Updates on inclusive charmed and bottomed meson decays from lattice

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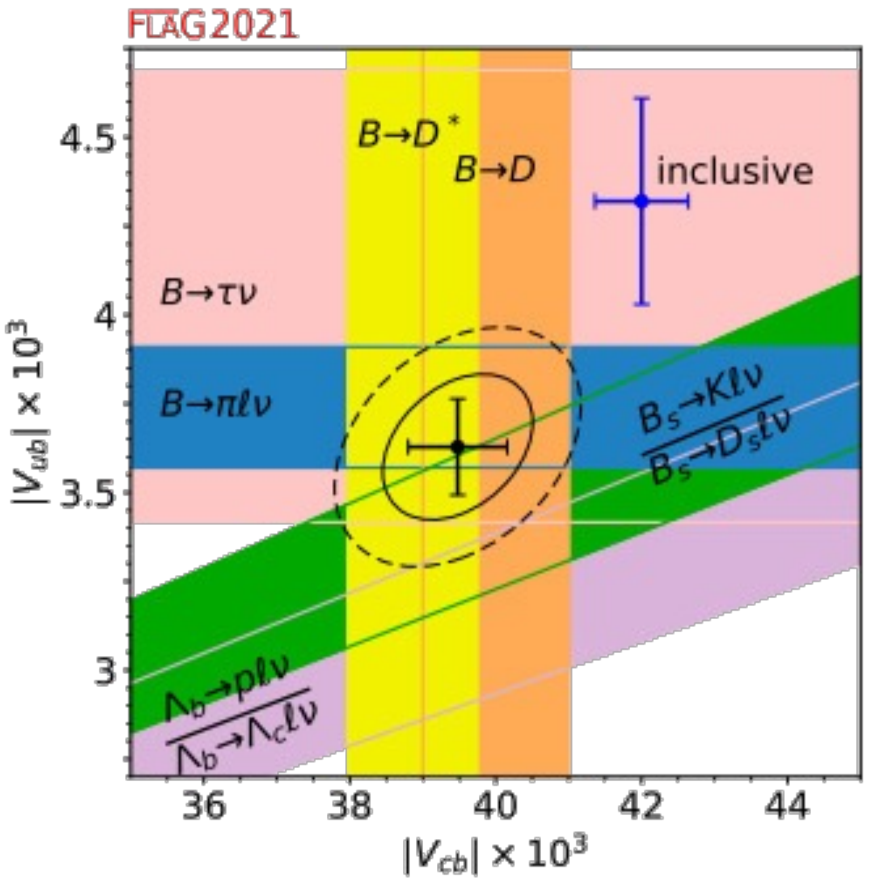
Motivation



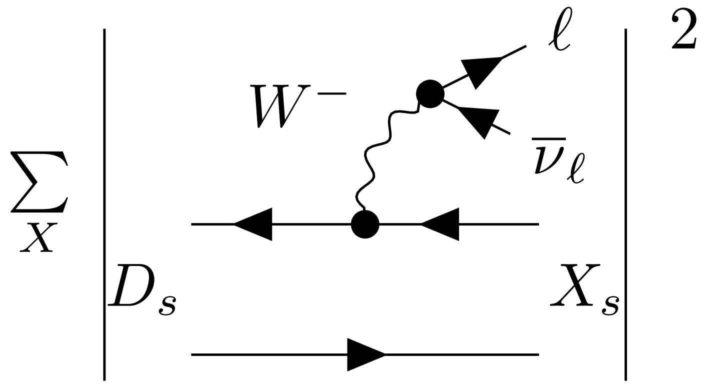
[Y. Aoki et al., arXiv:2111.09849]

- $\sim 3\sigma$ discrepancy between inclusive and exclusive determination (blue vs. black cross)
- Why lattice? → fully nonperturbative theoretical approach to QCD

This Talk:

- Updates on ongoing projects in the analysis of inclusive decays from the lattice perspective [A. Barone et al., arXiv:2305.14092]

Inclusive semileptonic decay rate $D_s \to X_s \ell \nu_\ell$



$$\frac{d\Gamma}{dq^2 dq_0^2 dE_l} = \frac{G_F^2 |V_{cs}|^2}{8\pi^2} L_{\mu\nu} W^{\mu\nu}$$

$L_{\mu\nu}$: Leptonic tensor (analytically known)

$W^{\mu\nu}$: Hadronic tensor (nonperturbative QCD)

$$W^{\mu\nu} \sim \sum_{X_s} \langle D_s(\boldsymbol{p}) | J_\mu^\dagger(\boldsymbol{q}) | X_s(\boldsymbol{r}) \rangle \langle X_s(\boldsymbol{r}) | J_\nu(\boldsymbol{q}) | D_s(\boldsymbol{p}) \rangle \sim \rho(\omega)$$

Spectral density

Analytical approach:

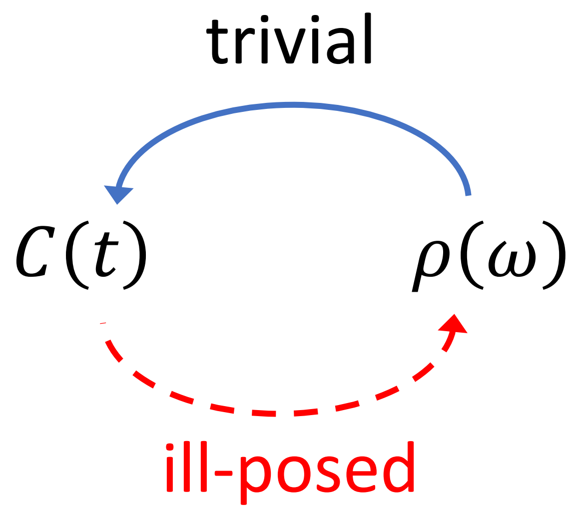
Operator-product expansion (OPE)

BUT

Full control over systematic errors requires nonperturbative methods

Challenges on the lattice:

- Large number of states
- Requires external states, i.e. ground state
- Extraction of $\rho(\omega)$ from correlator ill-posed problem (**inverse problem**)



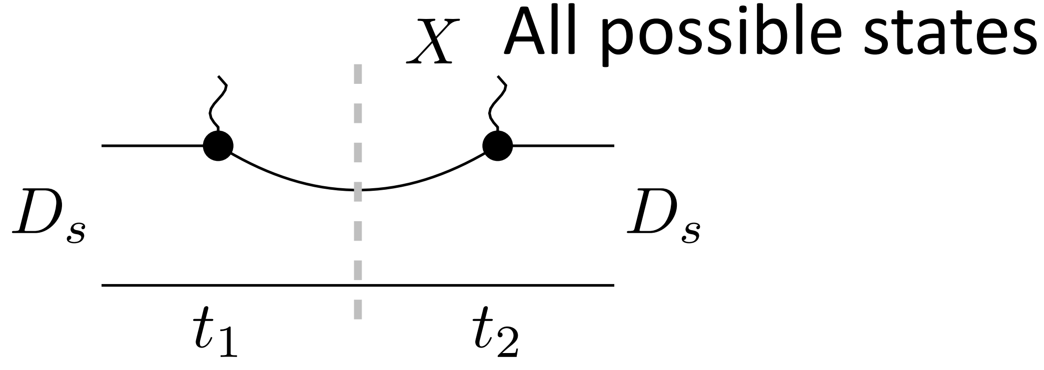
Idea [P. Gambino & S. Hashimoto arXiv:2005.13730]

Smeared spectral density

$$\rho_s(\omega)$$

Smearing ≙ phase space integral

Approximation using **4Pt function** correlation function



$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, [\,]_{\text{Lattice}}$$

Introduction

Inclusive Decays - Continuum

Total decay rate [arXiv:2211.16830]

$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q^2}}\, \bar{X}(\boldsymbol{q}^2)$$

$\bar{X}(\boldsymbol{q}^2)$ integral over energy of hadronic final states

$$\bar{X}(\boldsymbol{q}^2) \sim \int_{\omega_0}^{\infty} d\omega\, K^{(l)}(\omega, \boldsymbol{q}^2) \langle D_s(\boldsymbol{0}) | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) \delta\big(\hat{H} - \omega\big) \tilde{J}_\nu(\boldsymbol{q}) | D_s(\boldsymbol{0}) \rangle$$

Kernel function (analytically known)

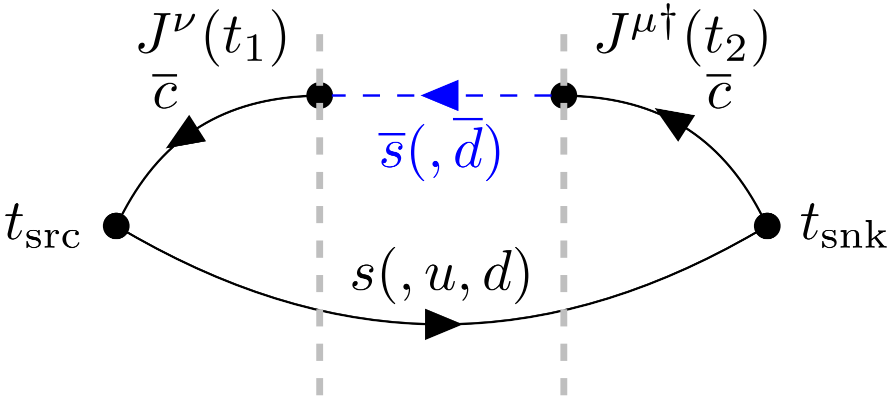
contains terms of power ω_X^l, with $l = 0,1,2$

Inclusive decays – Lattice

Non perturbative calculation of hadronic tensor

$$W^{\mu\nu}(\boldsymbol{q},\omega) \sim \sum_{X_s} \langle D_s|\tilde{J}_\mu^\dagger|X_s\rangle \langle X_s|\tilde{J}_\nu|D_s\rangle$$

4pt correlation functions



- t_{src}, t_2, t_{snk} fixed
- $t_{src} \leq t_1 \leq t_2$
- $t = t_2 - t_1$

$$C(t) \sim \langle D_s|\tilde{J}_\mu^\dagger(-\boldsymbol{q})e^{-\hat{H}t}\,\tilde{J}_\nu(\boldsymbol{q})|D_s\rangle$$

Inclusive decays – Reconstruction from Lattice

Lattice Data

$$C(t) \sim \langle D_s | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) e^{-\hat{H}t} \, \tilde{J}_\nu(\boldsymbol{q}) | D_s \rangle$$

Combined with $\bar{X}(\boldsymbol{q}^2)$

$$\bar{X}(\boldsymbol{q}^2) = \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger K^{(l)}(\hat{H}, \boldsymbol{q}^2) \tilde{J}_\nu | D_s(\mathbf{0}) \rangle$$

Approximate Kernel in polynomials of $e^{-\hat{H}}$

$$K(\hat{H}, \boldsymbol{q}^2) = k_0 + k_1 e^{-\hat{H}} + \ldots + k_N e^{-N\hat{H}}$$

C(0) C(N)

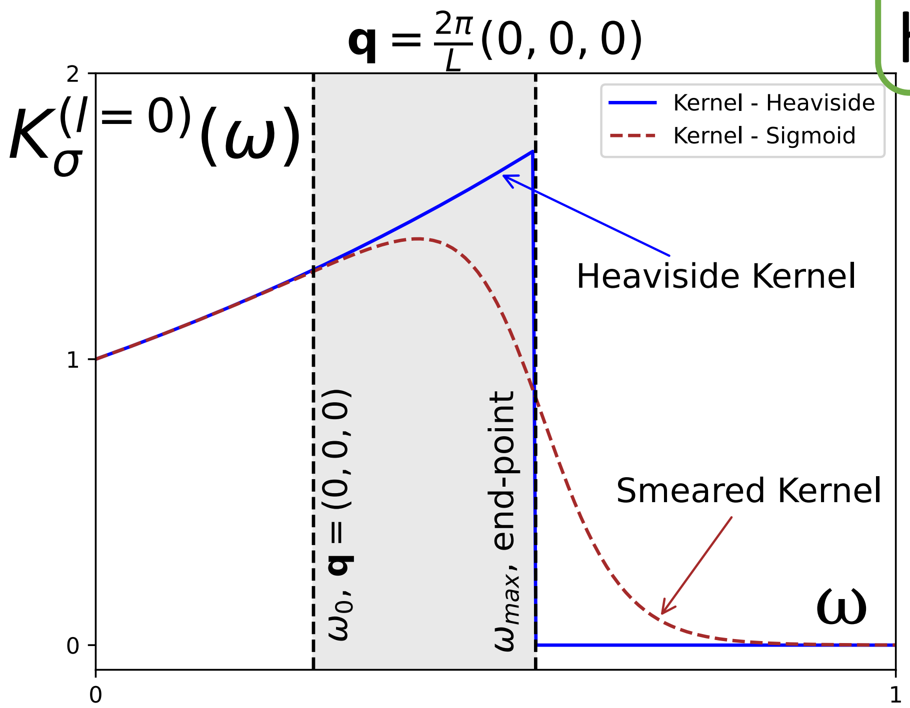
$$\bar{X}(\boldsymbol{q}^2) \sim k_0 \, \langle D_s | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) \, \tilde{J}_\nu(\boldsymbol{q}) | D_s \rangle + \cdots + k_N \, \langle D_s | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) e^{-N\hat{H}} \, \tilde{J}_\nu(\boldsymbol{q}) | D_s \rangle$$

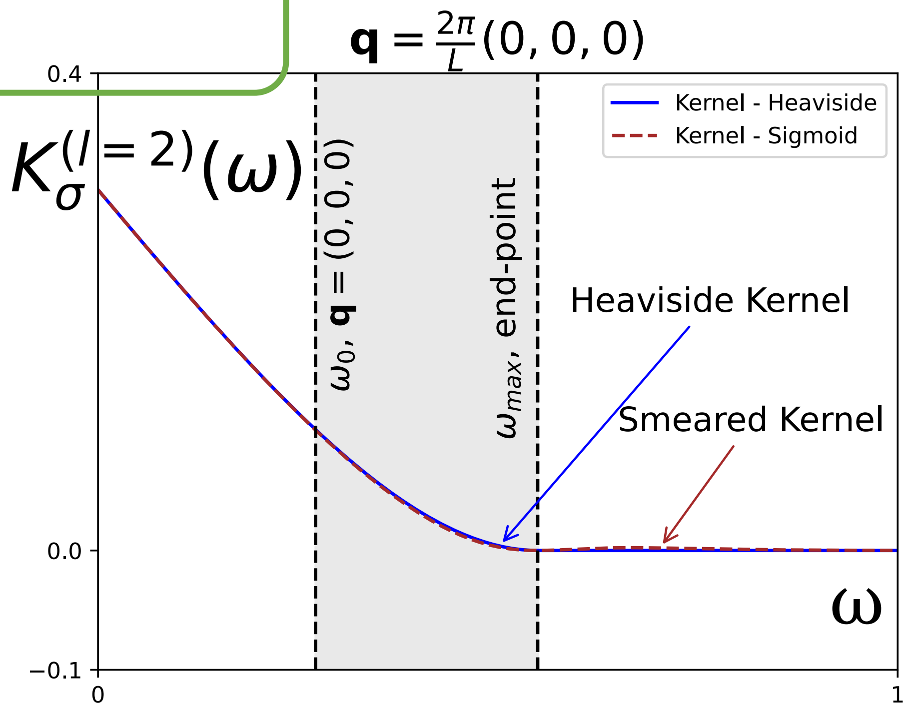
The Kernel function

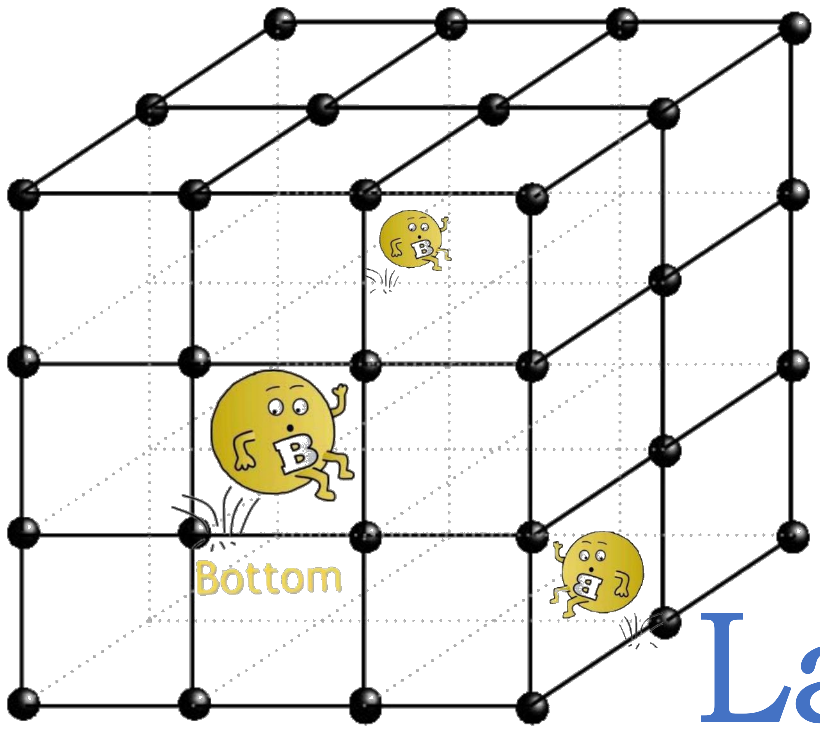
Sigmoid Function

$$K_\sigma^{(l)}(\omega) = e^{2\omega t_0} \left(\sqrt{\boldsymbol{q}^2}\right)^{2-l} \left(m_{D_s} - \omega\right)^l \theta_\sigma\left(m_{D_s} - \sqrt{\boldsymbol{q}^2} - \omega\right),$$

Momentum and energy of hadronic final state

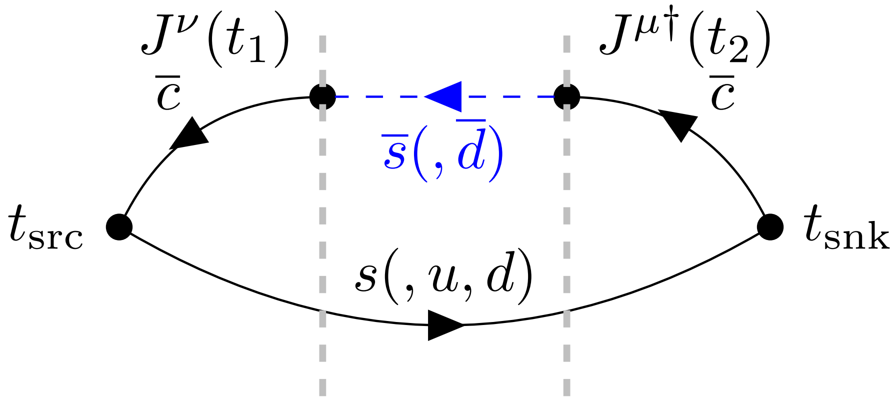






Lattice Setup

Simulations conducted on Fugaku using Grid [P. Boyle et al., https://github.com/paboyle/Grid] and Hadrons [A. Portelli et al., https://github.com/aportelli/Hadrons] software packages



Lattice setup:

- Lattice size: $48^3 \times 96$
- Lattice Spacing: $a = 0.055$ fm
- $M_\pi \simeq 300$ MeV

Simulation:

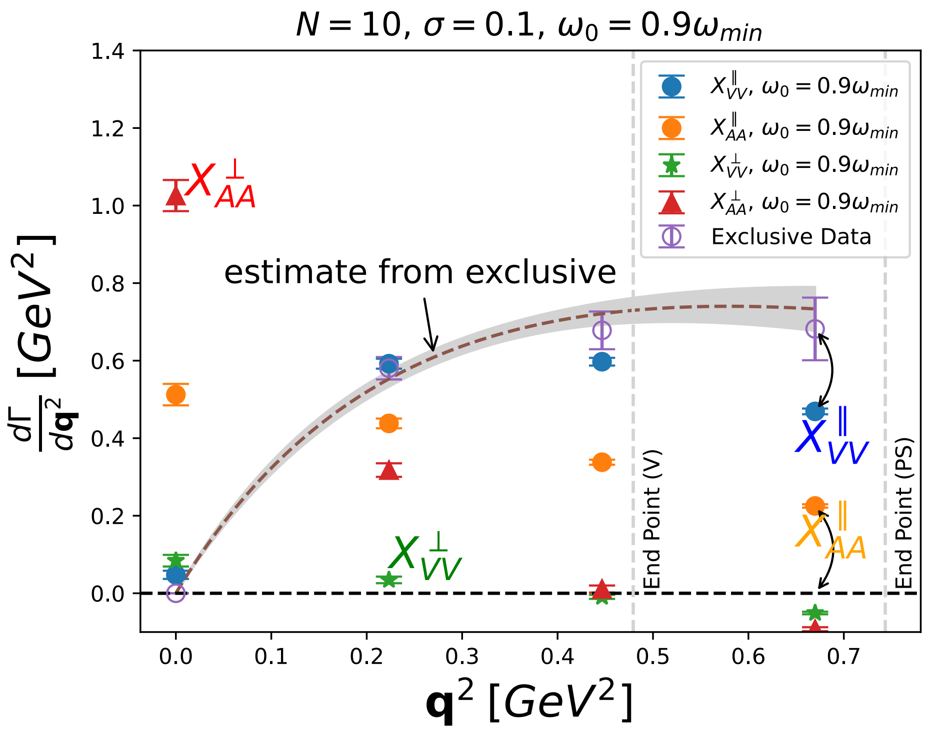
- 2+1 Möbius domain-wall fermions
- s, c quarks simulated at near-physical values
- Cover whole kinematical region $\boldsymbol{q} = (0,0,0) \to (1,1,1)$

First numerical results

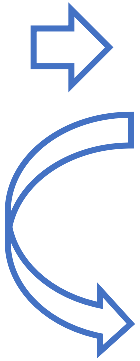
The differential rate $\bar{X} \sim \frac{d\Gamma}{d\boldsymbol{q}^2}$

$$\bar{X}_\sigma = \langle D_s(\mathbf{0}) | \tilde{J}_\mu^\dagger(-\boldsymbol{q}) K_\sigma(\hat{H}, \boldsymbol{q}^2) \tilde{J}_\nu(\boldsymbol{q}) | D_s(\mathbf{0}) \rangle$$

Using the smeared kernel we obtain



Decomposed $\bar{X}$ into channels of V and A; $\parallel$ and $\perp$

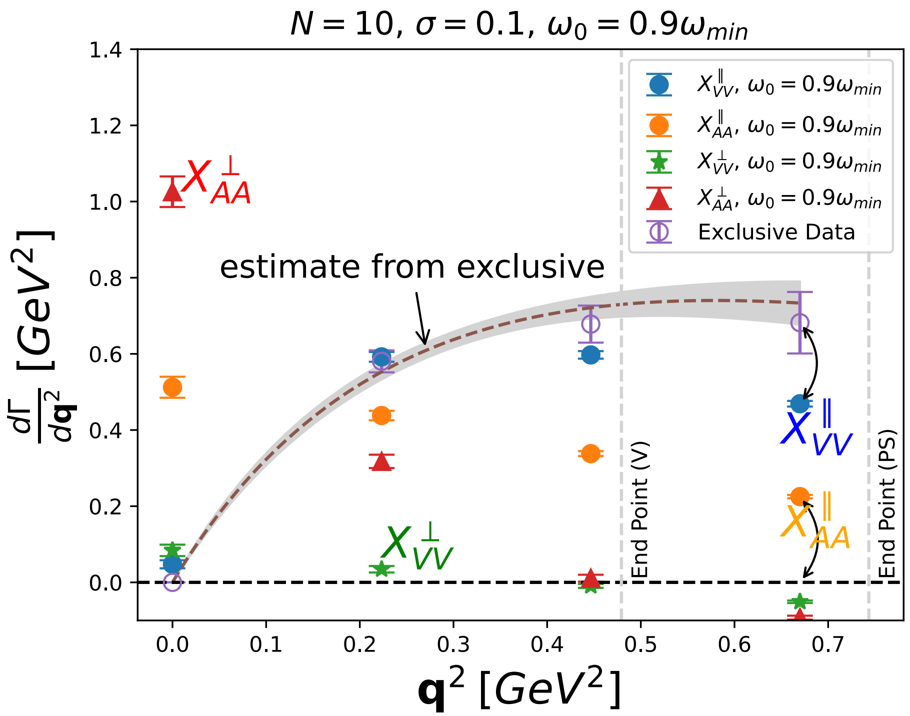


Comparision to the ground-state-only limit

$D \rightarrow K$ exclusive data; values seem to be in the right ballpark

Current points of interest

Systematic errors - Finite volume



Questions

- Error due to approximation? [arXiv:2211.16830]
- Infinite volume limit?
  - ➢ In finite volume spectral density is a sum of delta peaks

Translate into limits

- $\sigma \to 0$
- $V \to \infty$

Proper estimate requires

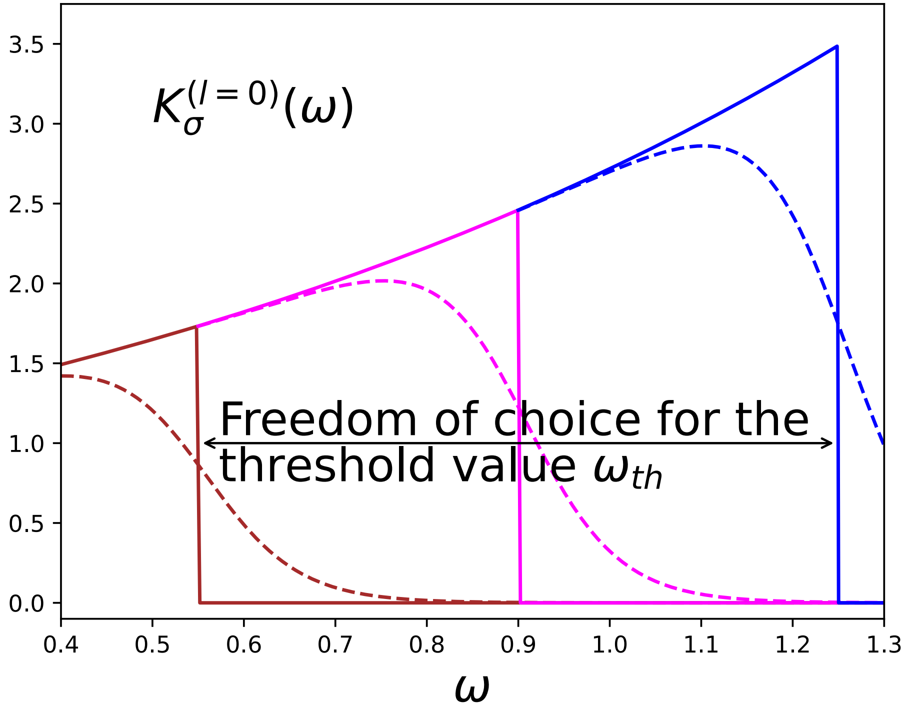
$$\lim_{\sigma\to 0} \lim_{V\to\infty} \bar{X}_{\sigma}(\boldsymbol{q}^2)$$

Necessary data not available

Model for the infinite volume limit

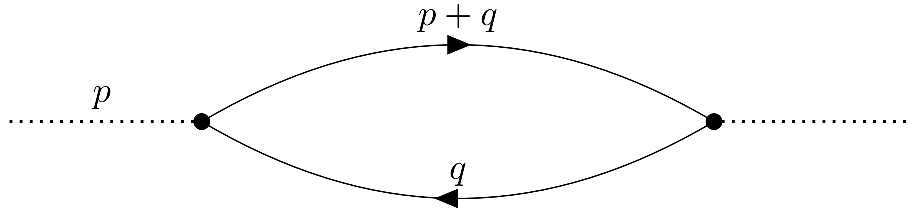
Introduce a model

- Include two-body final states
- Freely vary the upper limit of the energy integral ω_{th}



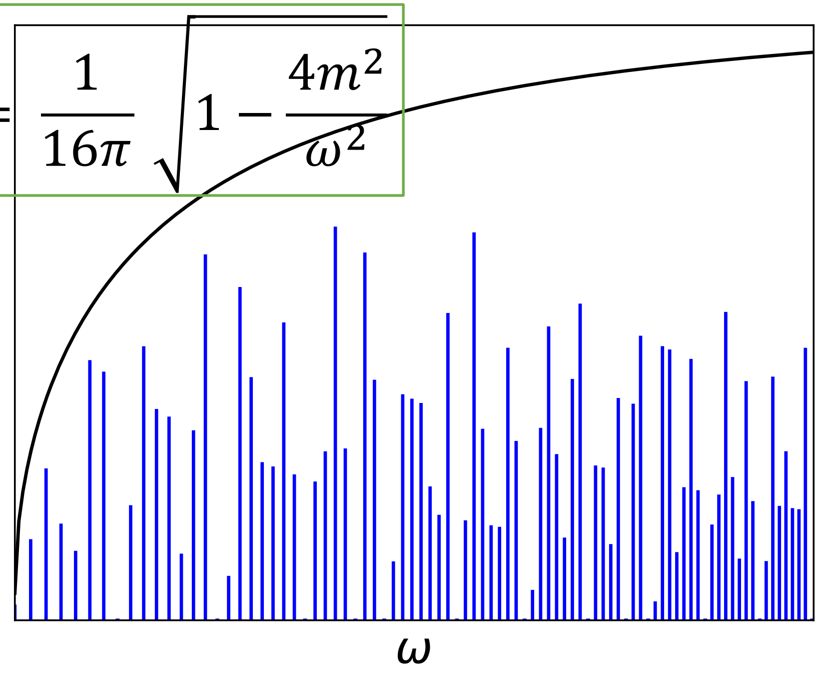
- Verify if the model reproduces the correct dependence on ω_{th}
- Estimate the $V \to \infty$ limit

Vacuum polarization ansatz

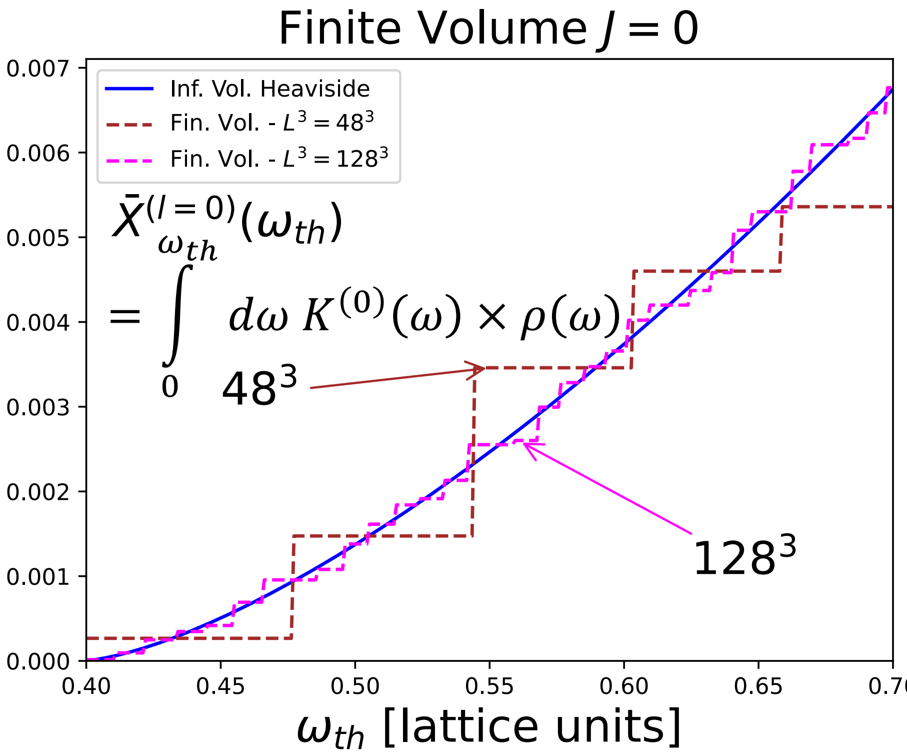


$$\mathrm{Im}[Diagram\,] \sim \pi \int \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\left(2\sqrt{m^2+\boldsymbol{q}^2}\right)^2}\ \delta(p_0 - 2\sqrt{m^2+\boldsymbol{q}^2})$$

$$\rho(\omega) = \frac{1}{16\pi}\sqrt{1-\frac{4m^2}{\omega^2}}$$



Infinite volume reconstruction

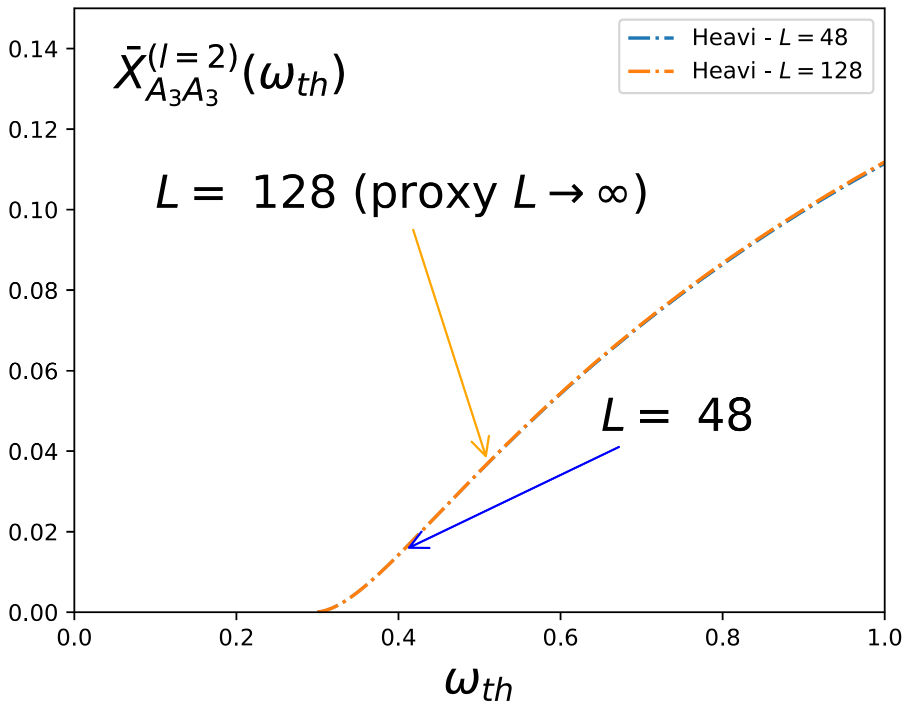


First application for $\bar{X}^{\perp}_{AA}(\mathbf{0})$

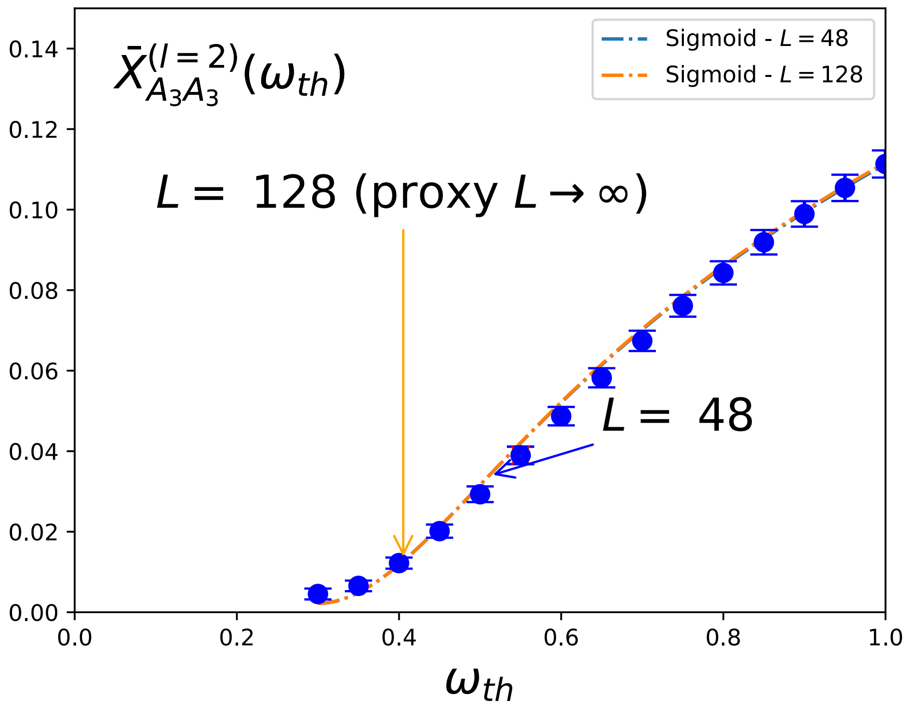
$\bar{X}^{\perp}_{AA}(\mathbf{0})$ only receives contribution from

$\langle D_s | A_3 A_3 | D_s \rangle$, with $l = 2$ in the kernel function

Heaviside



Smearing($\sigma = 0.1$) + lattice data



Using the model we estimate the limit $V \to \infty$, followed by $\sigma \to 0$ for $\bar{X}_{AA}^{\perp}(\mathbf{0})$

$$\bar{X}_{AA}^{\perp}(\mathbf{0}) \sim \text{Data}$$

Obtained from Model

$$+\,\text{Sigmoid}(\text{L}=128) - \text{Sigmoid}(\text{L}=48)$$

Infinite volume limit

$$+\,\text{Sigmoid}(\sigma=0) - \text{Sigmoid}\left(\sigma=\frac{1}{N}\right)$$

$\sigma \to 0$ limit

$$\Rightarrow \quad 0.0389(22) + 0.0001(0) + 0.0028(1) = 0.0418(22)$$

For the safest choice of $\boldsymbol{q}^2 = \mathbf{0}$:

- Negligible corrections from finite volume effects
- $\sigma \to 0$ limit gives a $\sim 7\%$ correction

BUT:

A worse picture is expected for higher $\boldsymbol{q}^2$

⇒ **Further studies required**

Studies of different observables

Preliminary setup to study other observables, e.g. *moments*

$$\Gamma \sim \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}(\boldsymbol{q}^2)$$

$$\langle (q^2)^n \rangle \sim \frac{1}{\Gamma} \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}_Q^{(n)}(\boldsymbol{q}^2)$$

$$\left\langle \left(M^2_{X_s}\right)^n \right\rangle \sim \frac{1}{\Gamma} \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}_H^{(n)}(\boldsymbol{q}^2)$$

$$\langle E_l^n \rangle \sim \frac{1}{\Gamma} \int_0^{\boldsymbol{q}^2_{max}} d\boldsymbol{q}^2 \sqrt{\boldsymbol{q}^2}\, \bar{X}_L^{(n)}(\boldsymbol{q}^2)$$

Same analysis strategy can be applied

Comparison to

- Other theory predictions, e.g. OPE [P. Gambino et al., arXiv:2203.11762]
- Experiments

➡ **Stay tuned for future updates**

Summary

Updates on the inclusive semileptonic decays for charmed and bottomed mesons

Systematic erros

- Studies into systematic errors induced due to finite volume corrections for the inclusive semileptonic decay rate for $D_s \to X_s \ell \nu_\ell$
  - $\boldsymbol{q}^2 = \mathbf{0}$: good estimate of infinite volume limit
  - Small corrections due to finite volume effects and $\sigma \to 0$ limits

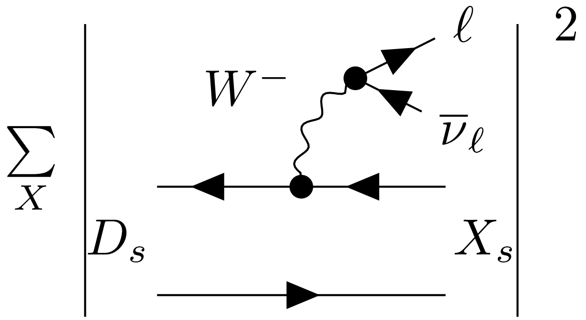
Observables - Moments

- Preliminary setup to study different observables that can be extracted from the inclusive semileptonic decays
  - Compare to continuum based theory predictions, e.g. OPE
  - Obtain more predictions which can be compared to experiments

BACKUP

Inclusive semileptonic decay rate

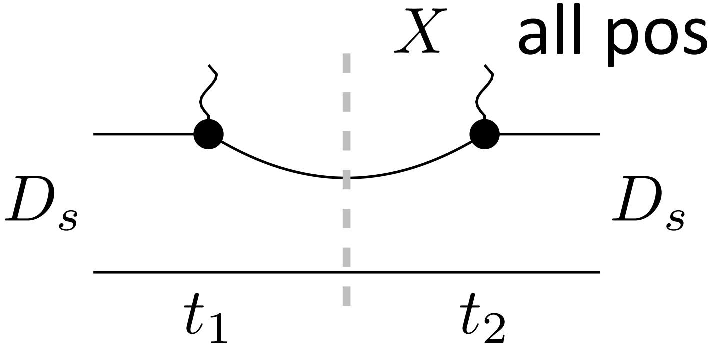
$D_s \to X_s \ell \nu_\ell$



$\sim \int d\omega\, K(\omega) \left\langle D_s \middle| J^\dagger\, \delta(\omega - \omega_X) J \middle| D_s \right\rangle$

Lattice: **4Pt function**

Contributions from all possible states



$\sim \left\langle D_s \middle| J^\dagger\, e^{-\hat{H}(t_2 - t_1)} J \middle| D_s \right\rangle$

$$\int d\omega_X\, K(\omega_X)\, [\ \ ]_{\text{Lattice}}$$

Determined by kinematics

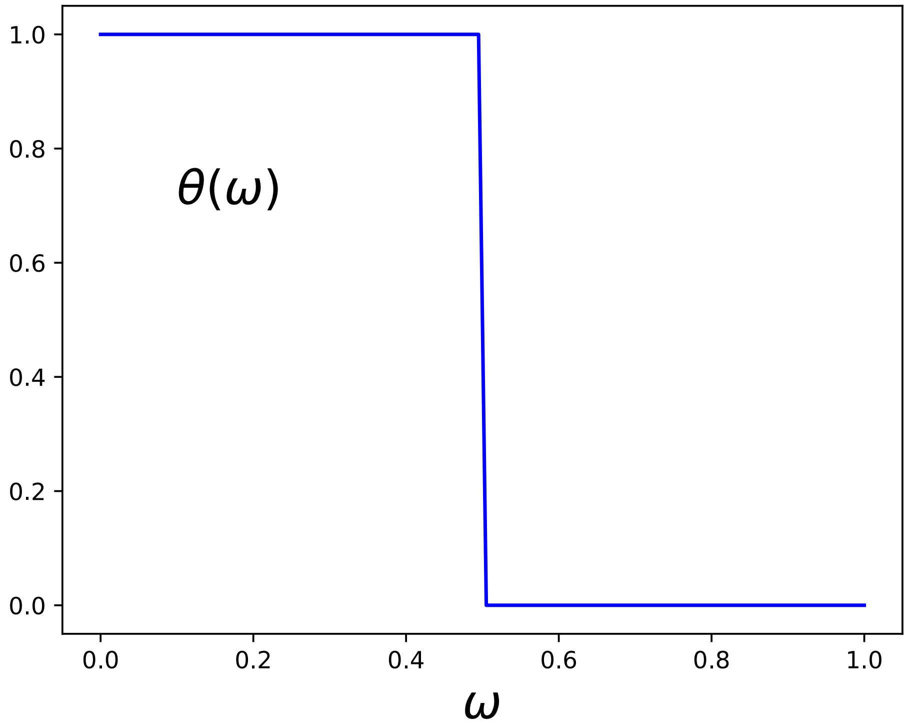
Problems:

1. Upper limit of the energy integral $\theta(\omega_{th} - \omega)$

2. In a finite volume we deal with a discrete set of states

⇨ Sources of systematic errors

1. Upper limit of the energy integral

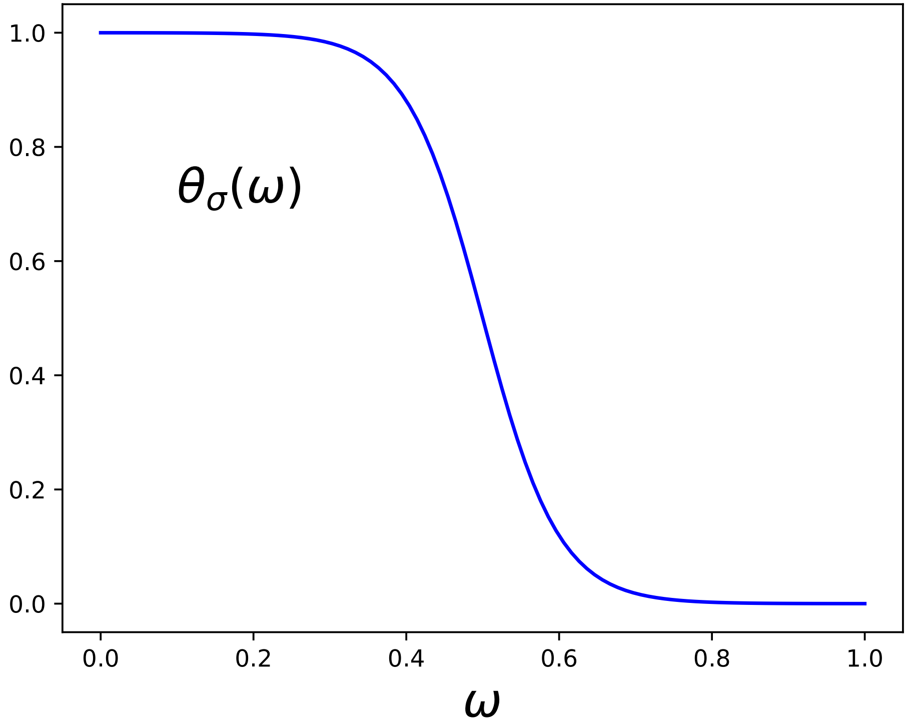


Approximation strategies [A. Barone et al., arXiv:2305.14092] :

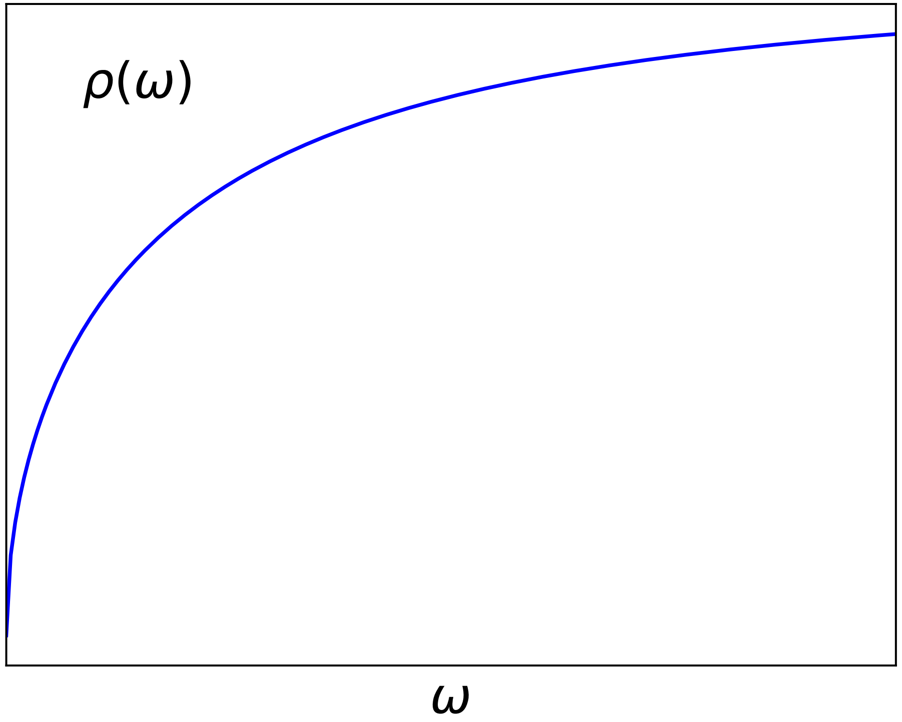
- Chebyshev approximation
- Backus-Gilbert approach

Direct approximation with $e^{-\omega(t_2-t_1)}$ not possible

Apply smearing



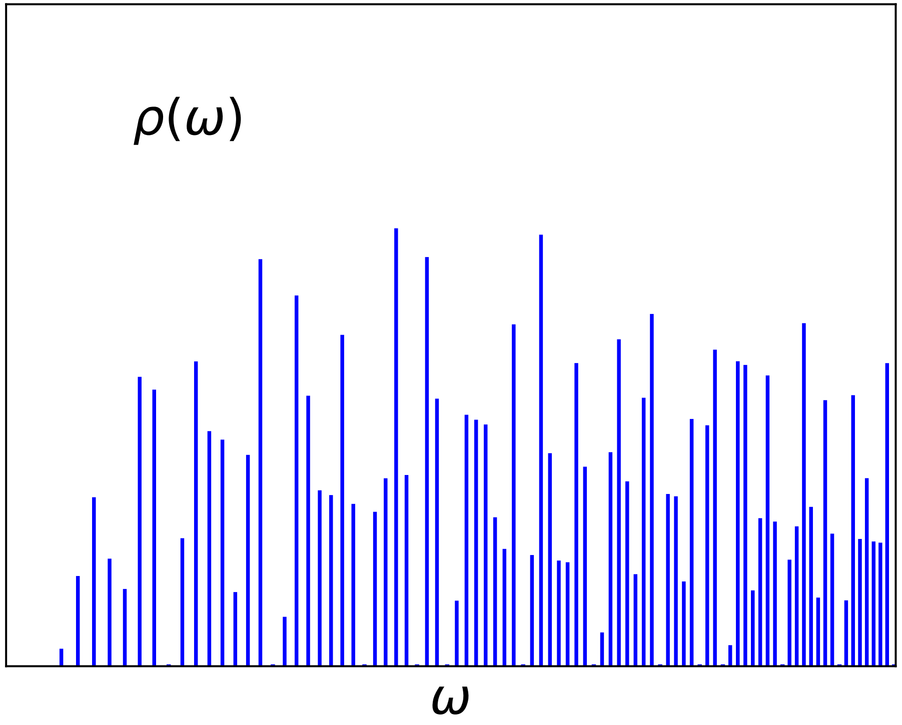
2. Discrete set of states



Develop and verify a modelling strategy to estimate the infinite volume limit

Problem

Finite volume



Polynomial reconstruction

Chebyshev approach

Standard Chebyshev polynomials

$$T_j(\omega) : [-1,1] \to [-1,1]$$

Construct an approximation in the range $[\omega_0, \infty]$, with $0 \leq \omega_0 < \omega_{min}$

Shifted Chebyshev

$$K(\omega) \simeq \sum_j c_j^* T_j^*(e^{-\omega})$$

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Backus-Gilbert approach

Minimize functional

Statistical error

$$F_{\mu\nu,\theta}[g] = A_{\mu\nu}[g] + \theta^2 B_{\mu\nu}[g]$$

Systematic error

$A_{\mu\nu}[g]$: polynomial approximation similar to Chebyshev

θ^2: controls balance between systematic and statistical errors

Polynomial reconstruction

Chebyshev approach

Standard Chebyshev polynomials

$$T_j(\omega) : [-1,1] \rightarrow [-1,1]$$

Construct an approximation in the range $[\omega_0, \infty]$, with $0 \leq \omega_0 < \omega_{min}$

Shifted Chebyshev

$$K(\omega) \simeq \sum_j c_j^* T_j^*(e^{-\omega})$$

Method employed in this analysis

Backus-Gilbert approach

Minimize functional

Statistical error

$$F_{\mu\nu,\theta}[g] = A_{\mu\nu}[g] + \theta^2 B_{\mu\nu}[g]$$

Systematic error

$A_{\mu\nu}[g]$: polynomial approximation similar to Chebyshev

θ^2: controls balance between systematic and statistical errors

→ A. Barone et al., arXiv:2305.14092
M. Hansen, et al, arXiv:1903.06476

Numerical Analysis – Procedure

The idea is to use the information from out model and fit this to our lattice data and then perform the infinite volume extrapolation based on the fitted data

Fit prescription

$$C(t) = A_0 e^{-E_0 t} + s\left[\sum_i A_i e^{-E_i t} F(E_i)\right]$$

Ground state

Overall contribution of excited states

Form factor motivated by the timelike kaon form factor

$$F(E_i) = \frac{1}{E_i^2 - m_J^2}$$

$J = 0$: "f_0" state (0^+)

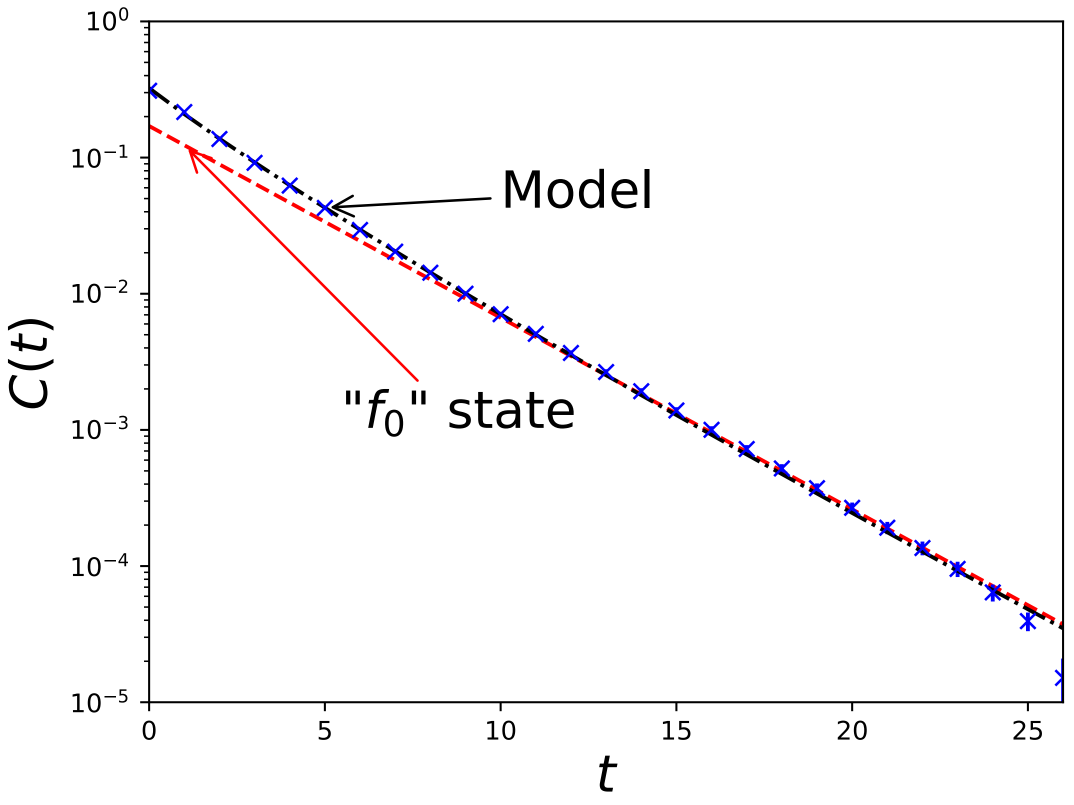
$J = 1$: ϕ meson (1^-)

Excited states

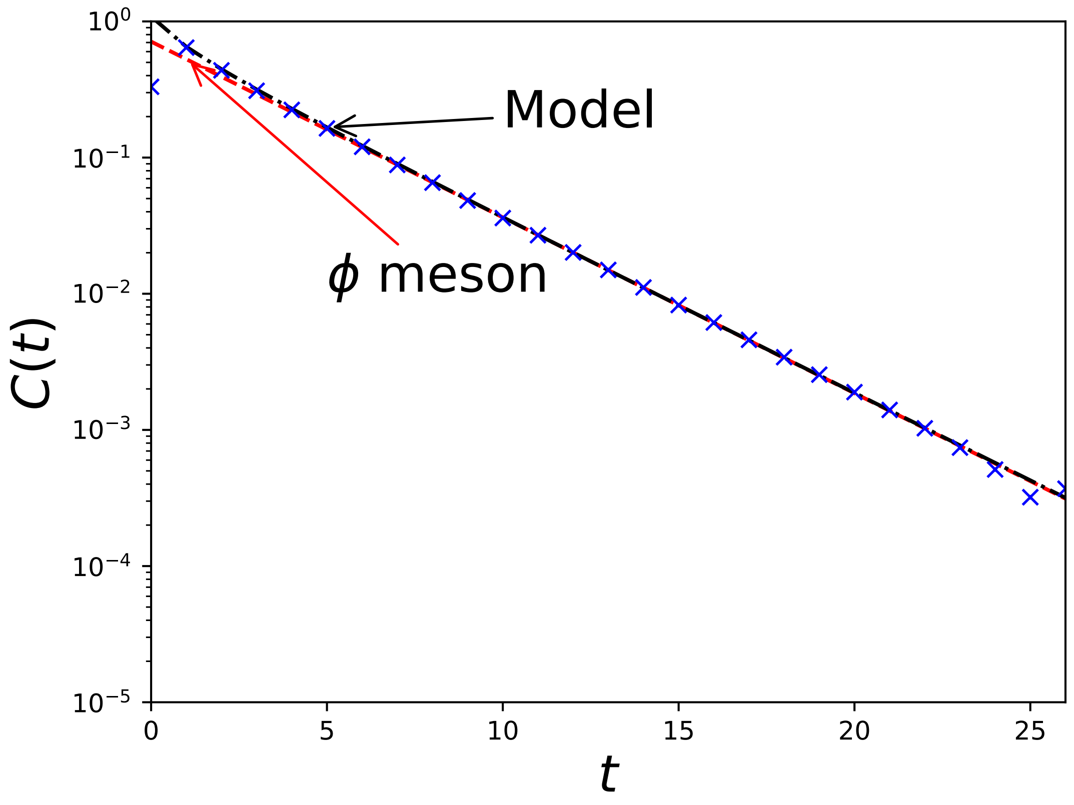
E_i and A_i extracted from the model

Numerical Analysis – Correlator fit

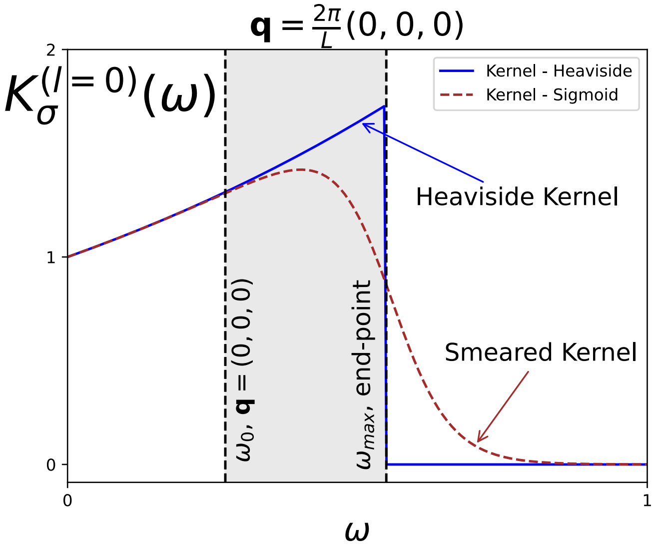
$\langle D_s | A_4 A_4 | D_s \rangle$



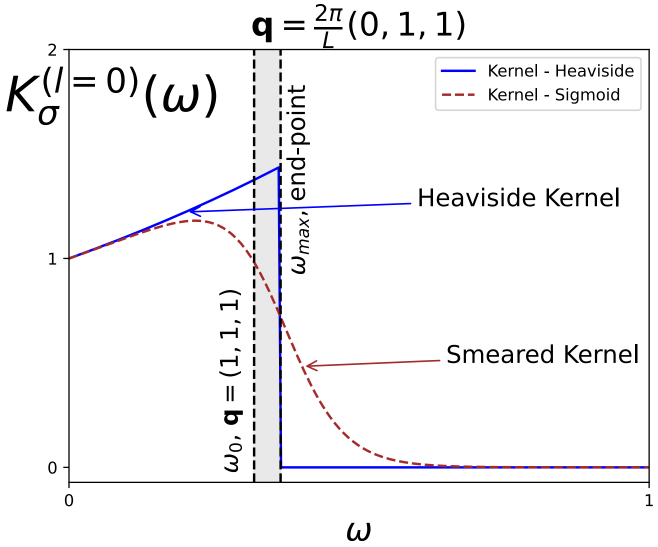
$\langle D_s | A_3 A_3 | D_s \rangle$



Finite volume – higher momentum

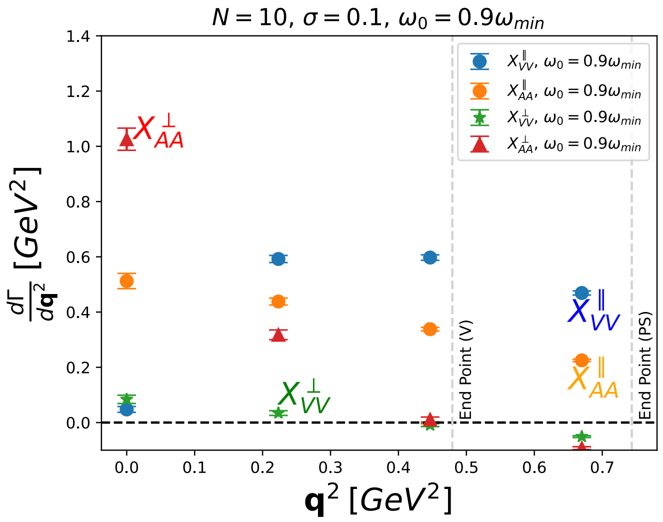


$$\boldsymbol{q} = (0,0,0) \rightarrow (0,1,1)$$



Approximation becomes harder; expect larger errors

At the same time



Total contribution to $\bar{X}(\boldsymbol{q}^2)$ becomes smaller

Total contribution to error budget?