Searching for the QCD critical point using Lee-Yang edge singularities

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The challenge faced by lattice QCD (LQCD)

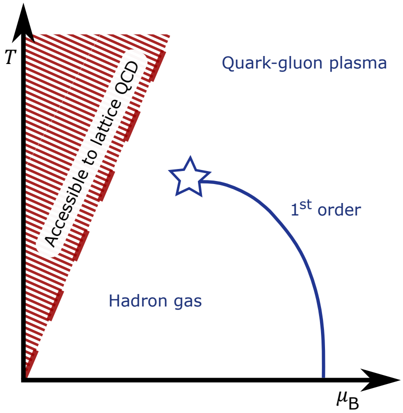
LQCD at $\mu_B = 0$: Straightforward, successful.

The sign problem: Introduction of $\mu_B \in \mathbb{R}$ makes Boltzmann factor complex; can no longer be interpreted as a probability.

Trick: μ_B pure imaginary avoids sign problem; can analytically continue to $\mu_B \in \mathbb{R}$[1,2].

Trick: Expand pressure P/T^4 in μ_B/T[3,4].

...there are others.



[1] P. de Forcrand and O. Philipsen, Nuclear Physics B, 642.1-2, 290–306 (2002).
[2] M. D'Elia and M.-P. Lombardo, Phys. Rev. D, 67.1, 014505 (2003).
[3] C. R. Allton et al., Phys. Rev. D, 66.7, 074507 (2002).
[4] R. V. Gavai and S. Gupta, Phys. Rev. D, 68.3, 034506 (2003).

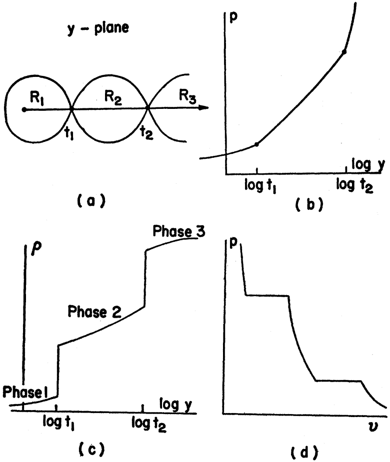
Where do the tricks work?

Tricks work where $\log \mathcal{Z}_{\rm QCD}$ is free of singularities/branch cuts.

Lee-Yang theorem[5]: Zeroes of the partition function that approach the real axis as $V \to \infty$ correspond to phase transitions.

Intuition: Indications of non-analyticities in P

- may hint at phase transitions
- or singularities in $\mathbb{C}$
- constrain validity of Taylor series



[5] C. N. Yang and T. D. Lee, Phys. Rev. 87.3, 404–409 (1952).

Lee-Yang edges and extended analyticity

Ising: Generically have branch cuts on imaginary axis. (Pinch real axis at T_c.)

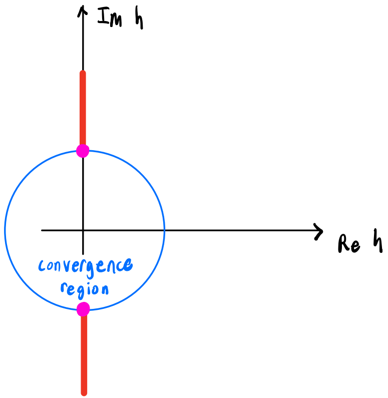
Lee-Yang edge (LYE): The singularities closest to real axis.

Extended analyticity conjecture[6]: LYE is the nearest singularity to the origin.

LYE position fixed at

$$z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

with $z \equiv th^{-1/\beta\delta}$ and critical exponents β, δ.



[6] P Fonseca and A Zamolodchikov, J. Stat. Phys. 110, 527–590 (2003).

Padé approximants

Want detailed information about singularities $\Rightarrow$ **rational functions**,

$$R_n^m(x) \equiv \frac{\sum_{i=0}^m a_i x^i}{1 + \sum_{j=1}^n b_j x^j}.$$

- Singularities captured or mimicked by zeros in denominator
- Useful for resummation (see e.g. Jishnu's talk)

Let f have a formal Taylor series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Padé approximant of order $[m, n]$: R_n^m with coefficients so that it equals the Taylor series up to order $m + n$. Gives relationship between coefficients a_i, b_j, c_k.

Padé approximants

Things to think about with Padé:

- Theorem: Unique when it exists
- Theorem: $[m,n]$ converges to f exactly as $m\to\infty$ when f has pole of order n
- Other properties deduced from numerical experiments
- Limited by number of known Taylor coefficients
- Only have up to 8^{th} order for $\log\mathcal{Z}_{\text{QCD}}$; difficultly increases drastically for higher orders[7]

[7] Computational requirements of HotQCD EoS exceed 2000 GPU-years and 2.4 PB.

Multi-point Padé approximants

Padé approximants you get by demanding[8]

$$R_n^m(x) = f^{m+n}(x) \equiv \sum_{i=0}^{m+n} c_k x^k.$$

Multi-point Padé: The R_n^m satisfying

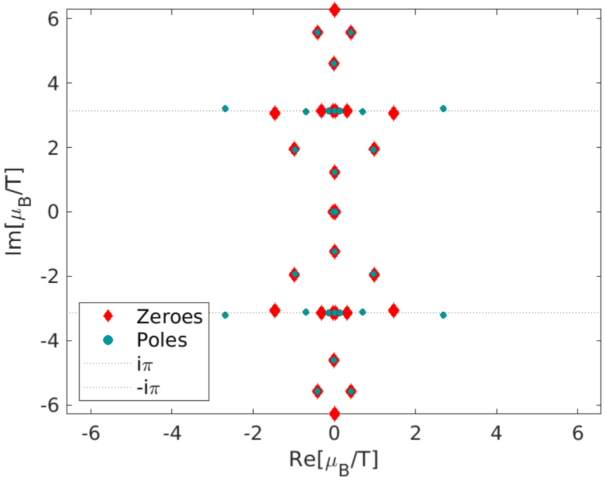
$$R_n^m(x_1) = f^{m+n}(x_1), \quad R_n^m(x_2) = f^{m+n}(x_2), \quad \dots, \quad R_n^m(x_N) = f^{m+n}(x_N)$$

for N known points x_ℓ. Some pros/cons:

- Need fewer Taylor coefficients!
- Less seems to be known about them...

[8] One expects corresponding relationships among derivatives of R and f.

Extracting a LYE[9]



[9] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).

The strategy

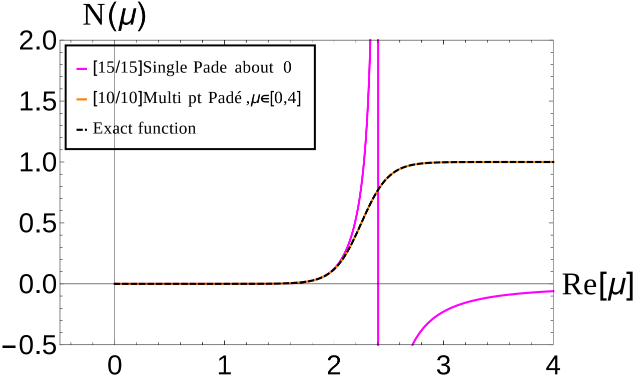
Roughly follow this procedure:

1. What transition are you interested in?
2. How should the singularities scale?
3. Find singularities with multi-point Padé.
4. Does scaling match expectation?
5. Analytically continue results to real μ_B.

But first: Is it trustworthy?

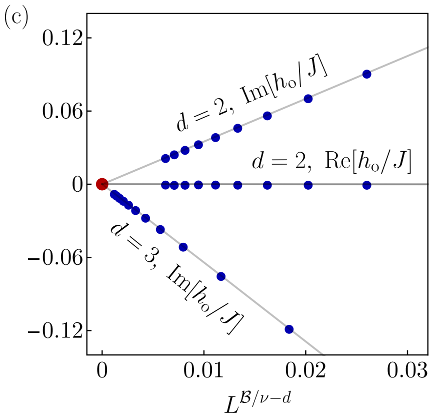
Test: 1-d Thirring model[10,11]

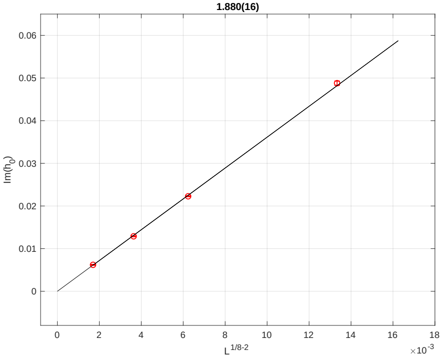
Number density $N(\mu)$ can be worked out exactly.



Multi-point captures the exact $N(\mu)$ well, outperforms single point.

[10] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).
[11] F. Di Renzo, S. Singh, and K. Zambello, Phys. Rev. D, 103.3, 034513 (2021).





Reproduces correct scaling and critical exponents extremely well.

[12] A. Deger and C. Flindt, Phys. Rev. Research, 1.2, 023004 (2019).
[13] F. Di Renzo and S. Singh *Lattice2022 proceedings*.

Test: The Roberge-Weiss transition[15]

$\mathcal{Z}_{\text{QCD}}$ at $\hat{\mu}_f = i\hat{\mu}_I$ has $\mathbb{Z}_3$ periodicity

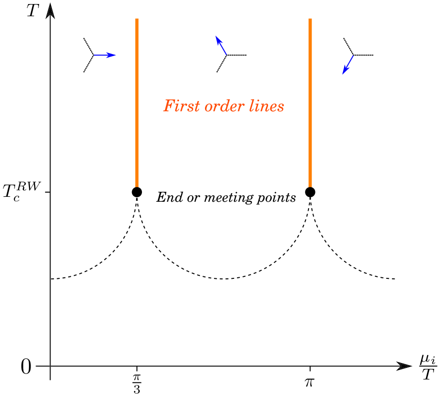
$$\hat{\mu}_I \to \hat{\mu}_I + 2\pi n/3$$

with $\hat{\mu} \equiv \mu/T$. First order lines separate phases distinguished by Polyakov loop

$$P \sim \sum_{\vec{x}} \text{tr} \prod_{\tau} U_4(\vec{x}, \tau).$$

Endpoint in 3-d, $\mathbb{Z}_2$ universality class. Critical exponents[14]:

$$\beta = 0.3264, \qquad \delta = 4.7898$$



[14] S. El-Showk et al., J Stat Phys, 157.4-5, 869–914 (2014).

[15] F. Cuteri et al., Phys. Rev. D, 106.1, 014510 (2022).

Test: The Roberge-Weiss transition[16,17]

Lattice setup:

- 2+1 dynamical HISQ quarks
- m_s/m_l fixed to physical value
- $N_\tau = 4$, 6 with $N_s/N_\tau = 6$

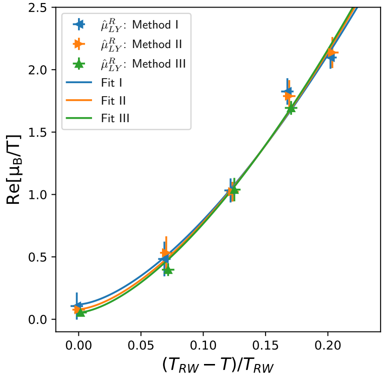
$$h \sim \hat{\mu}_B - i\pi \qquad t \sim T - T_{\rm RW}$$

$$z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

$$\mathrm{Re}\,\hat{\mu}_{\mathsf{LY}} = \pm\pi \left(\frac{z_0}{|z_c|}\right)^{\beta\delta} \left(\frac{T_{\rm RW} - T}{T_{\rm RW}}\right)^{\beta\delta}$$

$$\mathrm{Im}\,\hat{\mu}_{\mathsf{LY}} = \pm\pi$$

Taking $|z_c| = 2.032$ yields $z_0 \in [9.2, 9.5]$.



Taking $T_{\rm RW}^{N_\tau=4} = 201.4$ MeV yields $\beta\delta \approx 1.5635$, compare 1.563495(15).

Cont. est. $T_{\rm RW} = 207.1(2.4)$ MeV, compare 208(5) MeV.

[16] C. Bonati et al., Phys. Rev. D, 93.7, 074504 (2016).
[17] A. Connelly et al., Phys. Rev. Lett. 125.19, 191602 (2020).

Toward the CEP

Assuming multi-point Padé reliable, turn attention to CEP. Also in 3-d, $\mathbb{Z}_2$ universality class, so $\beta\delta \approx 1.5$. Exact mapping to Ising not yet known. Linear ansatz:

$$t = \alpha_t \Delta T + \beta_t \Delta \mu_B$$
$$h = \alpha_h \Delta T + \beta_h \Delta \mu_B,$$

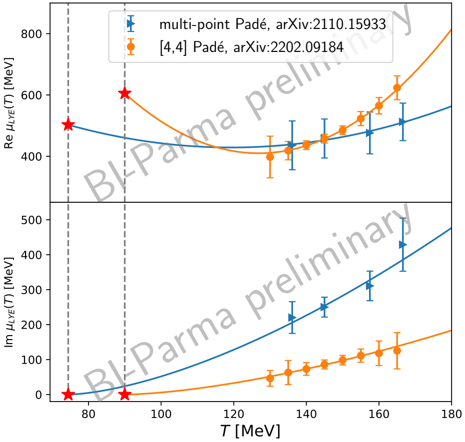
where $\Delta T \equiv T - T^{\text{CEP}}$ and $\Delta \mu_B \equiv \mu_B - \mu_B^{\text{CEP}}$, which leads to[18]

$$\mu_{\text{LY}} = \mu_B^{\text{CEP}} - c_1 \Delta T + i c_2 |z_c|^{-\beta\delta} \Delta T^{\beta\delta} + \mathcal{O}\left(\Delta T^2\right).$$

Expectation from lattice[19]: $\mu_B^{\text{CEP}}/T^{\text{CEP}} \gtrsim 3$. Norbert's talk: $\mu_B \gtrsim 400$ MeV.

[18] M. A. Stephanov, Phys. Rev. D, 73.9, 094508 (2006).
[19] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).

Toward the CEP



Some comments:

- Orange data smaller N_s/N_τ
- Orange data $\mu_S = 0$
- Orange data $N_\tau = 8$
- Blue data $\mu_s = \mu_\ell$
- Blue data $N_\tau = 6$
- Need lower T to control $\mathrm{Re}\,\mu_B$
- Not contradicting other estimates

Suggestion of CEP $T \sim 80$ MeV.

Summary and Outlook

- Tested on Thirring and Ising models
- Consistent with T_{RW} on coarse lattices
- Possible indication of CEP around $T \approx 80$ MeV
- In progress: Detailed analysis of finite size effects (smaller N_s simulations)
- In progress: Examination of chiral transition ($m_s/m_l = 80$ simulations)
- In progress: Continuum limit extrapolations ($N_\tau = 8$ simulations)
- Really need results at lower T

Thanks for your attention.