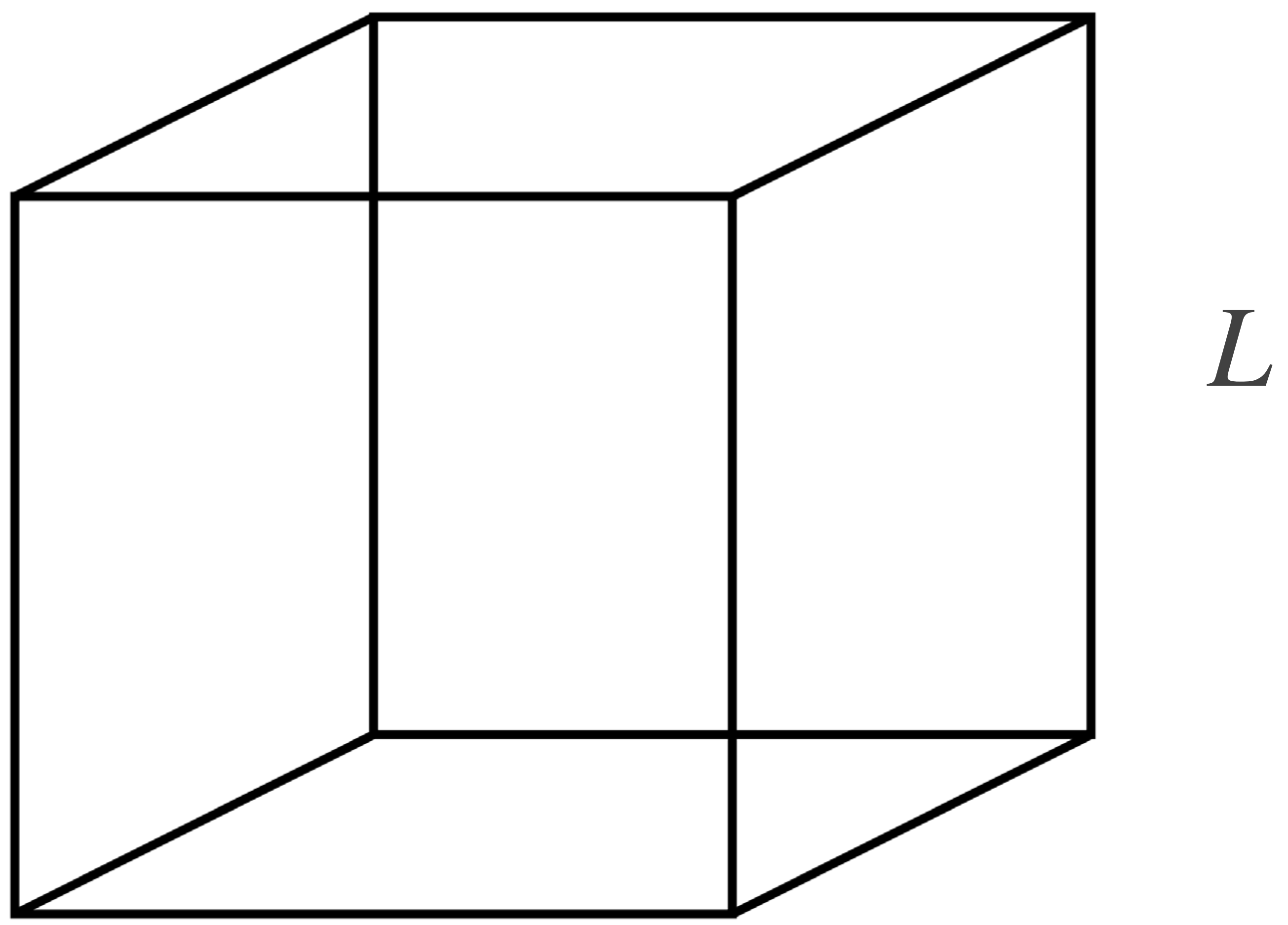
partial wave expansion:

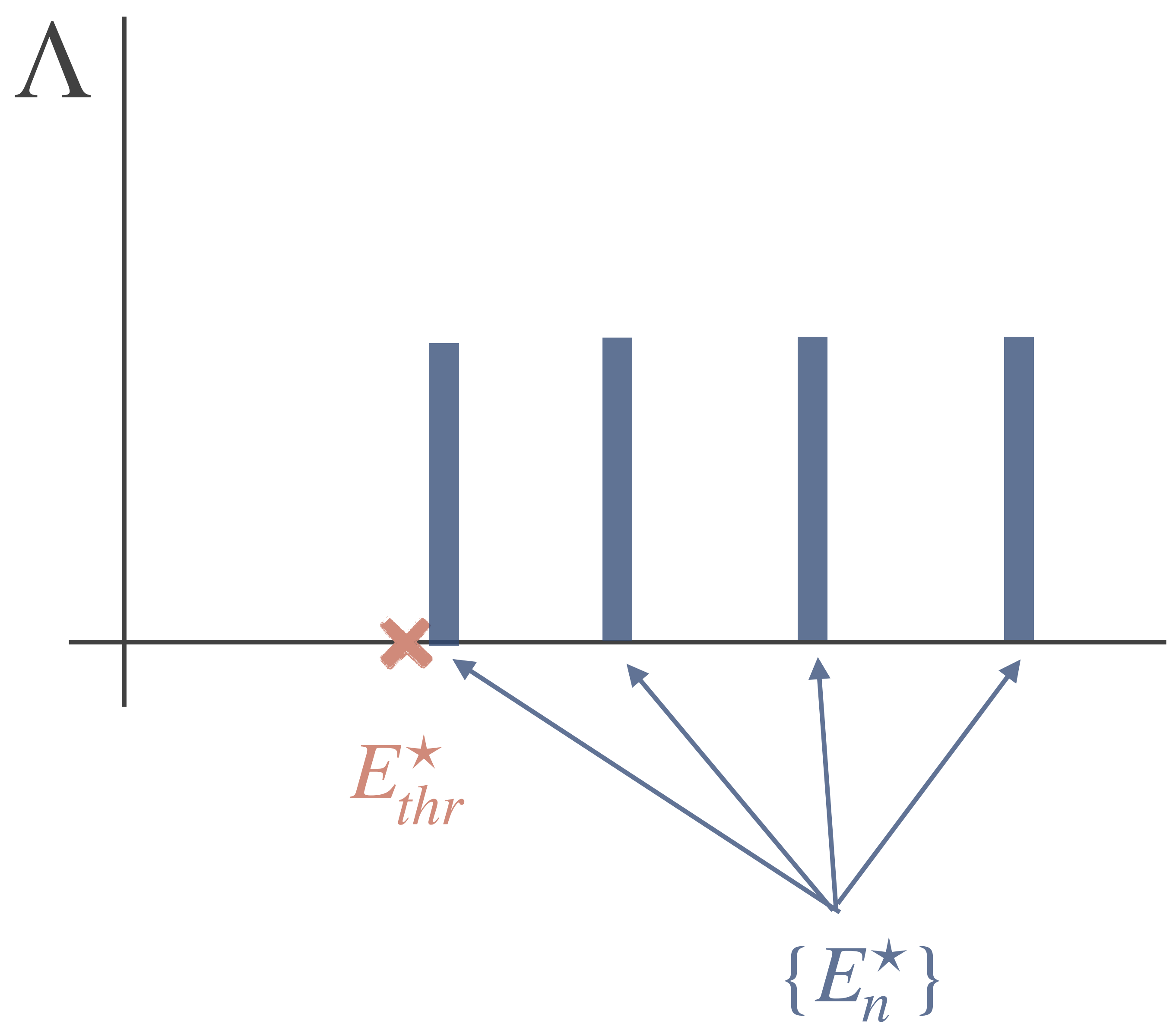
$$T = \sum_{\ell=0}^{\infty} T_\ell \sum_{m_\ell=-\ell}^{\ell} Y_{\ell m_\ell} Y^\star_{\ell m_\ell}$$

- $\pi\pi$ ($I=1$) scattering
  - $\ell = 0 \rightarrow$ nothing (Bose Symmetry)
  - $\ell = 1 \rightarrow \rho$ $\quad (-1)^{\ell - I}$
  - $\ell = 2 \rightarrow$ nothing (Bose Symmetry)
- $K\pi$ ($I = \frac{1}{2}$) scattering
  - $\ell = 0 \rightarrow K_0^\star(700)$, $K_0^\star(1430)$
  - $\ell = 1 \rightarrow K^\star(892)$, $K^\star(1410)$
  - $\ell = 2 \rightarrow K_2^\star(1430)$

!!!

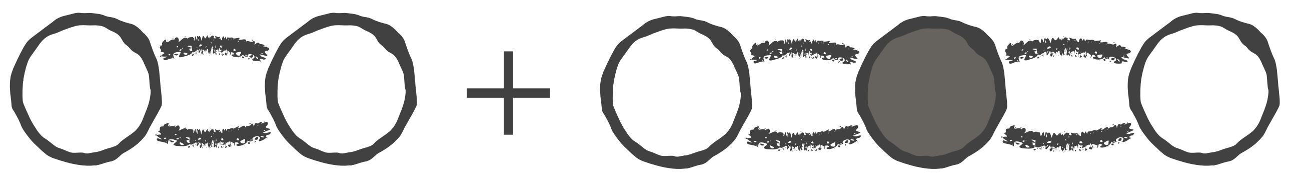
going to the lattice





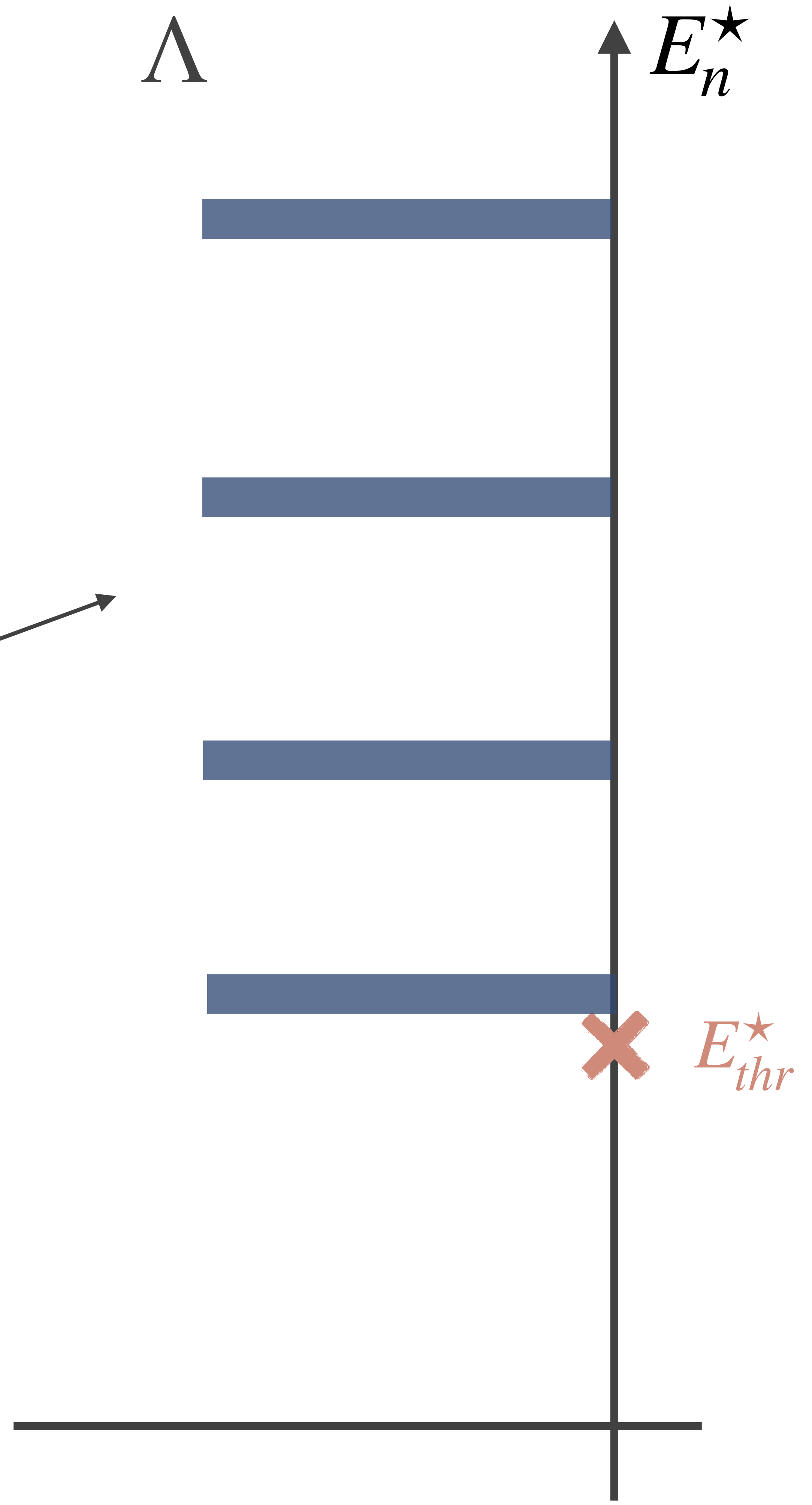
- discrete symmetries, Λ
- periodic BC
- can relate to scattering
- $m_{M1} = m_{M2}$ - no SP-wave mixing
- $m_{M1} \neq m_{M2}$ - SP-wave mixing

lattice states: energies

$C_L^{(2)} =$  $+ \ldots$

$$C_L^{(2)} = C_\infty^{(2)} - A' \frac{1}{F^{-1}(E^\star) + T(E^\star)} A$$

poles!

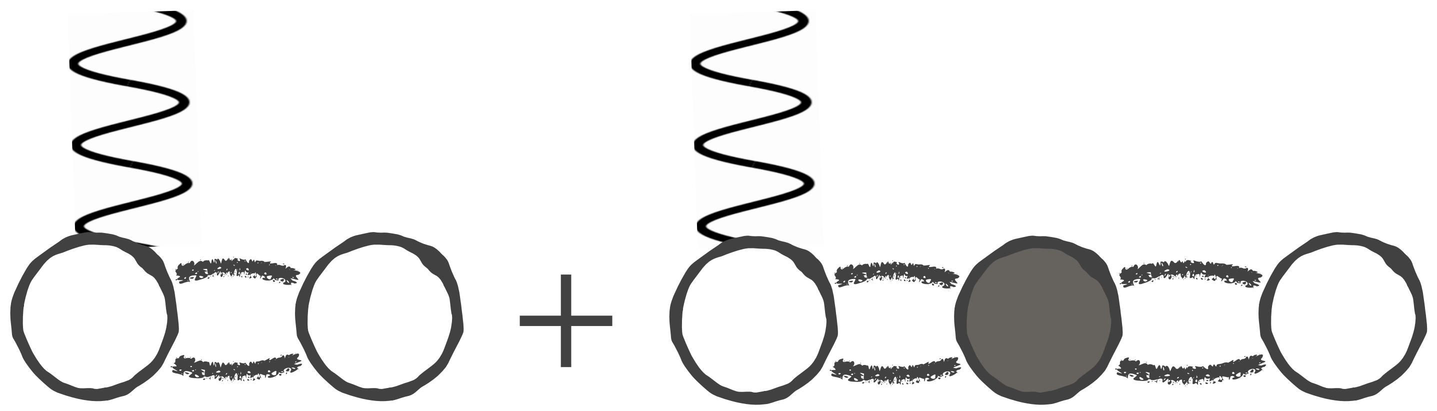


discrete spectrum where:

$$\det \left[F^{-1}(E^\star) + T(E^\star) \right] = 0$$

Luscher NPB354
Rummukainen, Gottlieb hep-lat/9503028
Kim, Sharpe, Sachrajda hep-lat/0507006
Leskovec, Prelovsek hep-lat/1202.2145
Briceno 1401.3312
Briceno, Dudek, Young 1706.06223
Woss, Wilson, Dudek 2001.08474
[and many more]

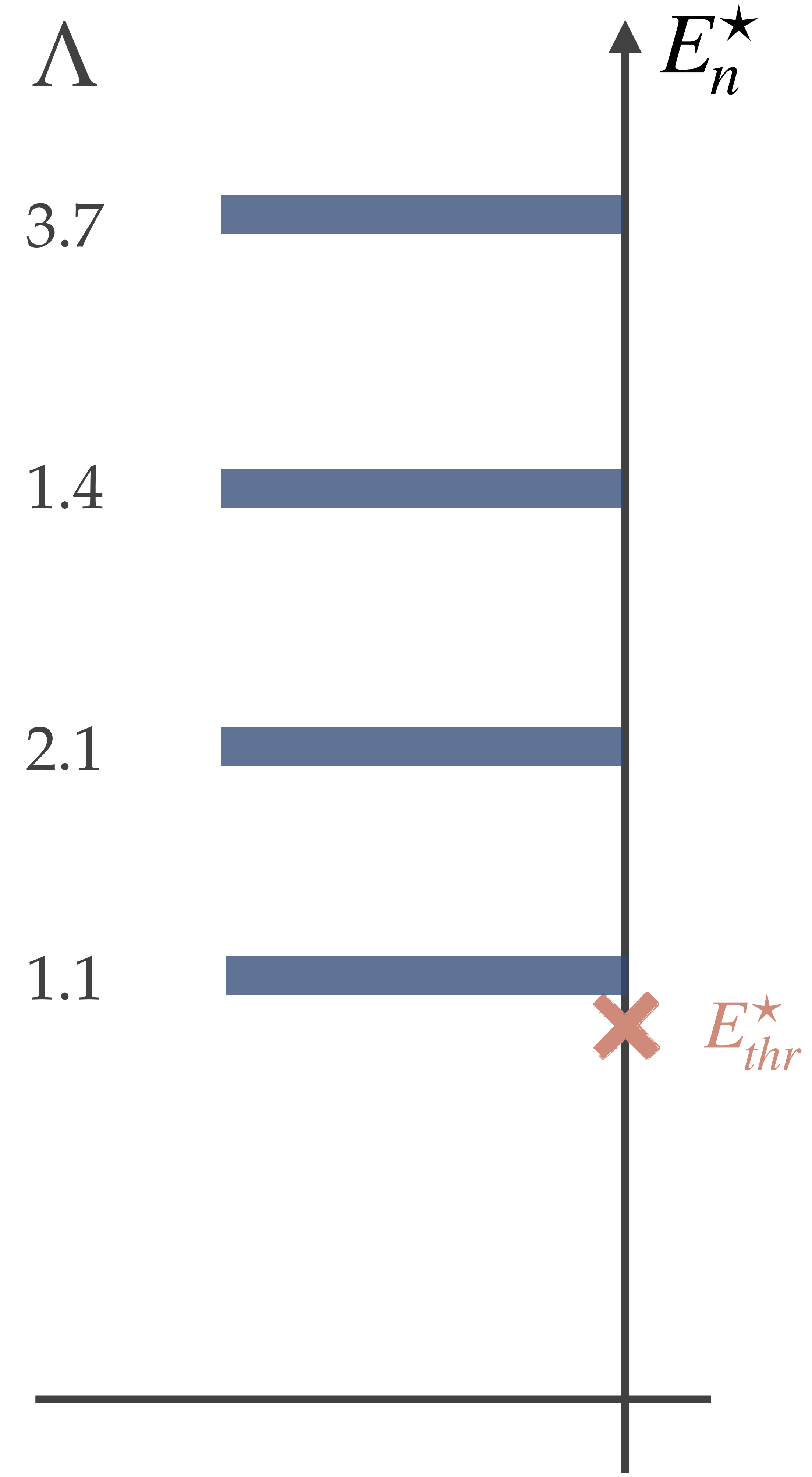
lattice states: normalization

$C_L^{(3)} =$  $+ \ldots$

$$C_L^{(3)} = C_\infty^{(3)} - A'RA$$

$$R = \lim_{E \to E_n} \frac{E - E_n}{F^{-1} + T}$$

residue of pole!



normalization of finite-volume states

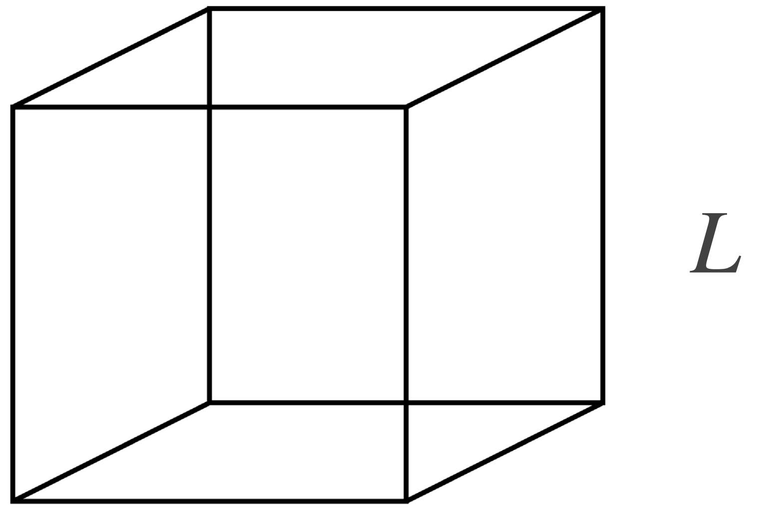
$$| E_n^{\Lambda\star} \rangle_L \sim \sqrt{R}\, | p_1 p_2 (E^\star = E_n^{\Lambda\star}) \rangle_\infty$$

the "Lellouch-Lüscher" factor

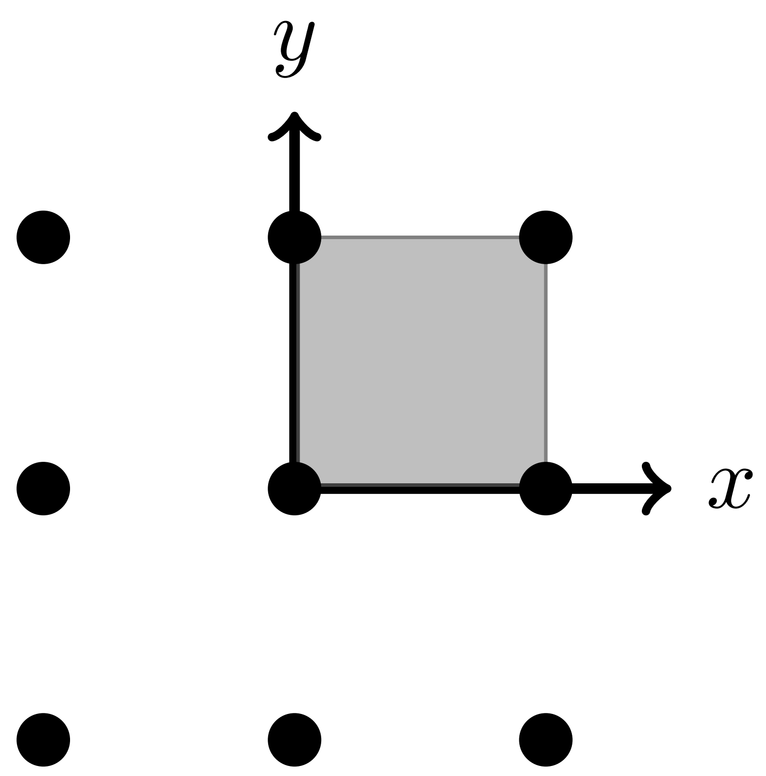
Lellouch, Luscher hep-lat/0003023
Lin, Sachrajda, Testa hep-lat/0104006
…
Briceno, Hansen, Walker-Loud 1406.5965
Briceno, Hansen 1502.04314
Briceno, Dudek, LL 2105.02017

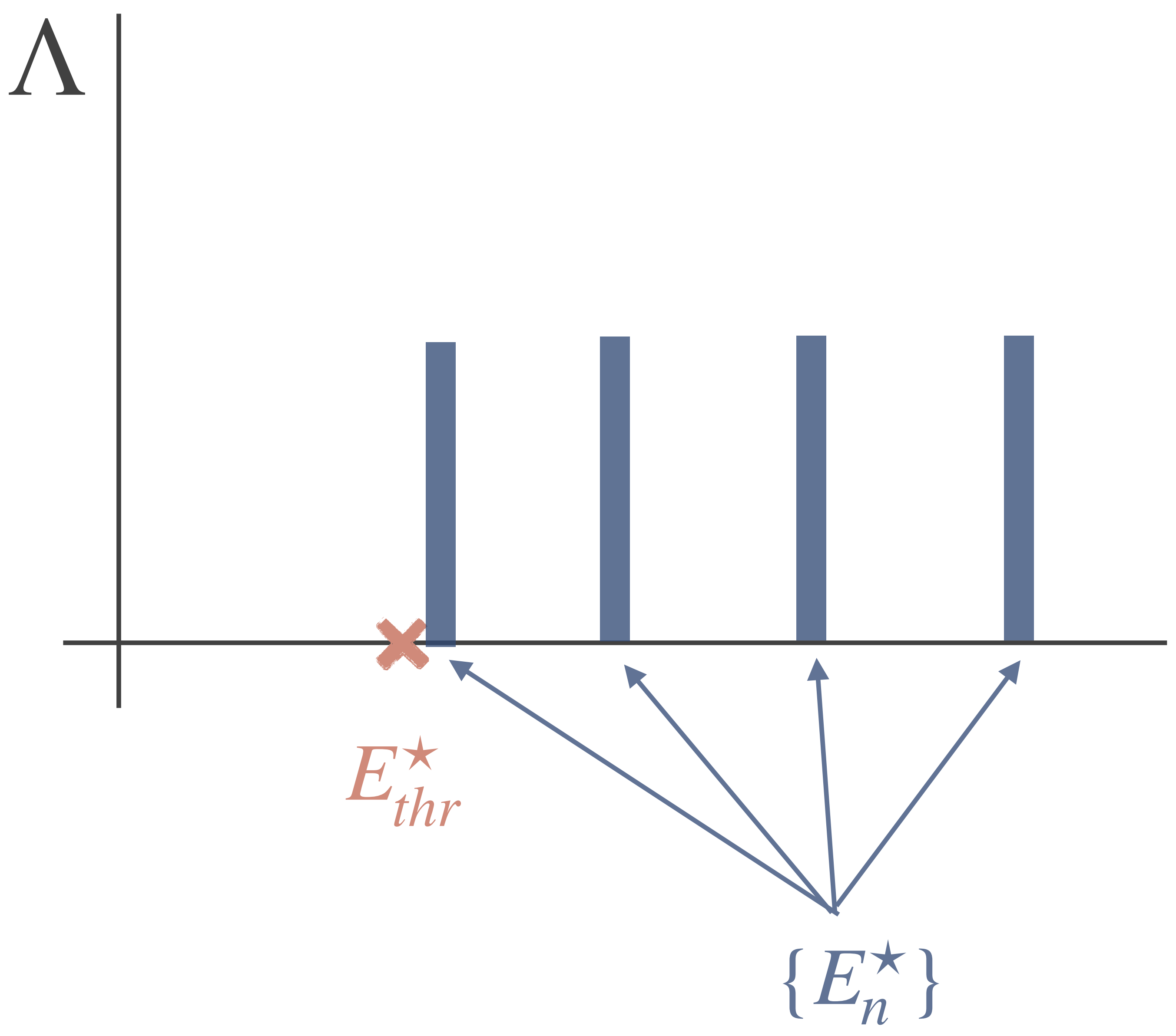
going to the lattice: ρ

finite volume:



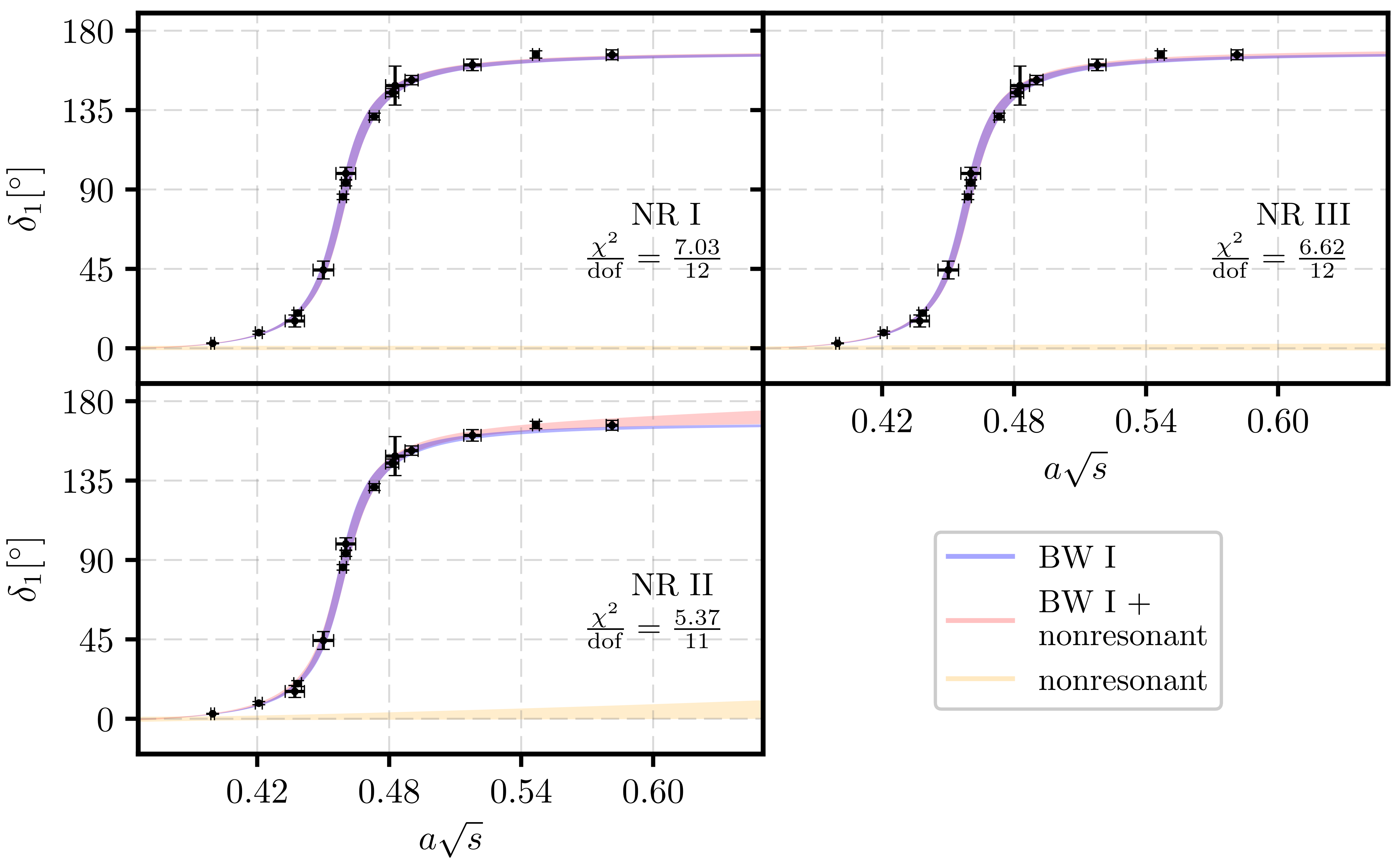
- many-to-one mapping from IV
- no PW mixing

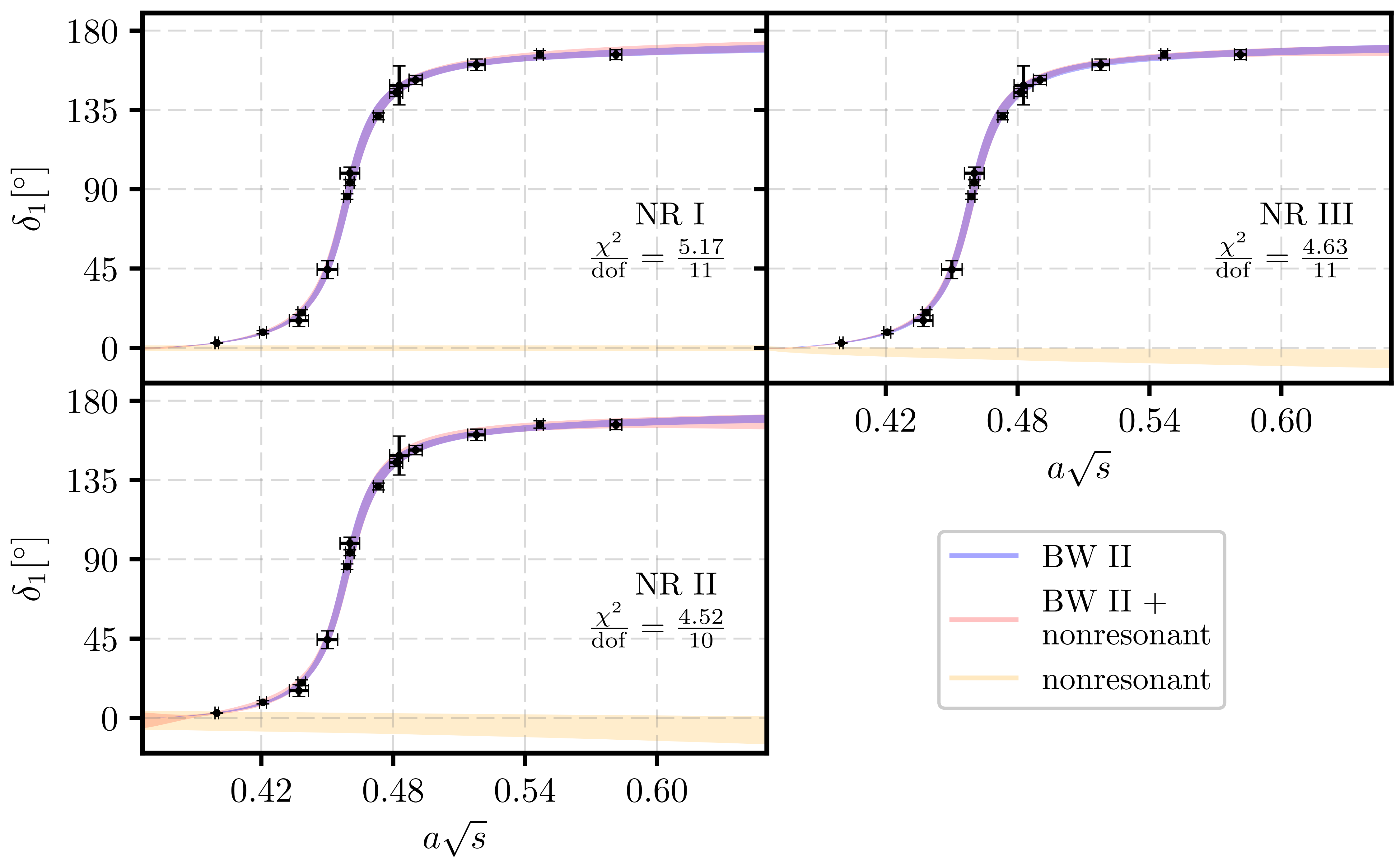




$\vec{p}_{K\pi}$	$(LG^{p_{\vec{K}\pi}})$	(Λ)	J
$(0,0,0)$	O_h	T_1^-	$1^-, 3^-, \ldots$
$(0,0,1)$	D_{4h}	A_2^-	$1^-, 3^-, \ldots$
$(0,0,1)$	D_{4h}	E^-	$1^-, 3^-, \ldots$
$(0,1,1)$	D_{2h}	B_1^-	$1^-, 3^-, \ldots$
$(0,1,1)$	D_{2h}	B_2^-	$1^-, 3^-, \ldots$
$(0,1,1)$	D_{2h}	B_3^-	$1^-, 3^-, \ldots$
$(1,1,1)$	D_{3d}	A_2^-	$1^-, 3^-, \ldots$
$(1,1,1)$	D_{3d}	E^-	$1^-, 3^-, \ldots$

$$T = \frac{E^\star \Gamma_i}{m_R^2 - E^{\star 2} - i E^\star \Gamma_i}$$





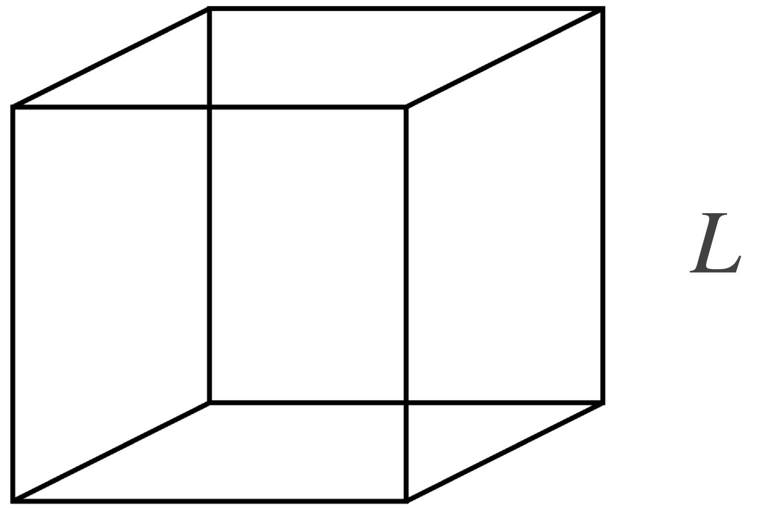
$$\Gamma_I = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{p^3}{E^{\star 2}}$$

$$\Gamma_{II} = \frac{g_{\rho\pi\pi}^2}{6\pi} \frac{p^3}{E^{\star 2}} \frac{1 + (k_R r_0)^2}{1 + (k r_0)^2}$$

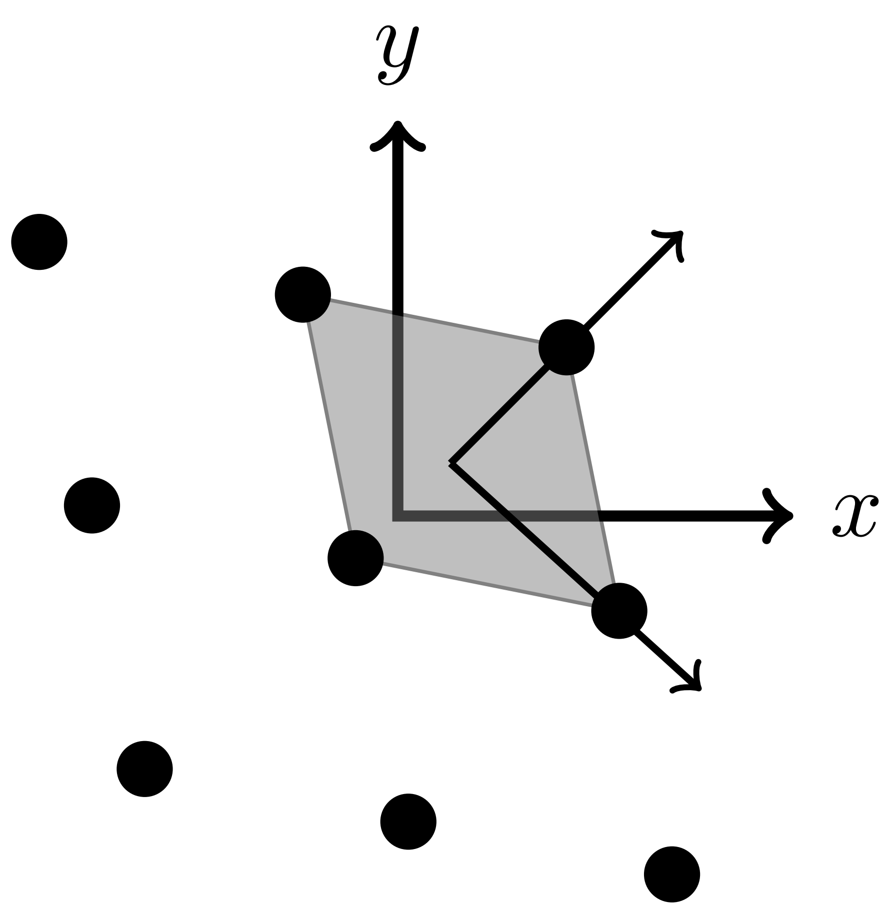
Alexandrou, LL, Meinel et al. 1704.05439

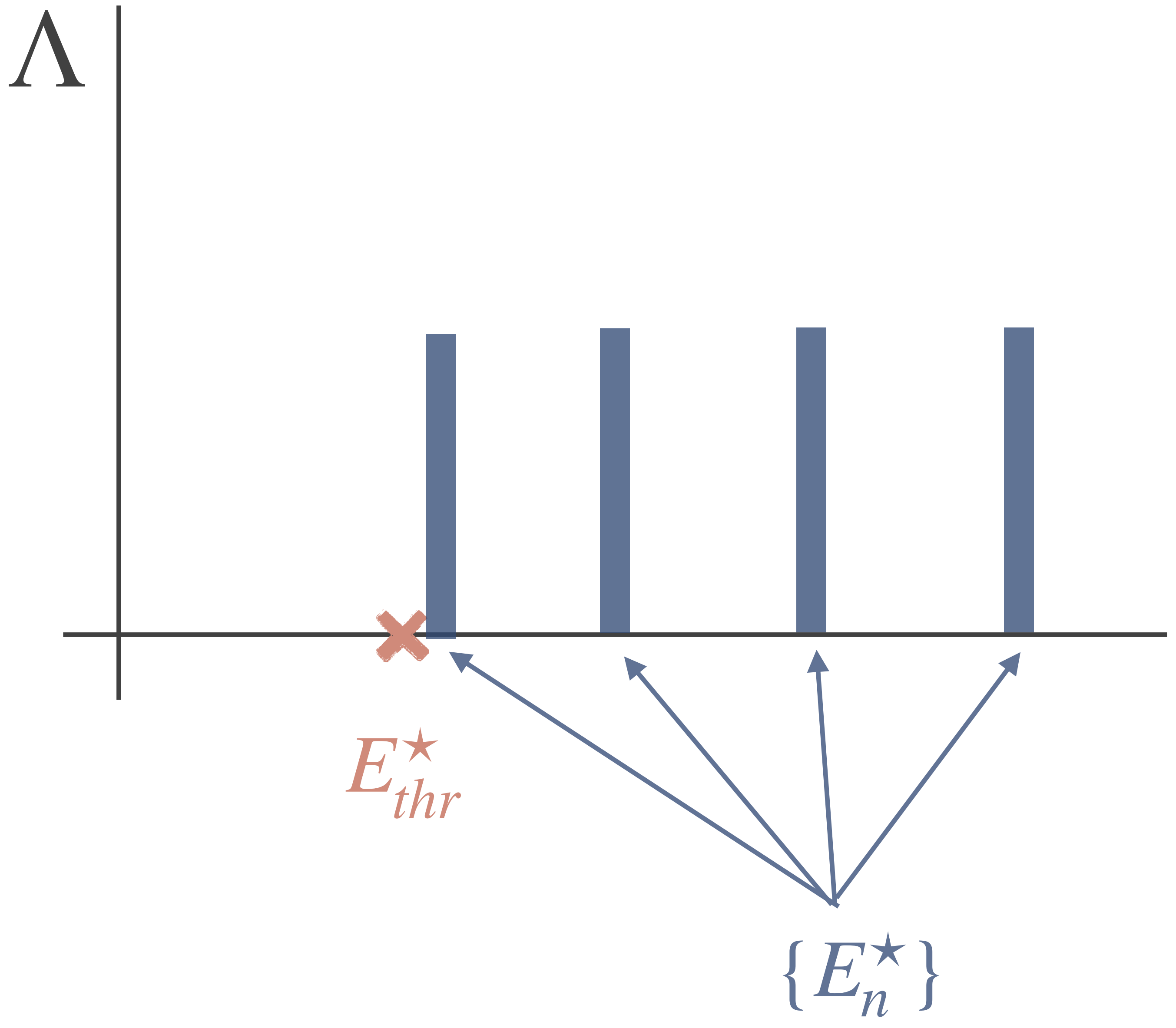
going to the lattice: $K^\star(892)$

finite volume:



- many-to-one mapping from IV
- PW mixing!



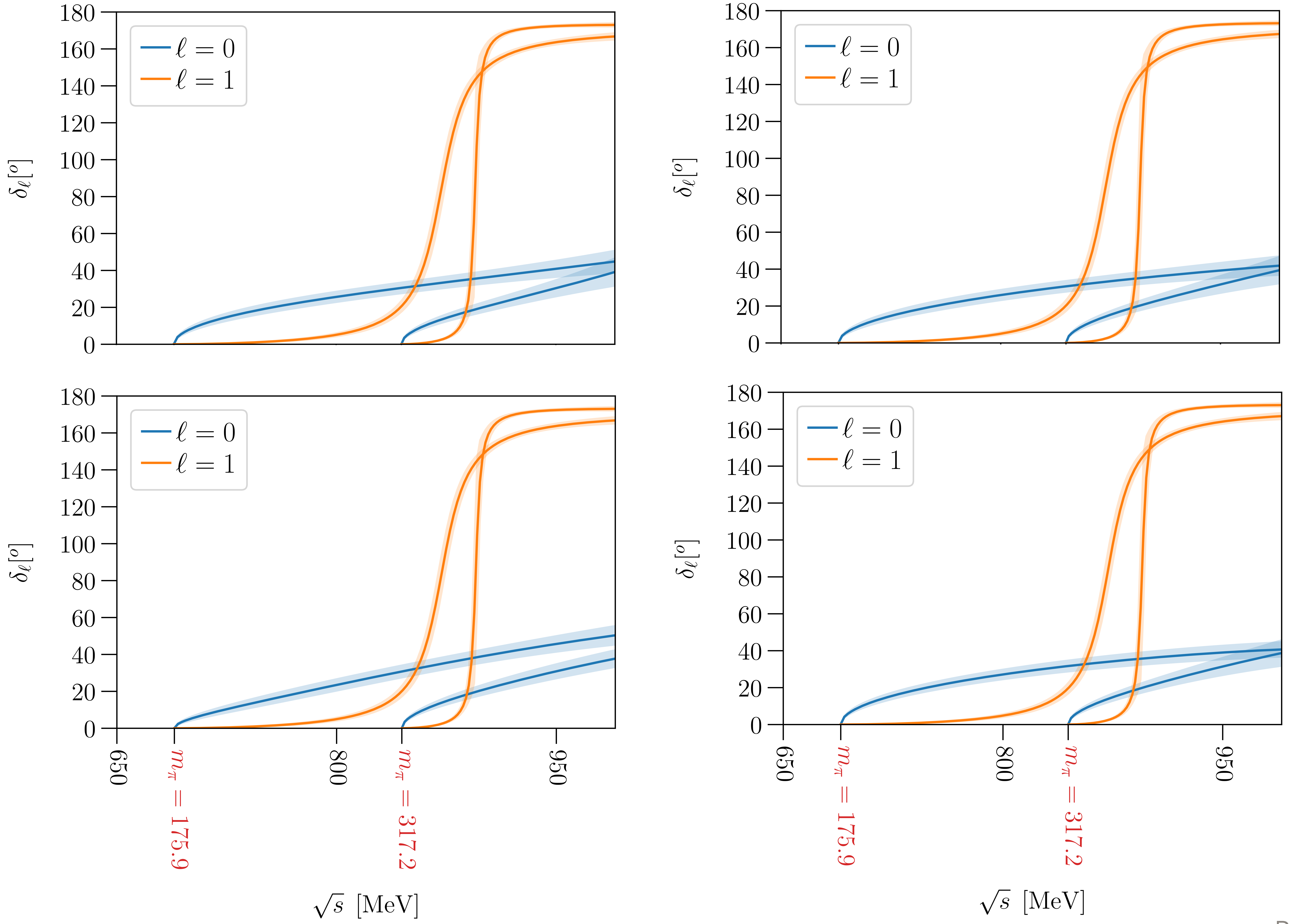


- S-wave only
- S- and P-wave
- P-wave only

$\vec{p}_{K\pi}$	$(LG^{p_{\vec{K}\pi}})$	(Λ)	J
$(0,0,0)$	O_h	A_1^+	$J=0,1,\dots$

$\vec{p}_{K\pi}$	$(LG^{p_{\vec{K}\pi}})$	(Λ)	J
$(0,0,1)$	C_{4v}	A_2	$J=0,1,\dots$
$(0,1,1)$	C_{2v}	B_3	$J=0,1,\dots$
$(1,1,1)$	C_{3v}	A_2	$J=0,1,\dots$

$\vec{p}_{K\pi}$	$(LG^{p_{\vec{K}\pi}})$	(Λ)	J
$(0,0,0)$	(O_h)	T_1^+	$J=1,3,\dots$
$(0,0,1)$	C_{4v}	E	$J=1,2,\dots$
$(0,1,1)$	C_{2v}	B_1	$J=1,2,\dots$
$(0,1,1)$	C_{2v}	B_2	$J=1,2,\dots$
$(1,1,1)$	C_{3v}	E	$J=1,2,\dots$

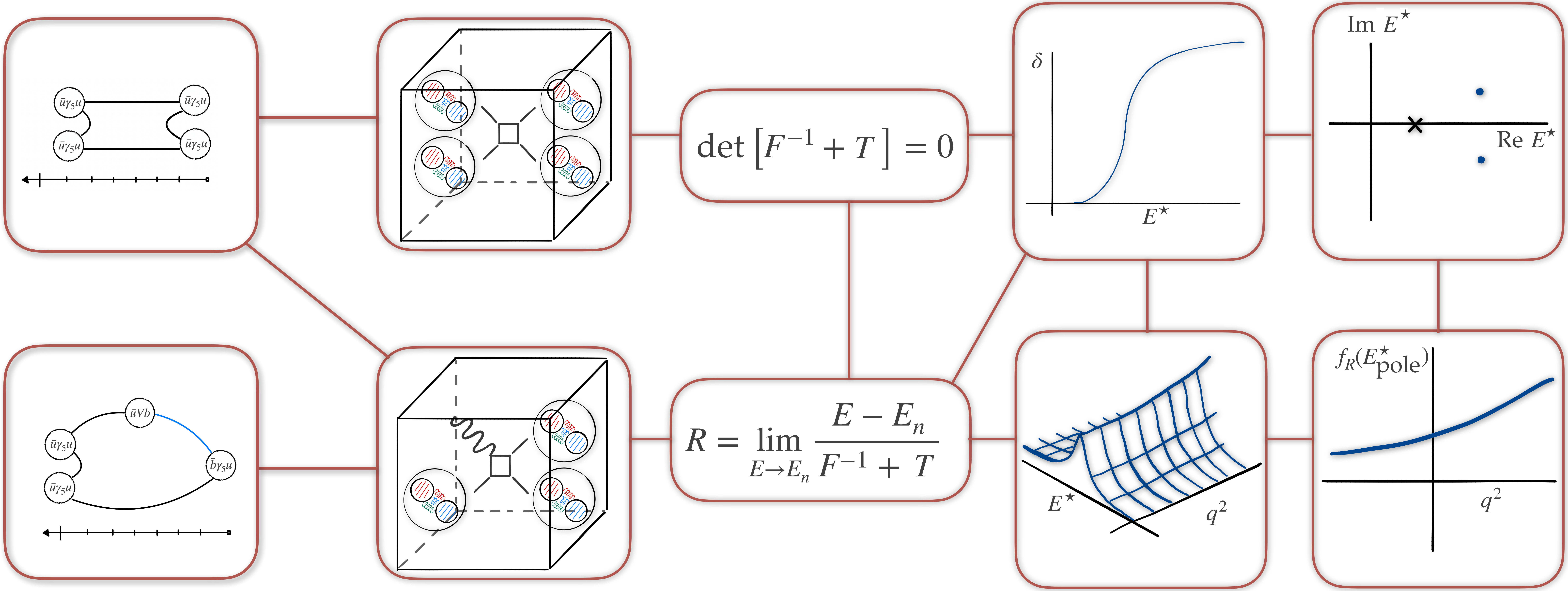


parameterizations:

- P-wave with B-W factors
- S-wave, 4 variations (K-matrix, LASS-like, K-matrix with Adler zero conformal map with Adler zero)

Rendon, LL, Meinel et al. 2006.14035

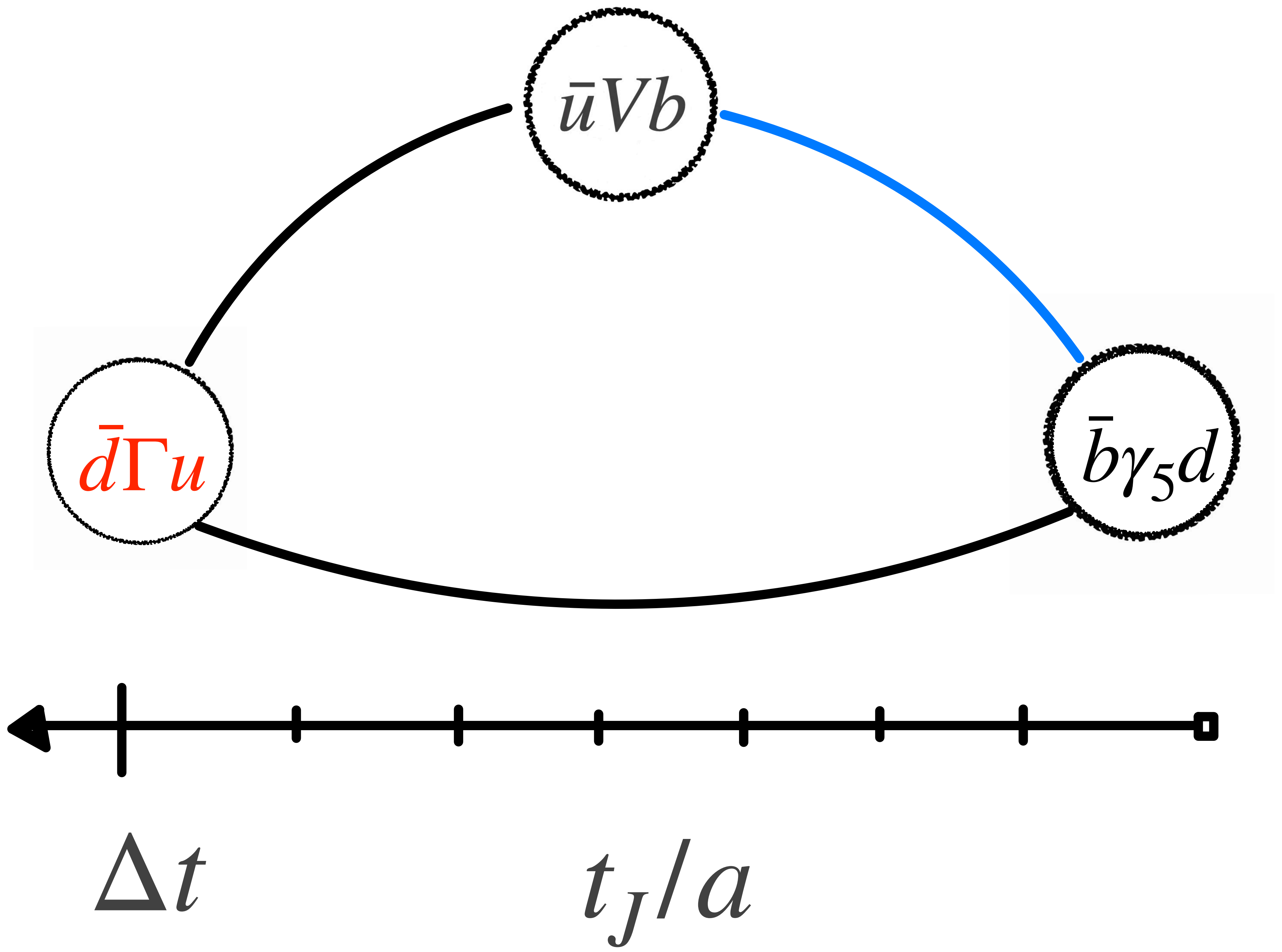
a "how-to" with $B \to \pi\pi(\rho)\ell\bar{\nu}$

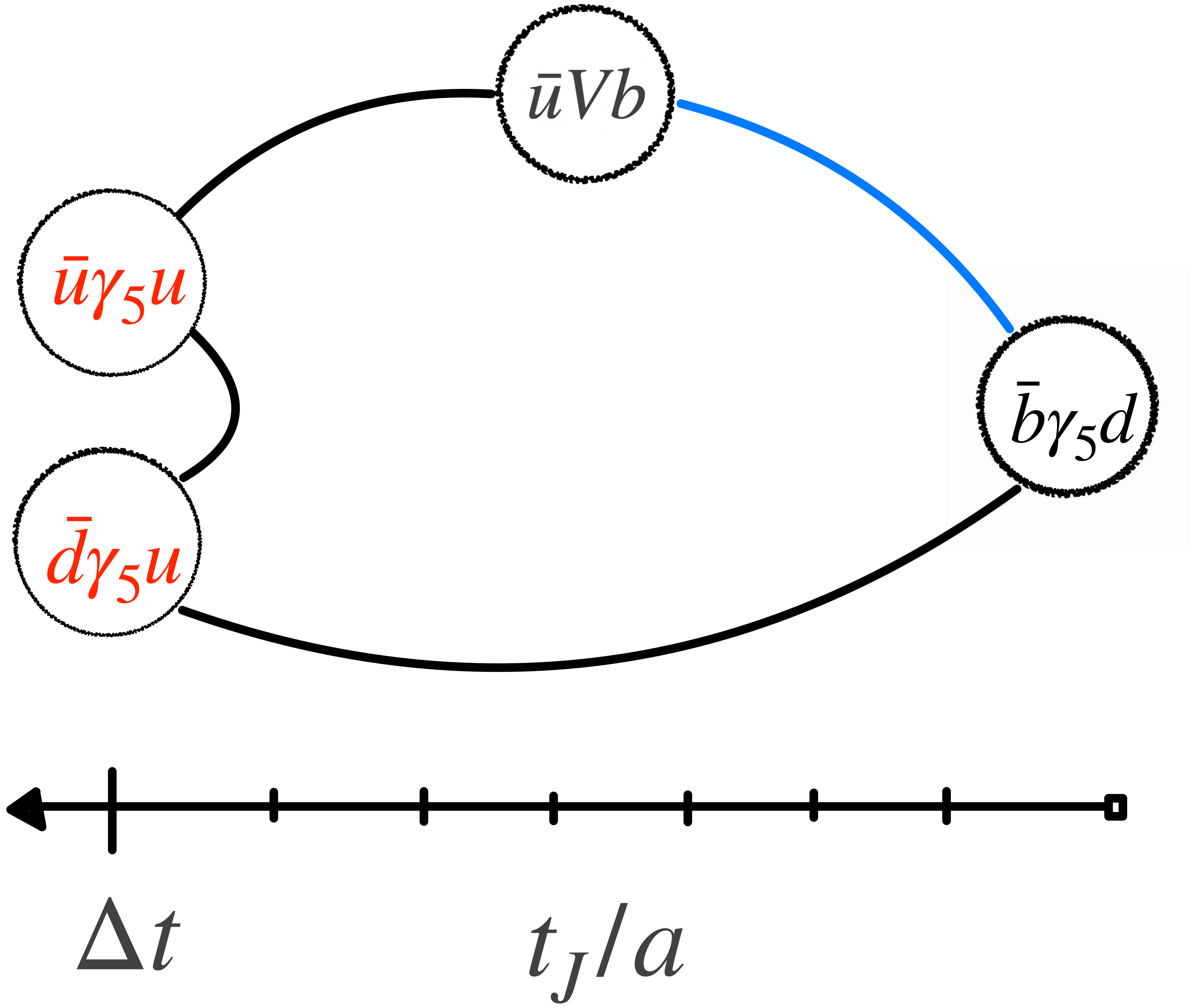


Briceño-Hansen-Walker-Loud approach

3-point functions

$$C_{3,i} = \langle\ O_i(\vec{p}, \Lambda)\ \ V^\mu\ \ O_B(\vec{p}_B)\ \rangle$$





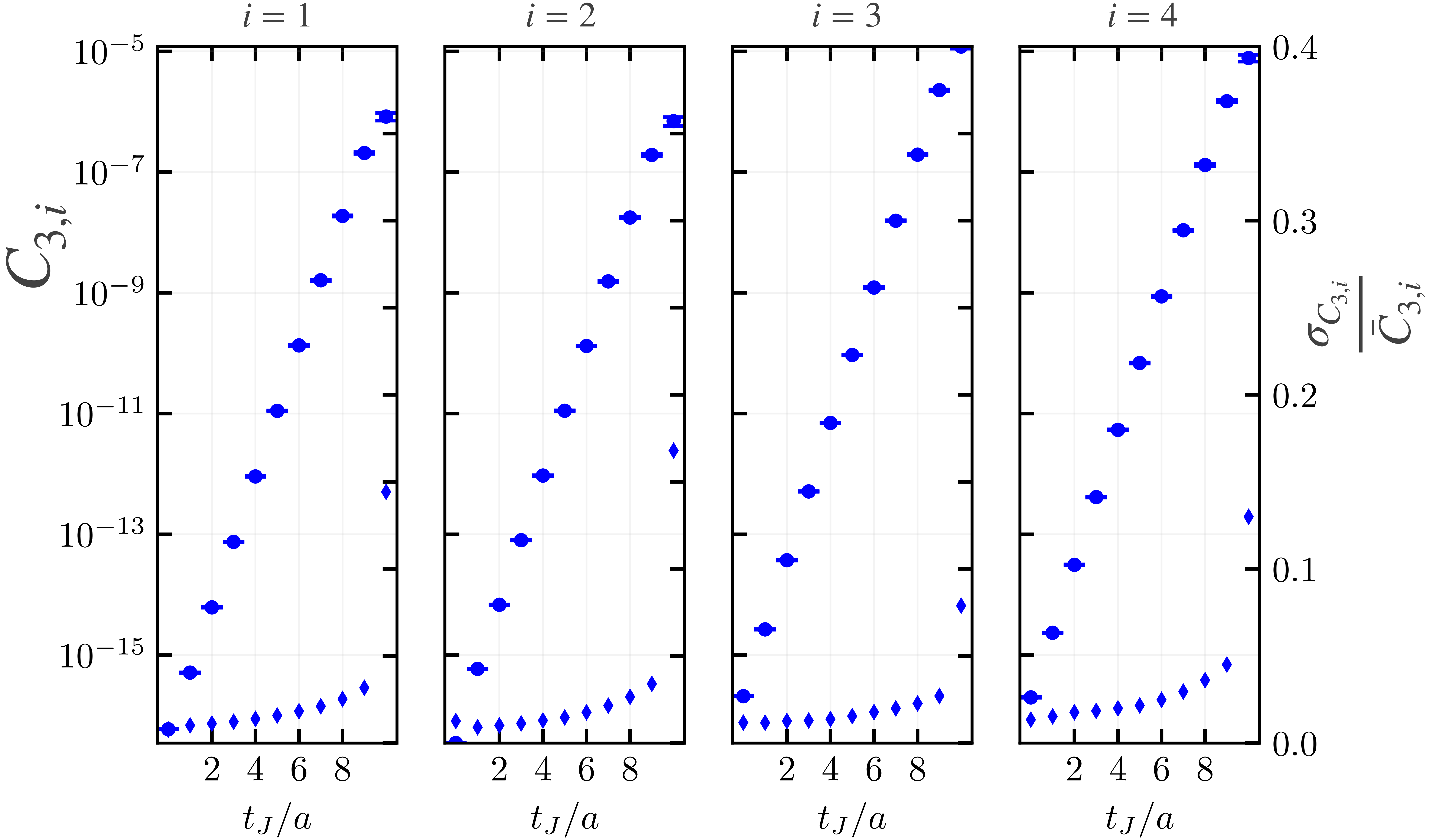
3-point function lattice data

$\vec{p} = \frac{2\pi}{L}[1,0,1]$
$\Lambda = B_2$

$\mu = z$

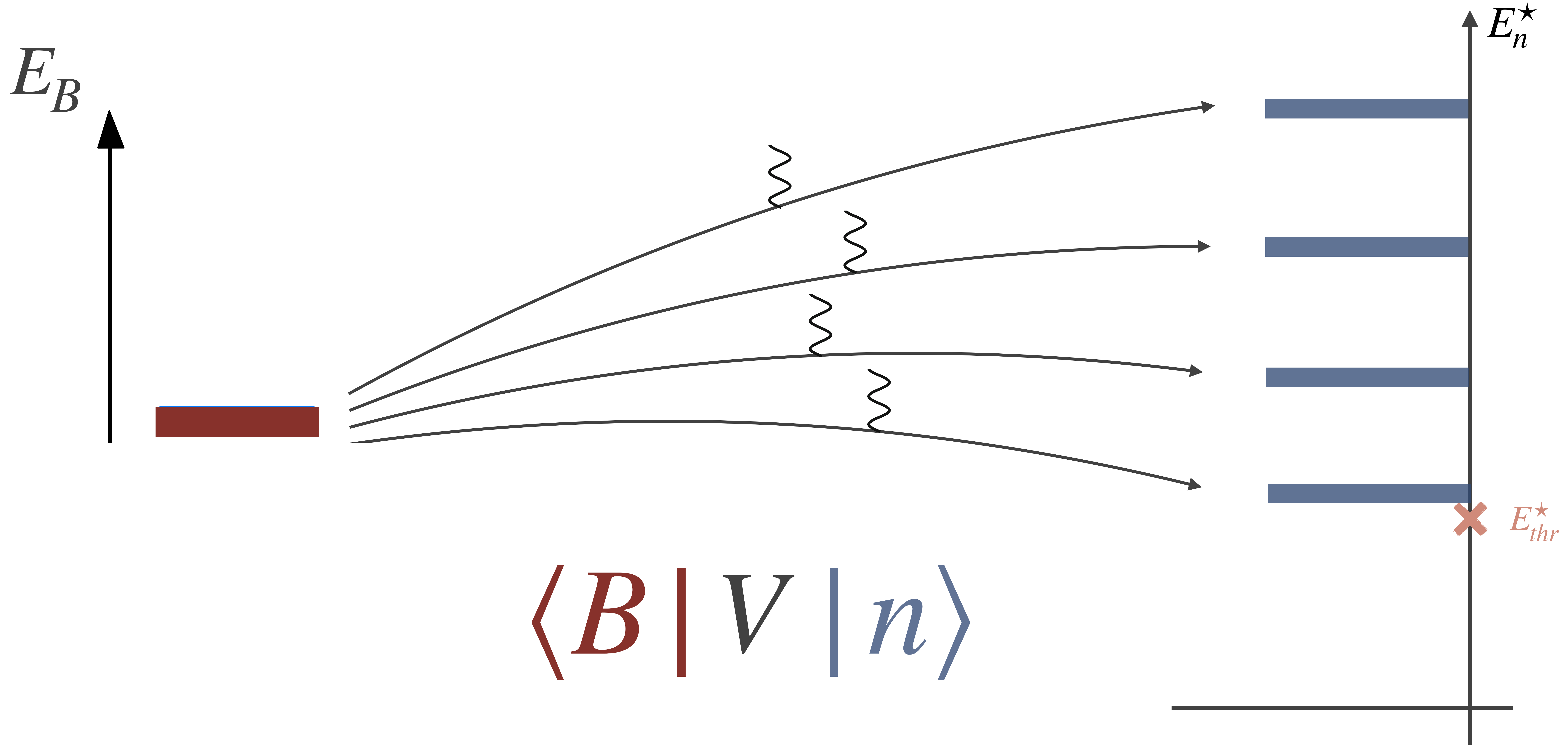
$\vec{p}_B = \frac{2\pi}{L}[0,0,1]$

$$C_{3,i} = < O_i(\vec{p}, \Lambda)\ V^\mu\ O_B(\vec{p}_B) \rangle$$

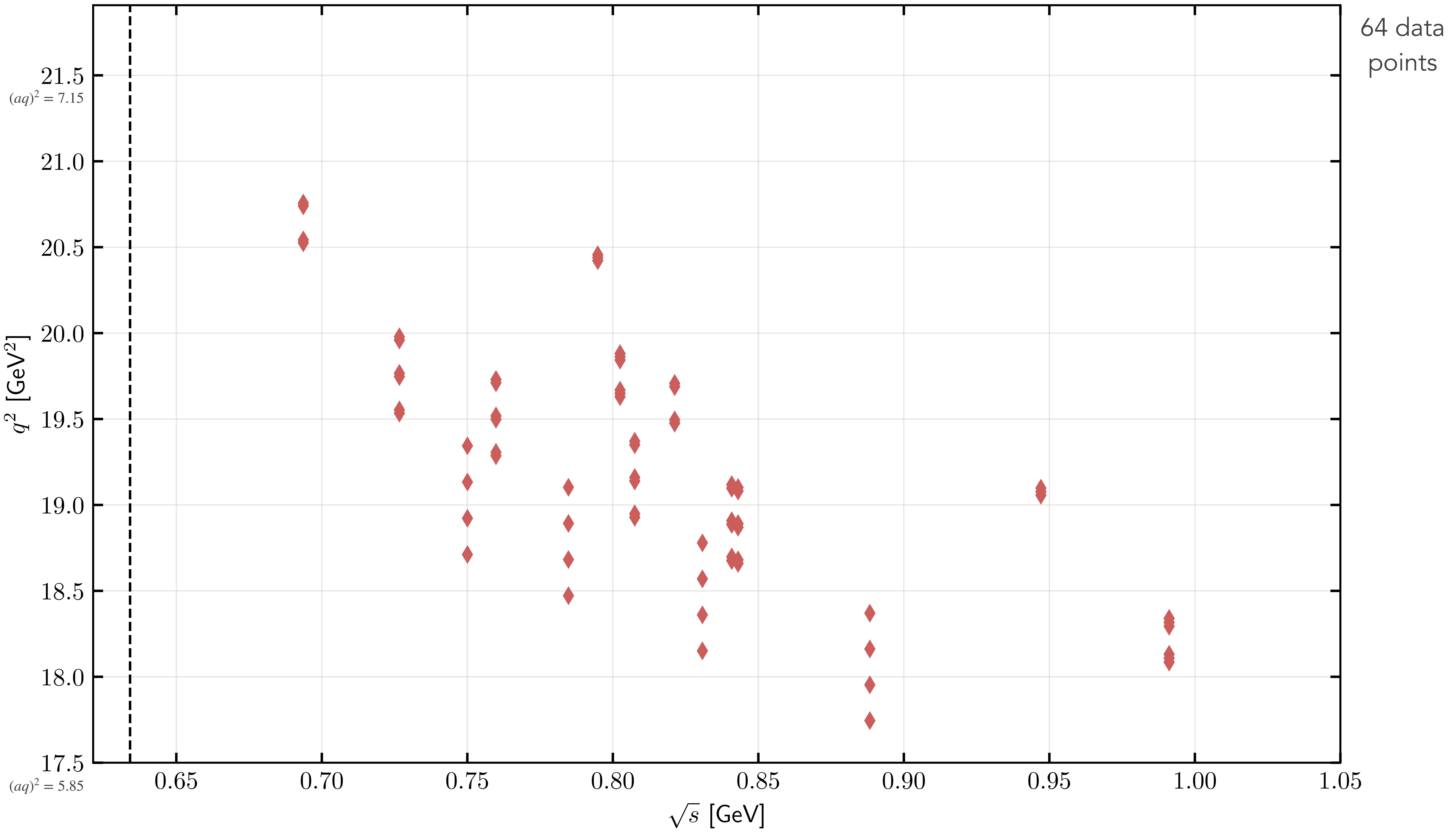


$$O_1 = \bar{u}\Gamma_{B_1}u$$
$$O_2 = \bar{u}\gamma_t\Gamma_{B_1}u$$
$$O_3 = \pi(\vec{p}_1)\pi(\vec{p}_2)$$
$$O_4 = \pi(\vec{p}_1)\pi(\vec{p}_2)$$

matrix elements



$$\langle B \,|\, V \,|\, n \rangle$$



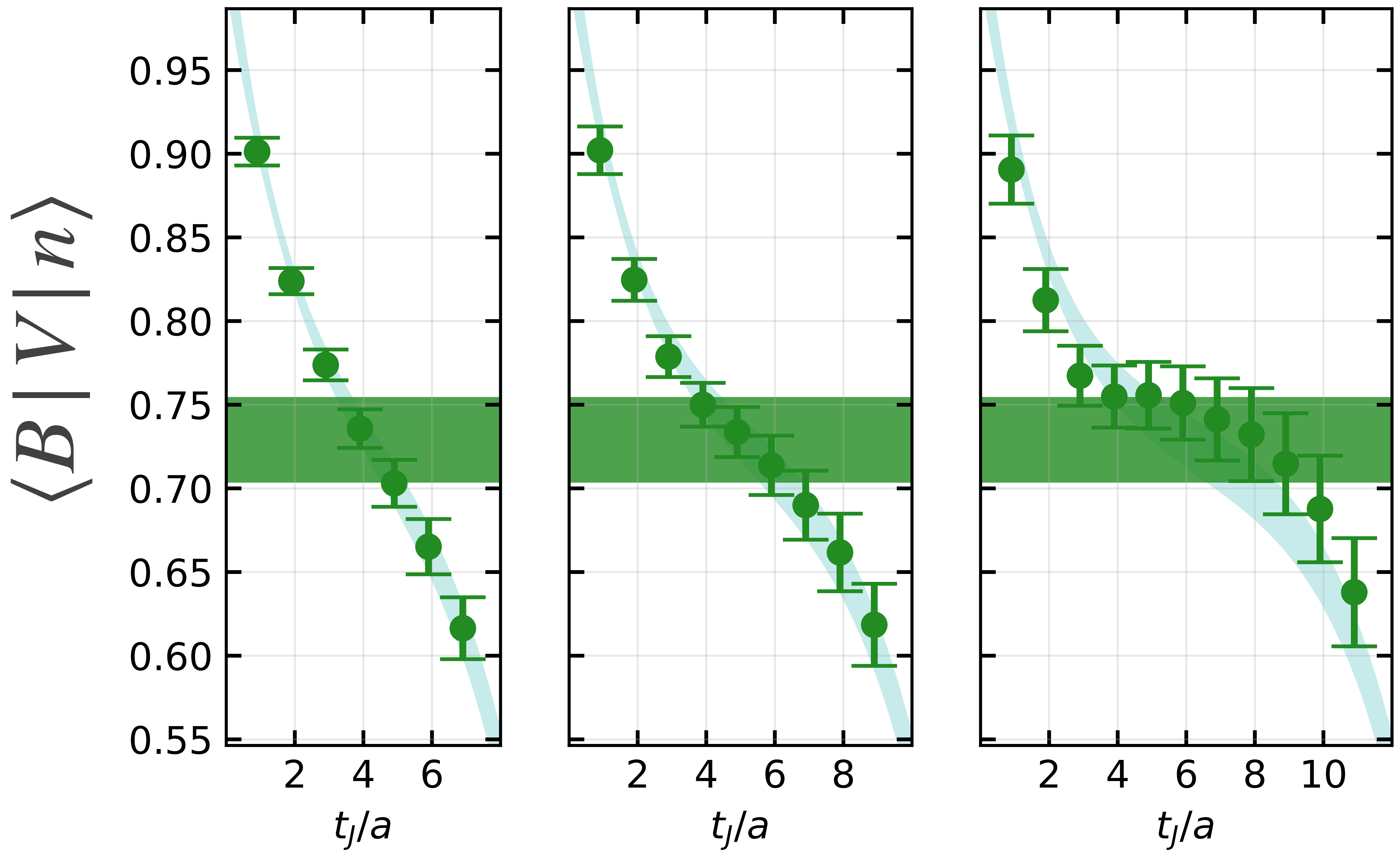
64 data points
21.5
$(aq)^2 = 7.15$
21.0
20.5
20.0
19.5
19.0
18.5
18.0
17.5
$(aq)^2 = 5.85$
q^2 [GeV^2]
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
1.05
$\sqrt{s}$ [GeV]

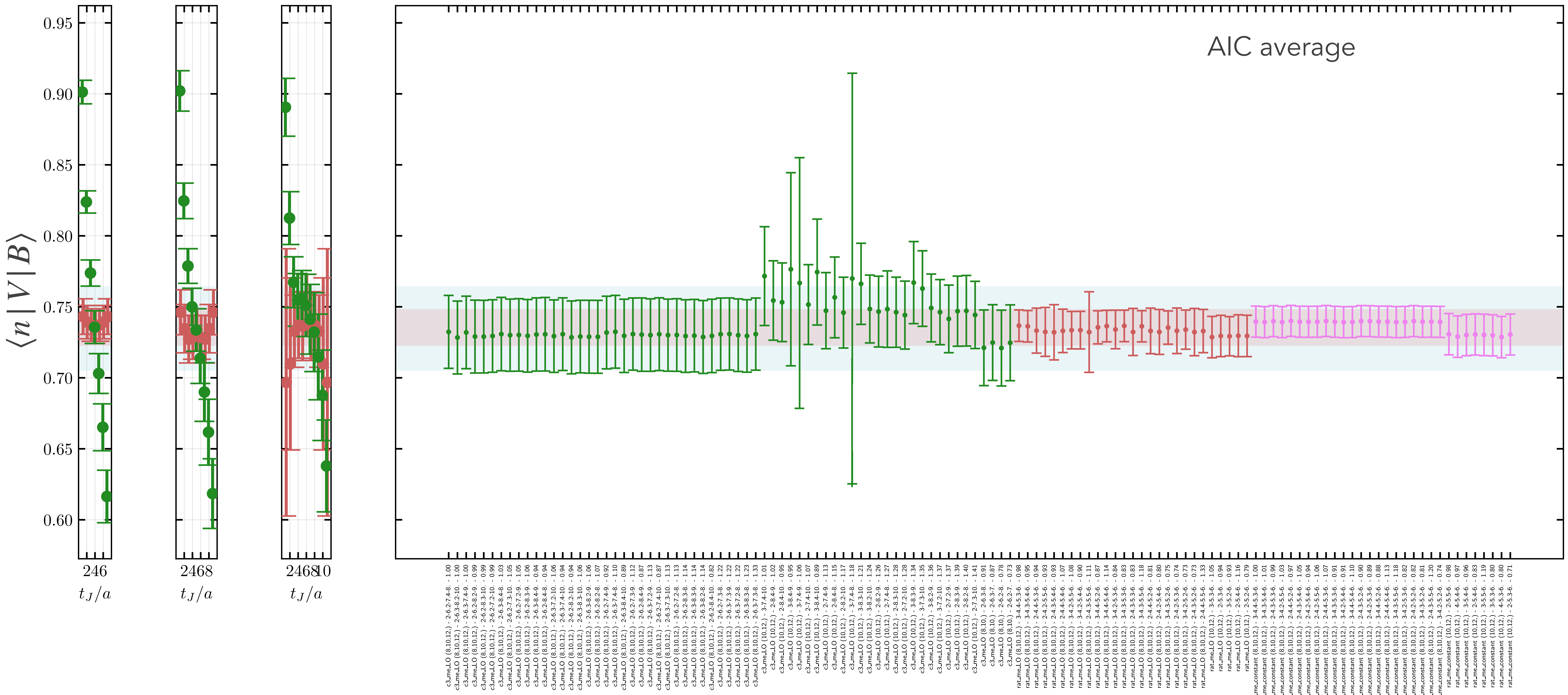
state projection

$$C_{3,i} = \sum_{n \in [\pi\pi]} Z_i^n \langle B | V | n \rangle Z_B \frac{e^{-E_n(\Delta t - t)} e^{-E_B t}}{2E_n 2E_B}$$

$$\boxed{C_3^n = u_i^n C_{3,i}}$$

state-projected 3-pt function





a multitude of models…

Jay & Neil, 2008.01069

transition amplitude

Boyd, Grinstein, Lebed hep-ph/9412324
Bourrely, Caprini, Lellouch 0807.2722
Alexandrou, LL, Meinel et al. 1807.08357

$q = p_f - p_i$

$E^\star = 2\sqrt{m_\pi^2 + k^2}$

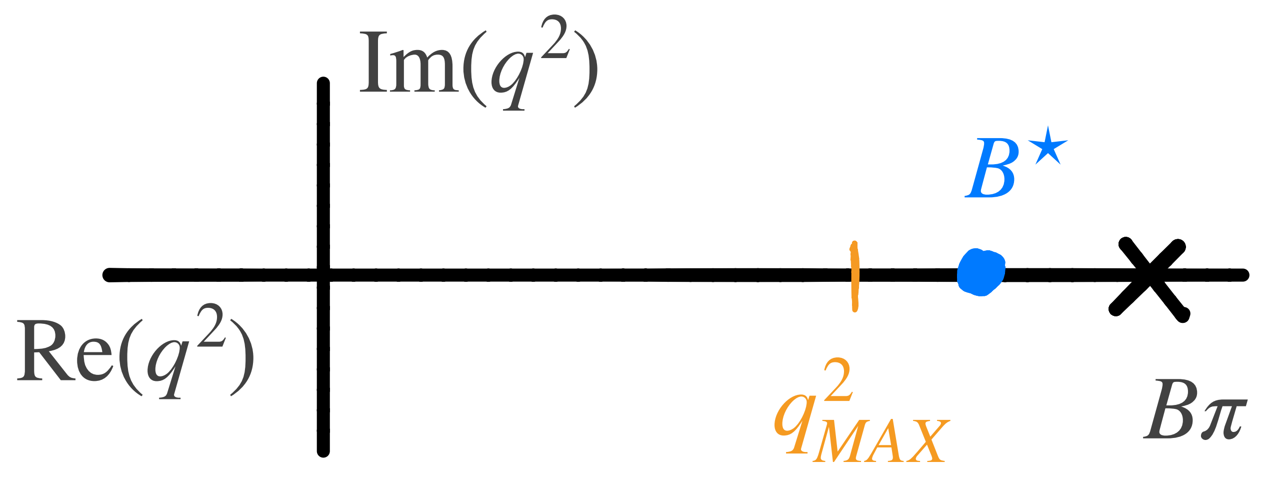
$$\langle \pi\pi, E^\star \,|\, V \,|\, B, p_B \rangle_\infty = \frac{2iV(E^\star, q^2)}{m_B + 2m_\pi} \epsilon_{\mu\nu\alpha\beta}\, \varepsilon^{\nu *}\, p^\alpha p_B^\beta$$

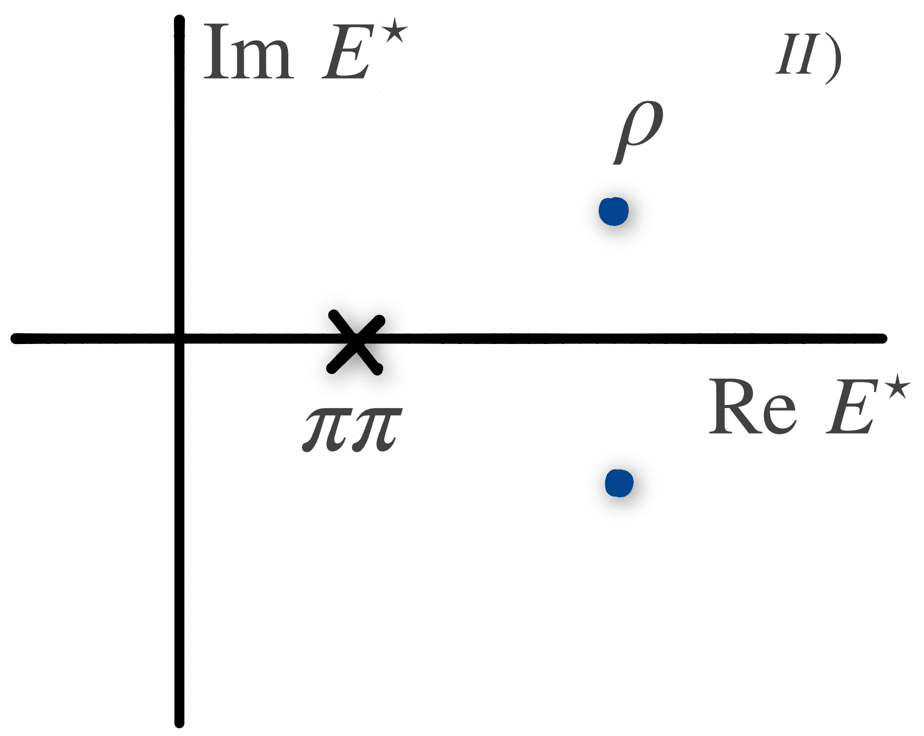
$$V(E^\star, q^2) = F(E^\star, q^2)\frac{T(E^\star)}{k}$$

$$T = \frac{E^\star \Gamma_i}{m_R^2 - E^{\star 2} - iE^\star \Gamma_i}$$

$$\Gamma_I = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{p^3}{E^{\star 2}}$$

$$\Gamma_{II} = \frac{g_{\rho\pi\pi}^2}{6\pi}\frac{p^3}{E^{\star 2}}\frac{1 + (k_R r_0)^2}{1 + (k r_0)^2}$$

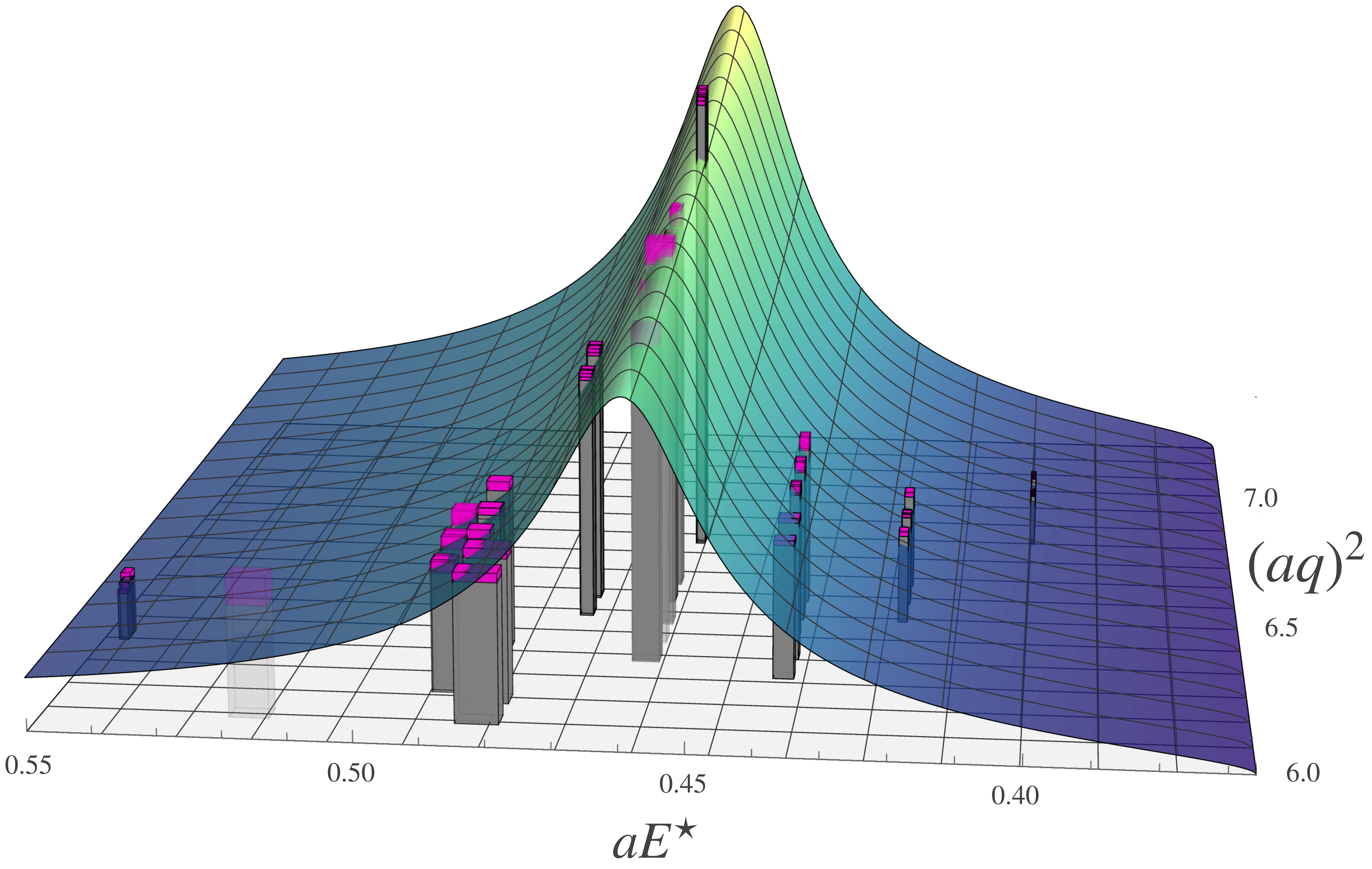




$$F(E^\star, q^2) = \frac{1}{1 - \frac{q^2}{m_P^2}} \sum_{n,m} A_{n,m}\, z^n(q^2)\, (E^{\star 2} - E_{thr}^2)^m$$

$$\langle B \,|\, V \,|\, n \rangle_L = \sqrt{R_n}\; \langle B, p_B \,|\, V \,|\, \pi\pi, E^\star \rangle_\infty$$

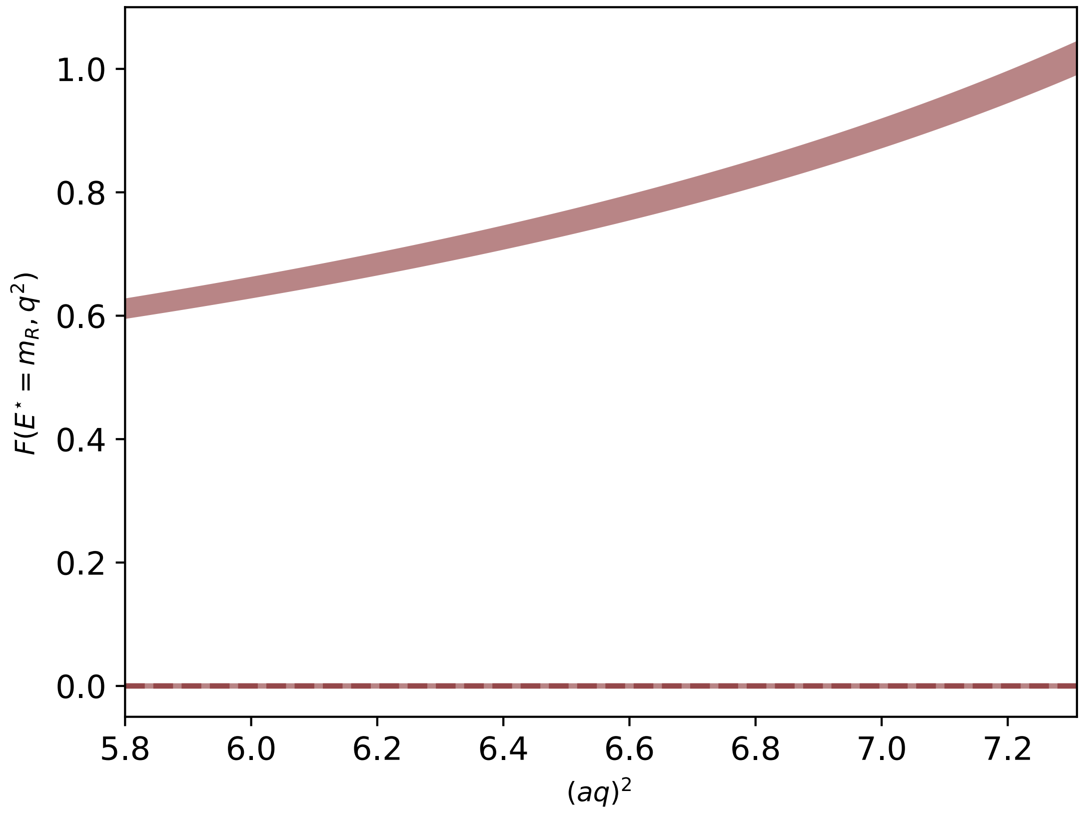
"Lellouch-Lüscher" factor



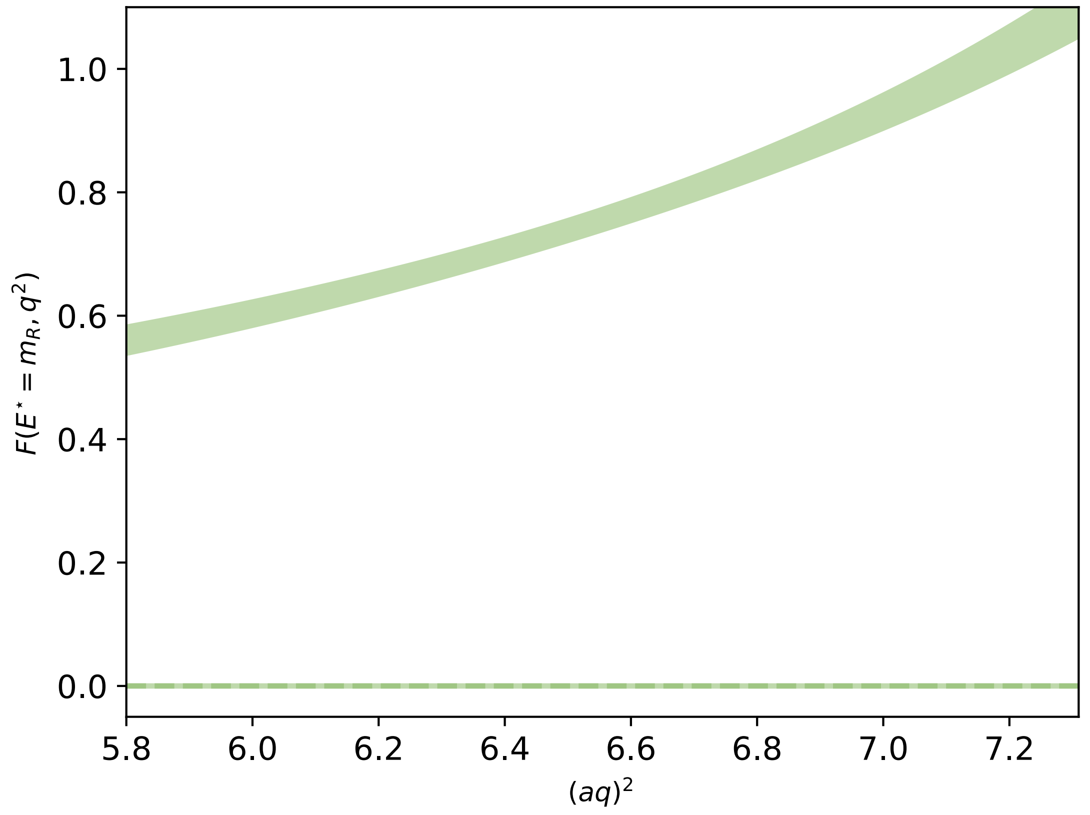
0.55
0.50
0.45
0.40
$aE^\star$
7.0
$(aq)^2$
6.5
6.0

details I

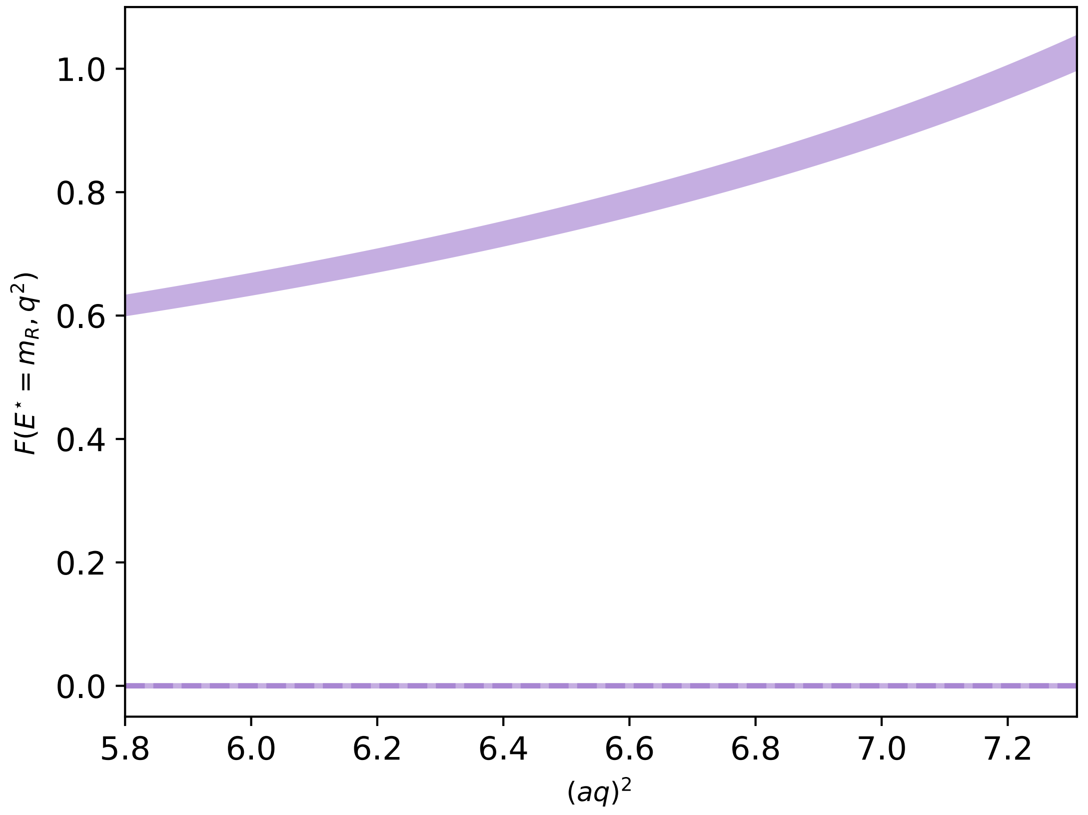
- FF3N0M0_TBWI
  - $\chi^2/\text{dof} = 74.3 / 63 \;= 1.179$
  - $A_{0,0} = 0.2413(66)$



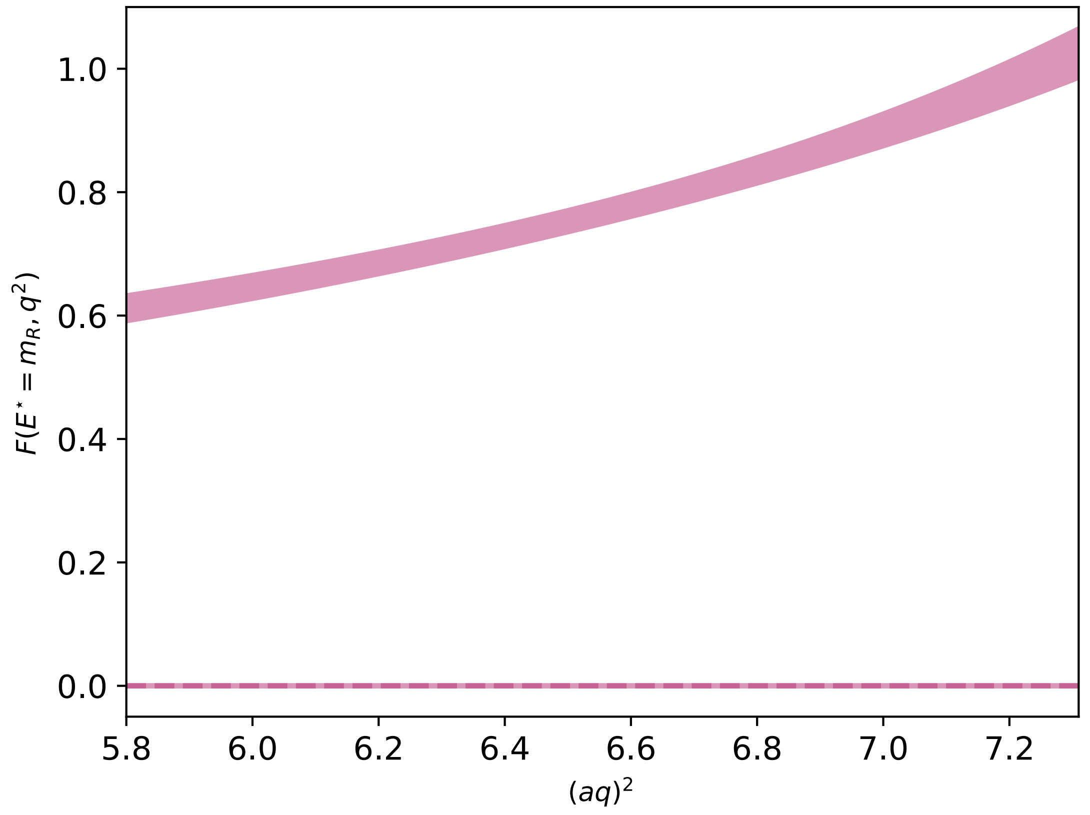
- FF3N1M0_TBWI
  - $\chi^2/\text{dof} = 66.8 / 62 = 1.07$
  - $A_{0,0} = 0.2256(87)$
  - $A_{1,0} = -0.41(18)$



- FF3N0M1_TBWI
  - $\chi^2/\text{dof} = 39.1/62 = 0.630$
  - $A_{0,0} = 0.279(12)$
  - $A_{0,1} = -0.061(13)$

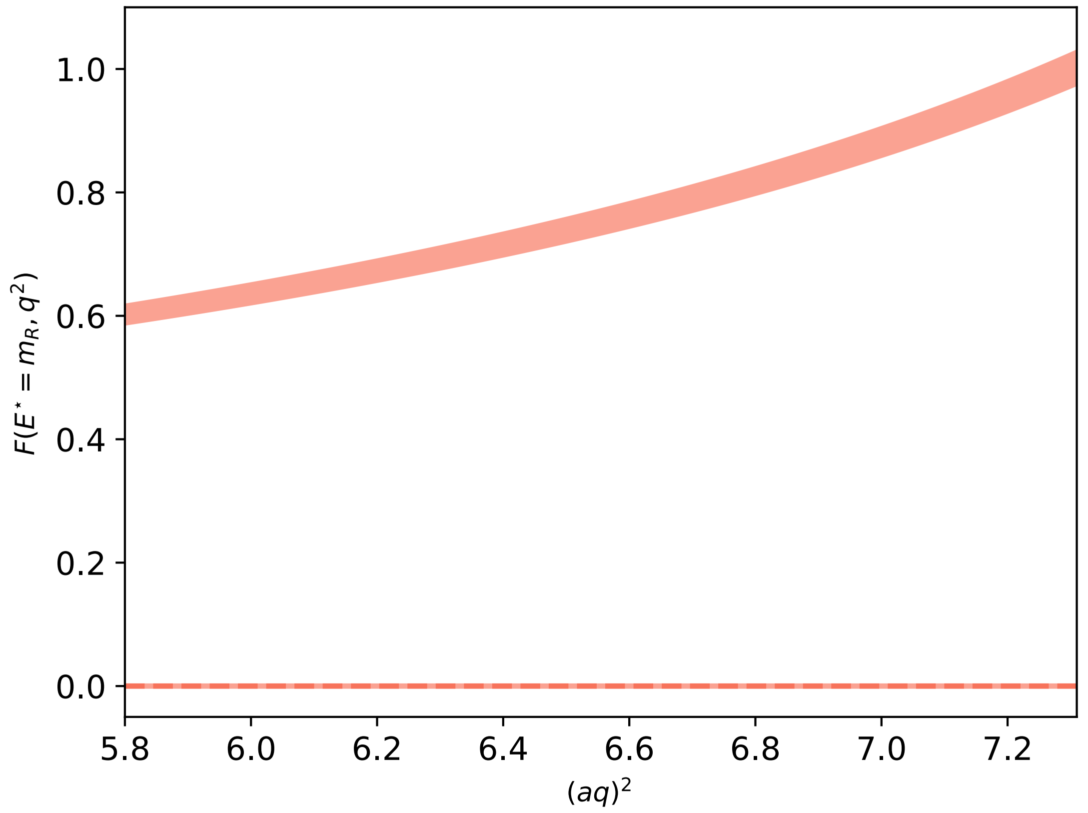


- FF3N1M1_TBWI
  - $\chi^2/\text{dof} = 29.9 / 60 = 0.499$
  - $A_{0,0} = 0.260(16)$
  - $A_{1,0} = -0.88(50)$
  - $A_{0,1} = -0.031(21)$
  - $A_{1,1} = 1.46(82)$

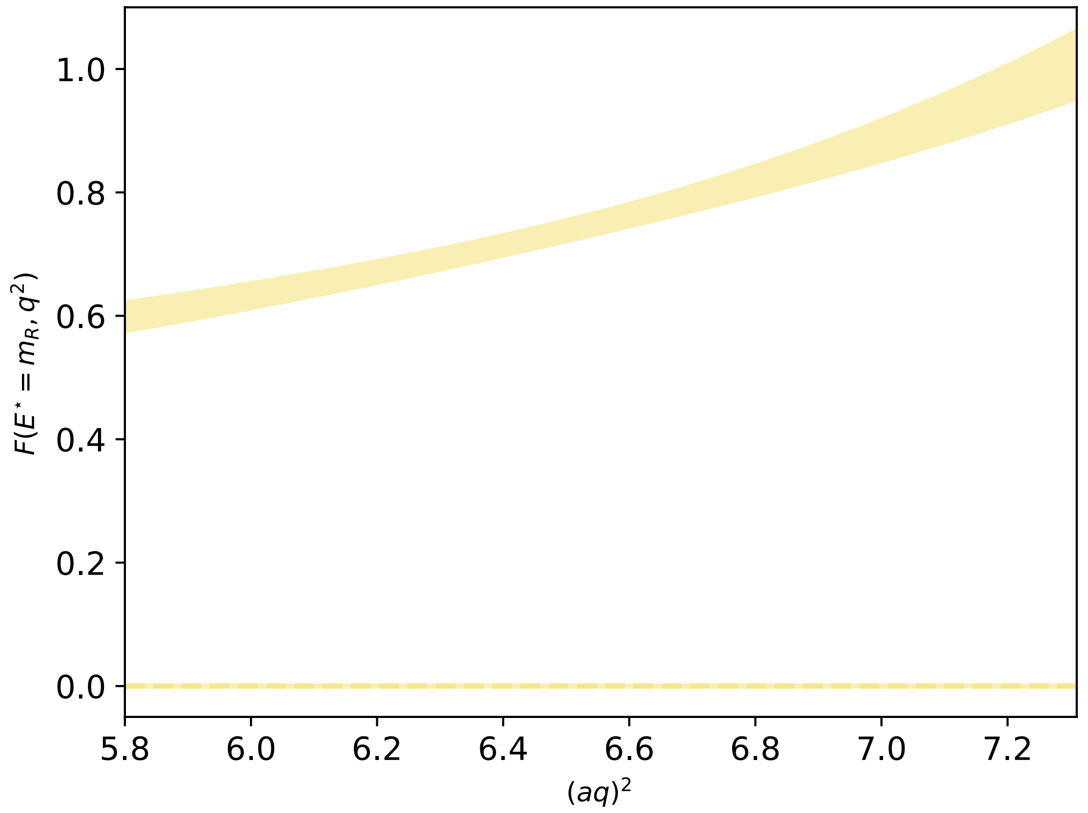


details II

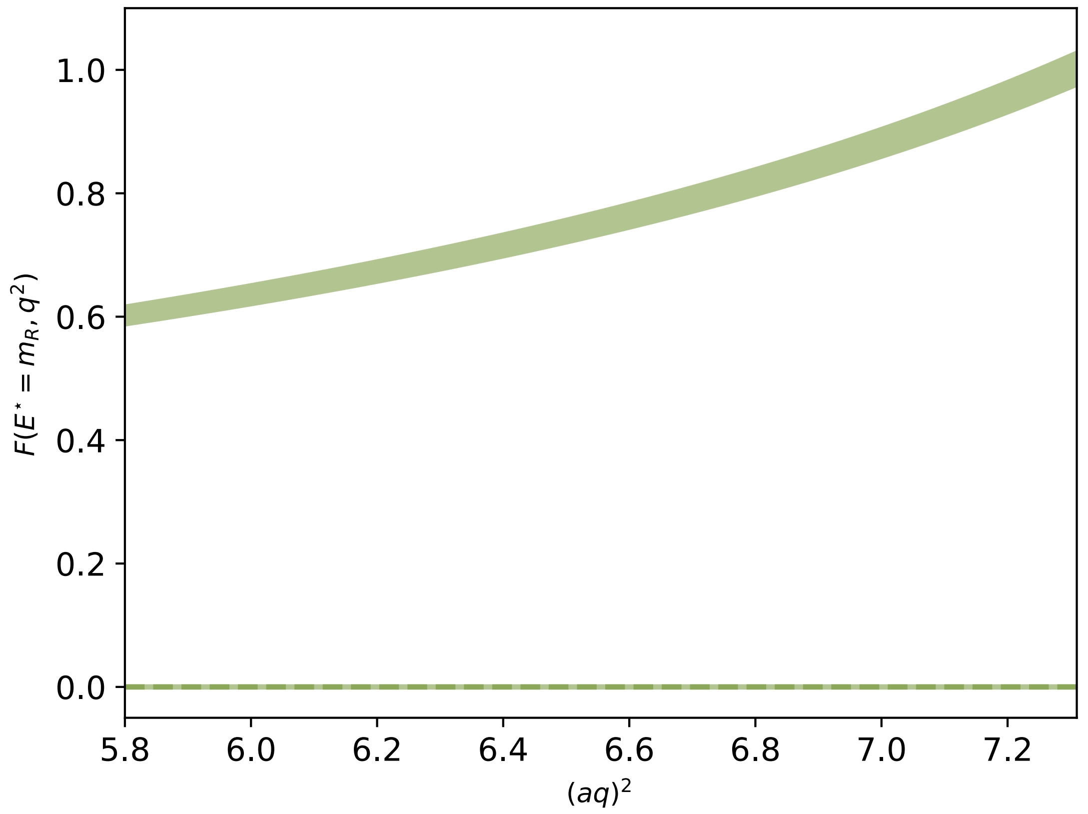
- FF3N0M0_TBWII
  - $\chi^2/\text{dof} = 34.3\,/\,63 = 0.545$
  - $A_{0,0} = 0.2376(71)$



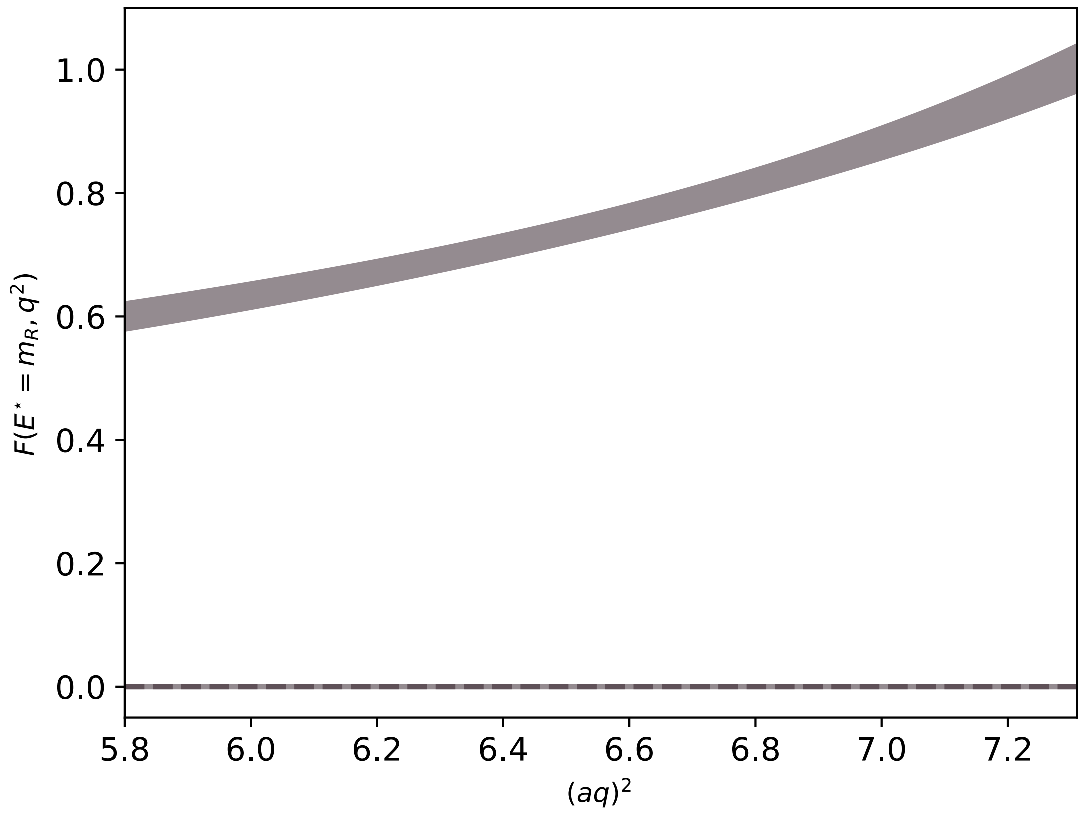
- FF3N1M0_TBWII
  - $\chi^2/\text{dof} = 34.0\,/\,62 = 0.549$
  - $A_{0,0} = 0.2365(89)$
  - $A_{1,0} = -0.03(22)$



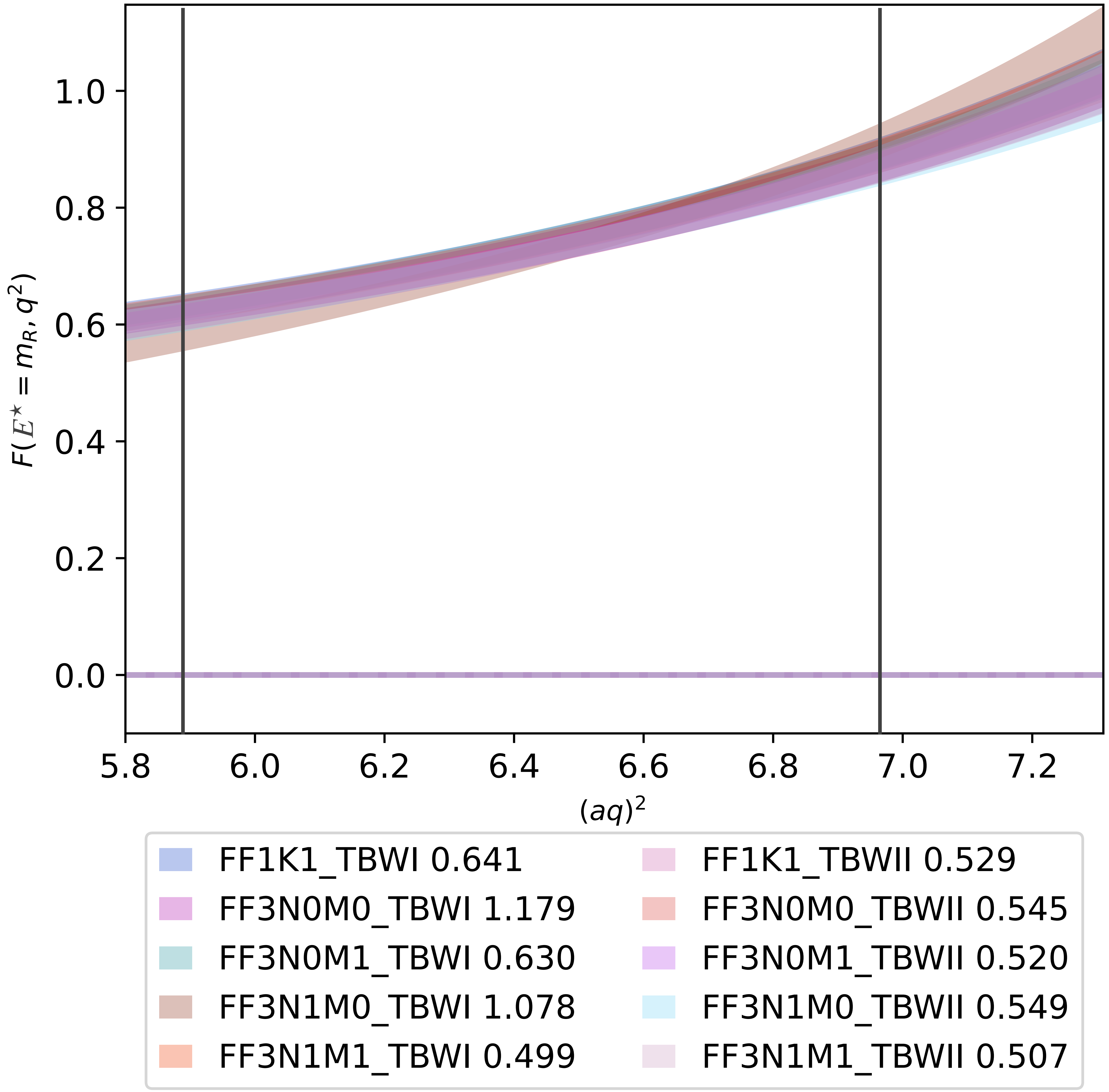
- FF3N0M1_TBWII
  - $\chi^2/\text{dof} = 32.2\,/\,62 = 0.520$
  - $A_{0,0} = 0.240(23)$
  - $A_{0,0} = -0.003(33)$



- FF3N1M1_TBWII
  - $\chi^2/\text{dof} = 30.4\,/\,60 = 0.507$
  - $A_{0,0} = 0.232(20)$
  - $A_{1,0} = -0.27(60)$
  - $A_{0,1} = 0.008(28)$
  - $A_{1,1} = 0.45(1.02)$

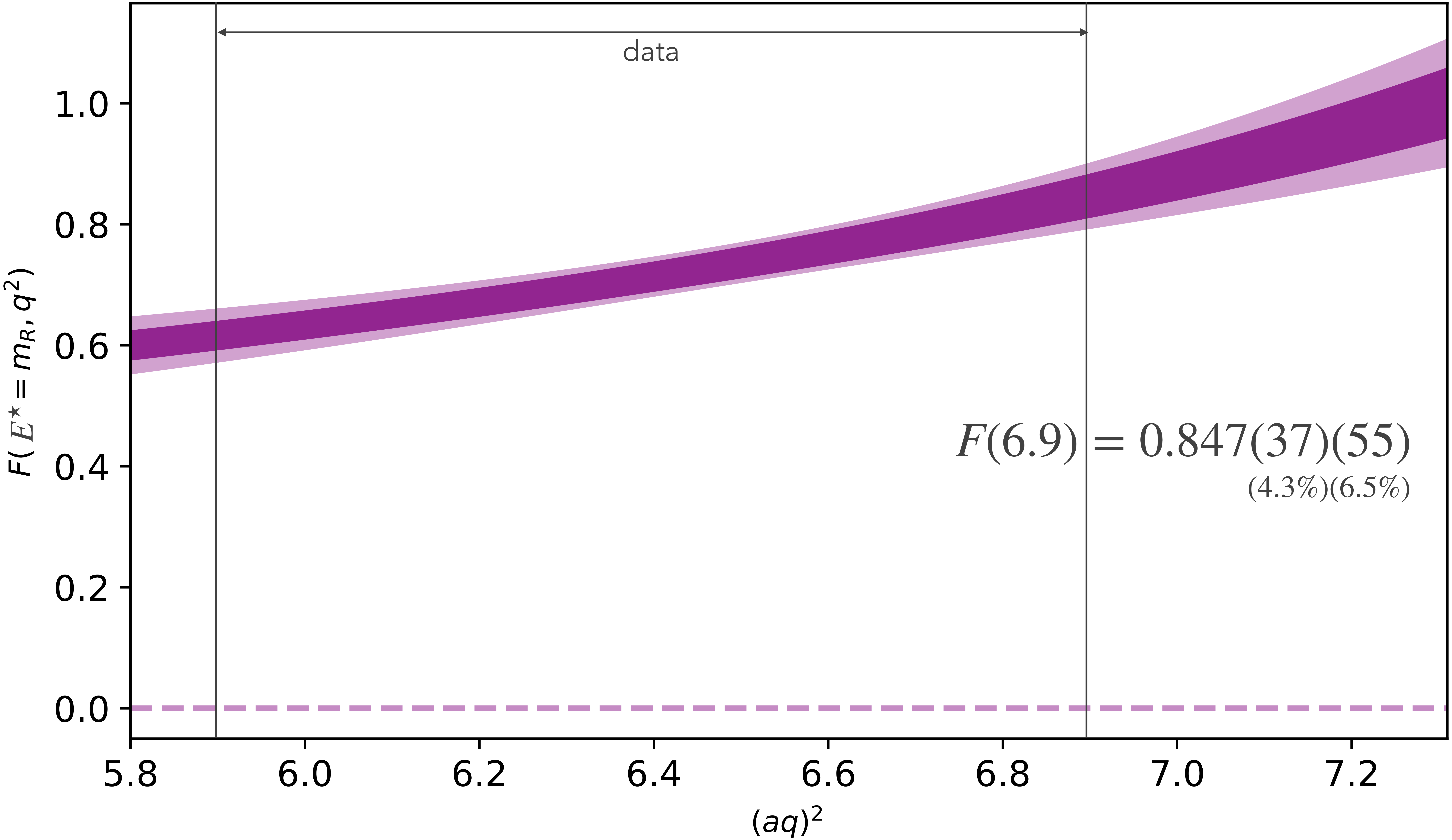


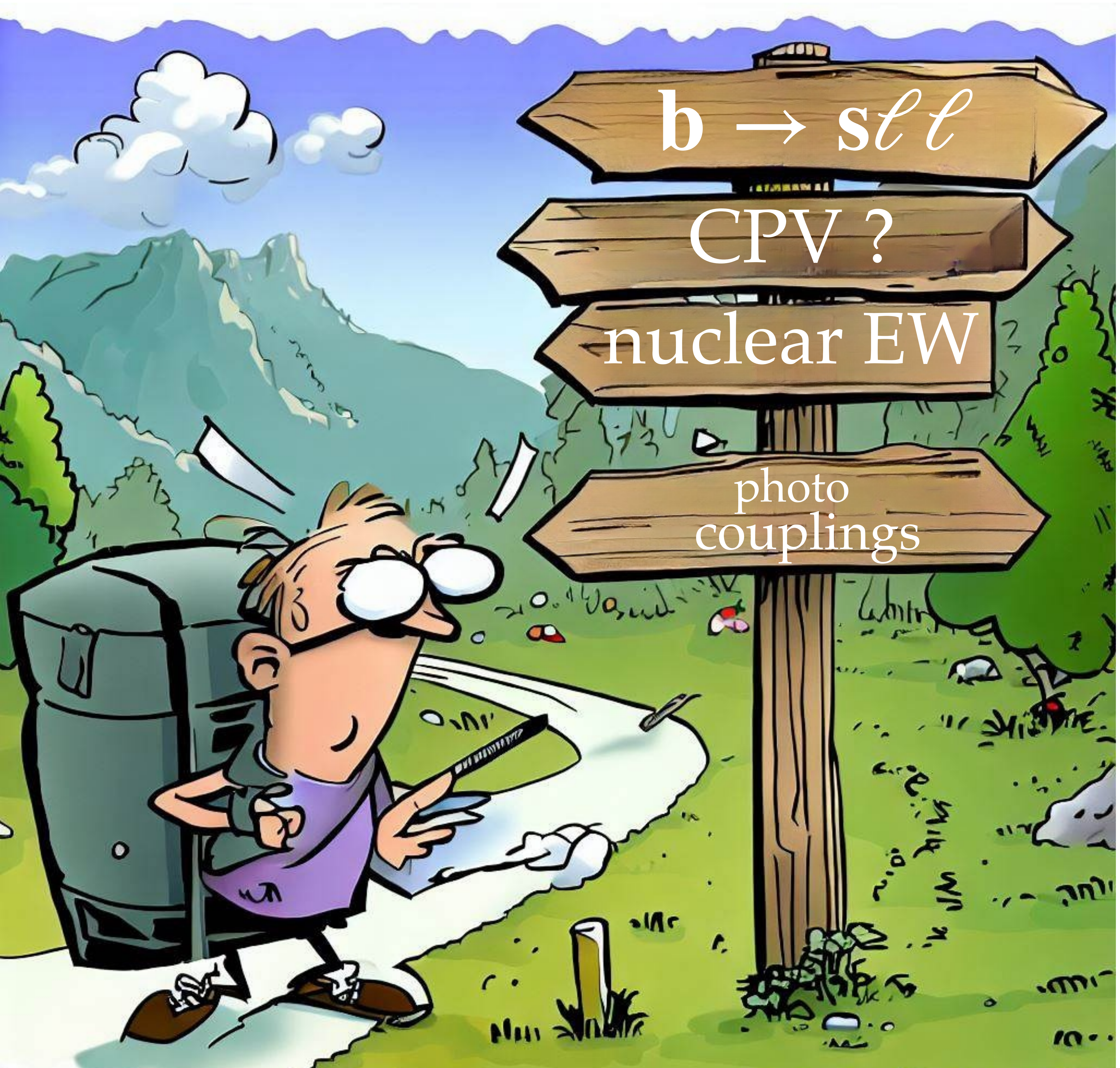
different parameterizations



- $E^\star$ dependence statistically significant, albeit small
- different $E^\star$ dependence in BWI and BWII
- FF3N1M1_TBWII - chosen as central

combining the parameterizations





- Vector transition amplitude example
- similar uncertainties as FLAG!
- a great start to a (more) complete understanding of SM
- missing:
  - m_π continuation to physics!
  - $a \to 0$ limit!
- then… cf Florian Herren
  @ Tue: 18:27

comment

Florian Herren @CKM2023:Tue-18.27

Formalism

Unitarity bounds

- Derivation of BGL ([Boyd, Grinstein, Lebed PRL 74 (1995) 4603-4606;PRD 56 (1997) 6895-6911;...]) can be generalized to multi-hadron final states
- However, a z-expansion is not straightforward, due to the dependence of the FFs on 2 variables
- Weak interaction and final state interactions can be factorized ([Watson PR 88 (1952) 1163-1171])
- Approximate weak interaction dependence on invariant mass
- Corrections can be systematically incorporated

$$\chi^L_{(J)}(Q^2) \equiv \left.\frac{\partial \Pi^L_{(J)}}{\partial q^2}\right|_{q^2=Q^2} = \frac{1}{\pi}\int_0^\infty \mathrm{d}q^2 \frac{\mathrm{Im}\,\Pi^L_{(J)}(q^2)}{(q^2-Q^2)^2} ,$$

$$\chi^T_{(J)}(Q^2) \equiv \left.\frac{1}{2}\frac{\partial^2 \Pi^T_{(J)}}{\partial (q^2)^2}\right|_{q^2=Q^2} = \frac{1}{\pi}\int_0^\infty \mathrm{d}q^2 \frac{\mathrm{Im}\,\Pi^T_{(J)}(q^2)}{(q^2-Q^2)^3}$$

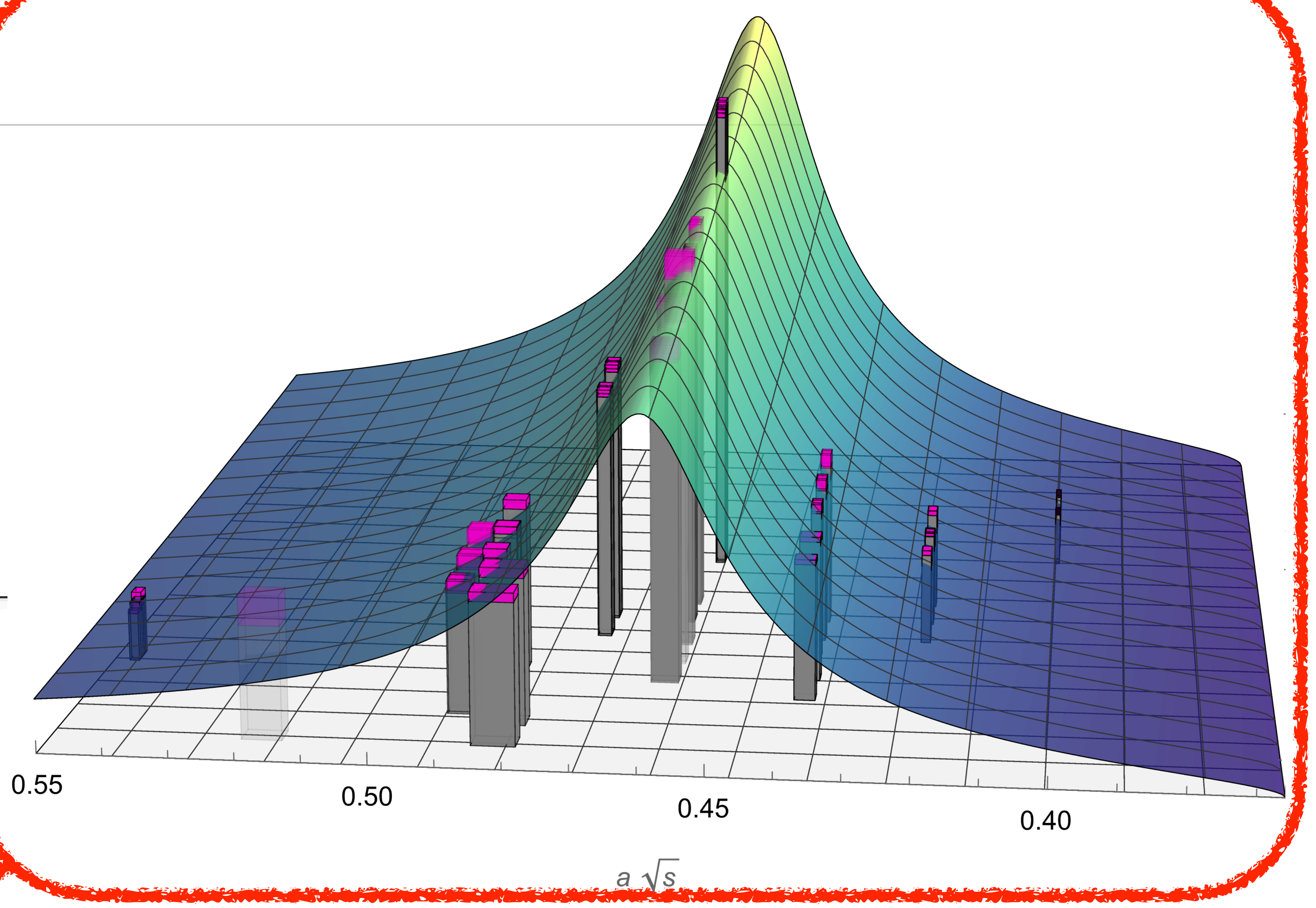
$$\mathrm{Im}\,\Pi^L_A \supset \frac{1}{32\pi^3}\frac{M_B^4}{q^4}\sum_{l=0}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\, M^2_{D\pi} W^{2l+1}\frac{1}{2l+1}|\mathcal{F}_2^{(l)}(q^2, M^2_{D\pi})|^2 ,$$

$$\mathrm{Im}\,\Pi^T_A \supset \frac{1}{96\pi^3}\frac{M_B^4}{q^4}\sum_{l=0}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\, M^2_{D\pi} \frac{W^{2l+1}}{\lambda(M_B^2, M^2_{D\pi}, q^2)}\frac{1}{2l+1}|\mathcal{F}_1^{(l)}(q^2, M^2_{D\pi})|^2$$

$$+ \frac{1}{96\pi^3}\frac{M_B^4}{q^2}\sum_{l=1}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\, M^2_{D\pi} \frac{W^{2l+1}}{\lambda(M_B^2, M^2_{D\pi}, q^2)}\frac{l+1}{l(2l+1)}|f^{(l)}(q^2, M^2_{D\pi})|^2$$

$$\mathrm{Im}\,\Pi^T_V \supset \frac{1}{96\pi^3}\frac{M_B^4}{q^2}\sum_{l=1}\int_{(M_D+m_\pi)^2}^{(\sqrt{q^2}-M_B)^2} \mathrm{d}M^2_{D\pi}\, M^2_{D\pi} W^{2l+1}\frac{l+1}{l(2l+1)}|g^{(l)}(q^2, M^2_{D\pi})|^2 .$$

$$f(q^2, M^2_{D\pi}) = \hat{f}(q^2, M^2_{D\pi})g(M^2_{D\pi}) \approx \left(\tilde{f}(q^2) + \mathcal{O}((M_R^2 - M^2_{D\pi})/M_B^2)\right) g(M^2_{D\pi})$$



well except for the right channels…