CPOD2022
Workshop on Critical Point and Onset of Deconfinement

Cumulants from global baryon number conservation with short-range correlations

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Supported by NCN, Grant 2018/30/Q/ST2/00101

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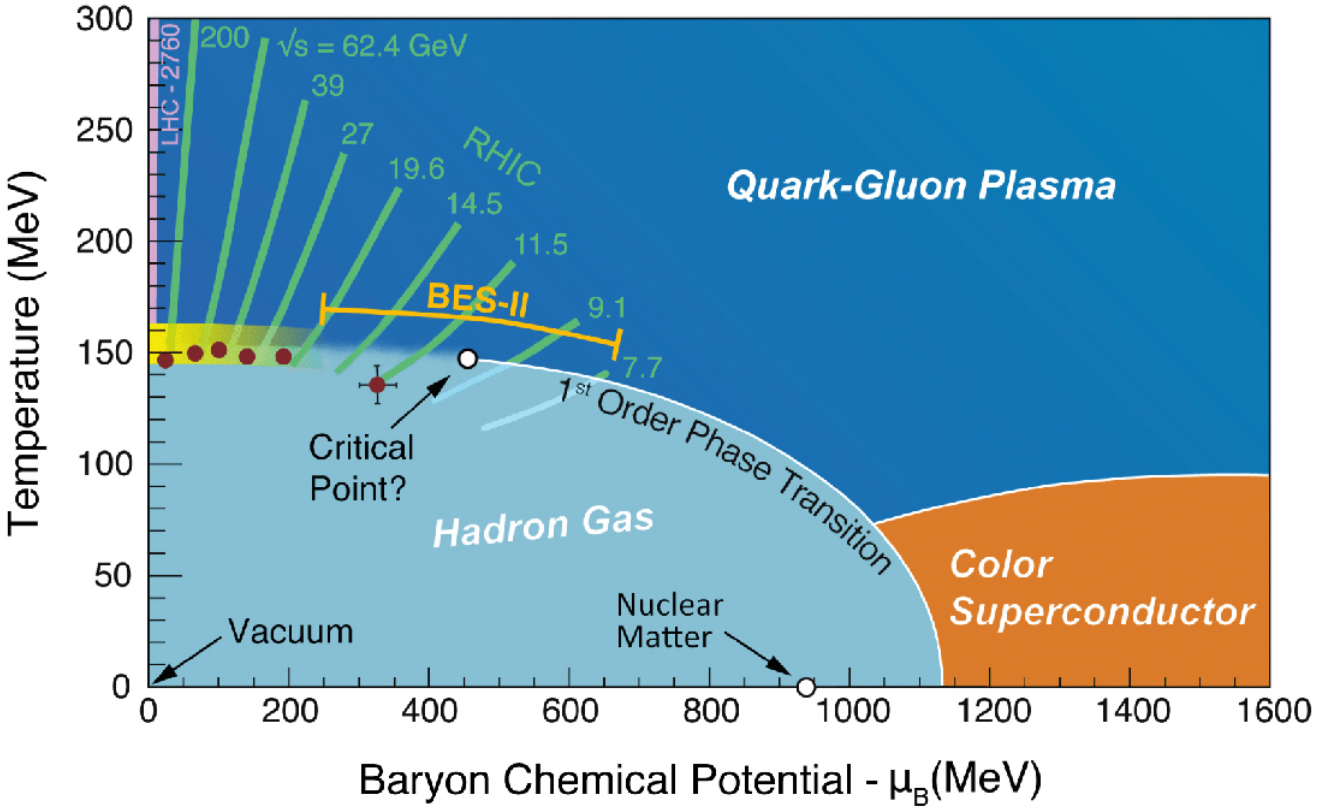
Outline

Motivation

The conjectured QCD phase diagram

- Most of this is only an educated guess based on effective models.
- Search for the critical point - conserved charges fluctuations.
- Experiments: heavy-ion collisions at different energies.



A. Bzdak, S. Esumi, V. Koch, J. Liao, M. Stephanov and N. Xu, Phys. Rept. **853**, 1-87 (2020)
A. Aprahamian, A. Robert, H. Caines, *et al.*, *Reaching for the horizon: The 2015 long range plan for nuclear science*

Cumulants vs factorial cumulants

Cumulants	Factorial cumulants
+ naturally appear in statistical mechanics and in Lattice QCD calculations, + often measured in experiments, – they mix correlation functions of different orders.	+ integrated genuine correlation functions $\Rightarrow$ easier to interpret, – not directly connected with statistical mechanics and rarely measured.
Generating func: $K(t) = \ln\left[\sum_{n=0}^{\infty} P(n) e^{tn}\right]$	Generating func: $G(z) = \ln\left[\sum_{n=0}^{\infty} P(n) z^n\right]$
Cumulants: $\kappa_i = \left.\frac{d^i K}{dt^i}\right\|_{t=0}$	Factorial cumulants: $\hat{C}_k = \left.\frac{d^k G}{dz^k}\right\|_{z=1}$

- Cumulants can be calculated from factorial cumulants:

$$\kappa_n = \sum_{k=1}^{n} S(n,k)\hat{C}_k \,,$$

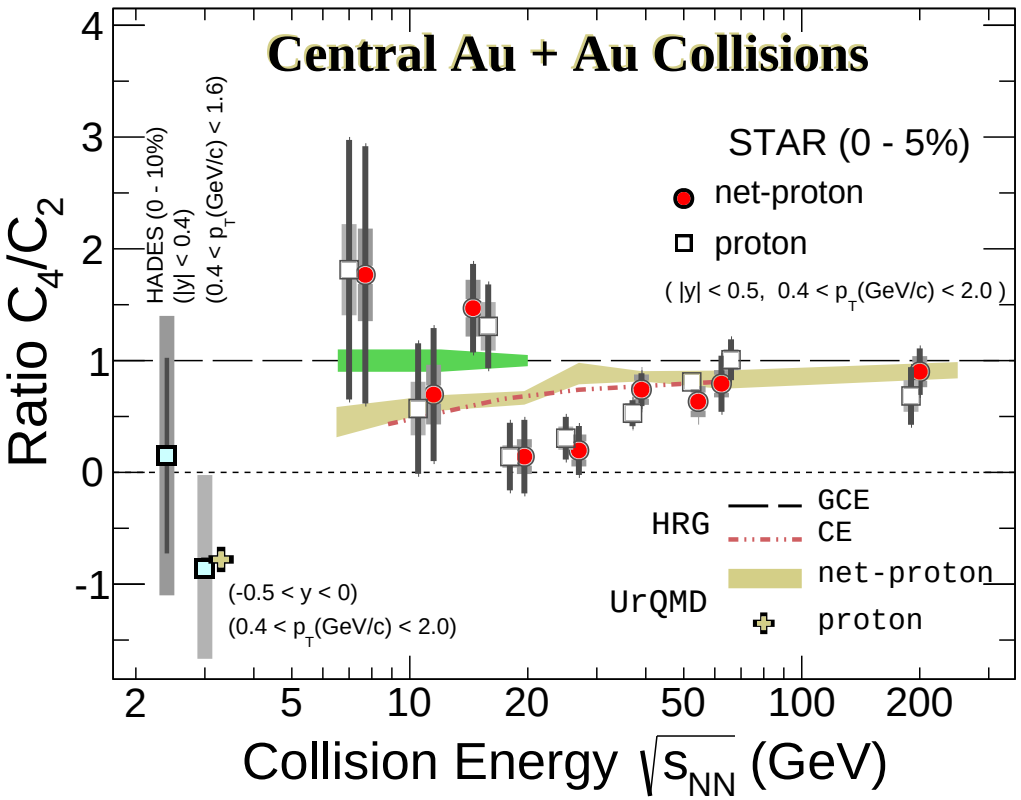
where $S(n,k)$ is the Stirling number of the second kind.

B. Friman and K. Redlich, [arXiv:2205.07332 [nucl-th]].

- For example,
$\kappa_1 = \langle n \rangle = \hat{C}_1,$
$\kappa_2 = \langle n \rangle + \hat{C}_2,$
$\kappa_3 = \langle n \rangle + 3\hat{C}_2 + \hat{C}_3,$
$\kappa_4 = \langle n \rangle + 7\hat{C}_2 + 6\hat{C}_3 + \hat{C}_4.$

STAR and HADES results

κ_4/κ_2 (denoted by STAR as C_4/C_2) depends non-monotonically on energy. Possible signature of the critical phenomena?



M. S. Abdallah *et al.* [STAR], Phys. Rev. Lett. **128**, no.20, 202303 (2022)
J. Adamczewski-Musch *et al.* [HADES], Phys. Rev. C **102**, no.2, 024914 (2020)

Effects contributing to fluctuations

- Fluctuations may reflect the desired phase transition
- ... but they may be due to other usual effects (background):
  - small fluctuations of the impact parameter (and thus the number of wounded nucleons)

    V. Skokov, B. Friman and K. Redlich, Phys. Rev. C **88**, 034911 (2013)
  - **global baryon number conservation**

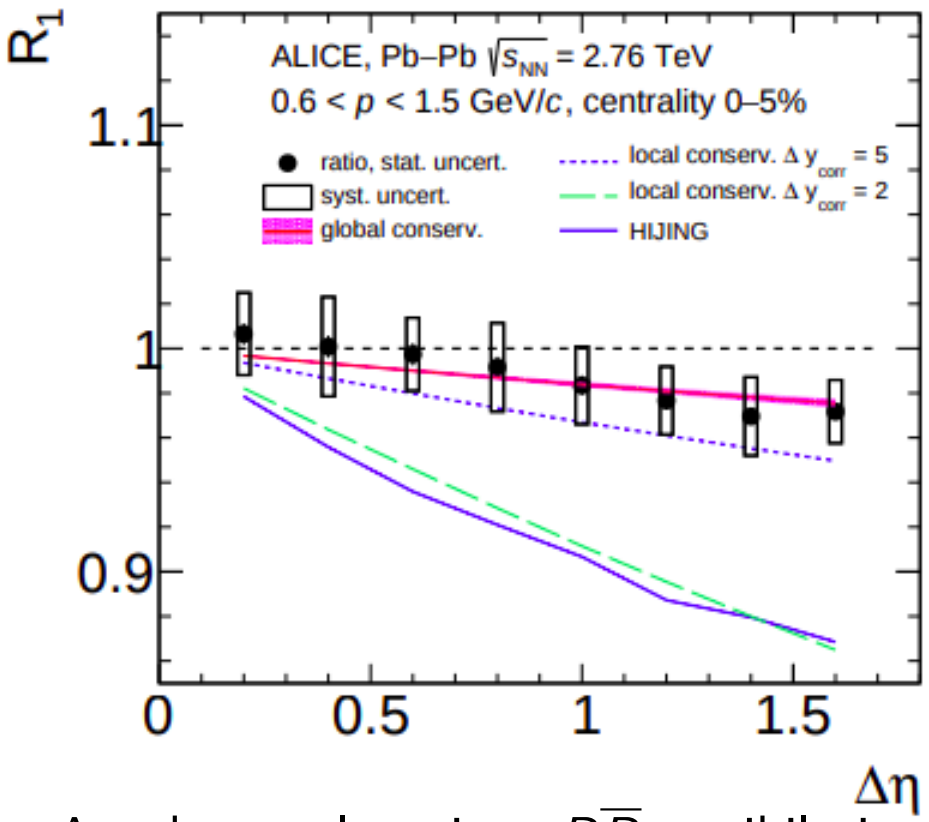
    Baryon number conservation modifies cumulants.

    A. Bzdak, V. Koch and V. Skokov, Phys. Rev. C **87**, no.1, 014901 (2013)
    V. Vovchenko, V. Koch and C. Shen, Phys. Rev. C **105**, no.1, 014904 (2022)
    P. Braun-Munzinger, B. Friman, K. Redlich, A. Rustamov and J. Stachel, Nucl. Phys. A **1008**, 122141 (2021)
- It's essential to know the "background" in order to extract the "signal".

ALICE results

- Global baryon number conservation is favored over local conservation and leads to long-range correlations.

S. Acharya *et al.* [ALICE], Phys. Lett. B **807**, 135564 (2020)



$R_1 = \frac{\kappa_2(n_p - n_{\bar{p}})}{\langle n_p + n_{\bar{p}} \rangle}$

- normalized 2nd net-proton cumulant

- Another explanation: $B\overline{B}$ annihilation + local baryon conservation

O. Savchuk, V. Vovchenko, V. Koch, J. Steinheimer and H. Stoecker, PLB **827**, 136983 (2022)

- It's necessary to study other cumulants and factorial cumulants.

Mixed factorial cumulants can verify global conservation

MB and A. Bzdak, Phys. Rev. C **102**, no.6, 064908 (2020)

- Mixed proton-antiproton factorial cumulants can be measured.
- They carry more information than net-proton cumulants.
- $\hat{C}^{(n,m)}$ - n-proton and m-antiproton factorial cumulant from global baryon number conservation.

$$\hat{C}^{(1,0)} = p\langle N_b\rangle_c$$

$$\hat{C}^{(2,0)} = -p^2\left(\langle N_b\rangle_c + \Delta\right)$$

$$\boldsymbol{\hat{C}^{(1,1)} = -p\bar{p}\Delta}$$

$$\hat{C}^{(3,0)} = p^3\left[2!\left(\langle N_b\rangle_c + \Delta + \tfrac{1}{2}\gamma\right)\right]$$

$$\hat{C}^{(2,1)} = p^2\bar{p}\,\gamma$$

$$\hat{C}^{(4,0)} = -p^4\left[3!\left(\langle N_b\rangle_c + \Delta + \tfrac{1}{2}\gamma\right) + \beta\right]$$

$$\hat{C}^{(3,1)} = -p^3\bar{p}\beta$$

$$\hat{C}^{(2,2)} = -p^2\bar{p}^2\left(\beta - \gamma\right)$$

Notation:

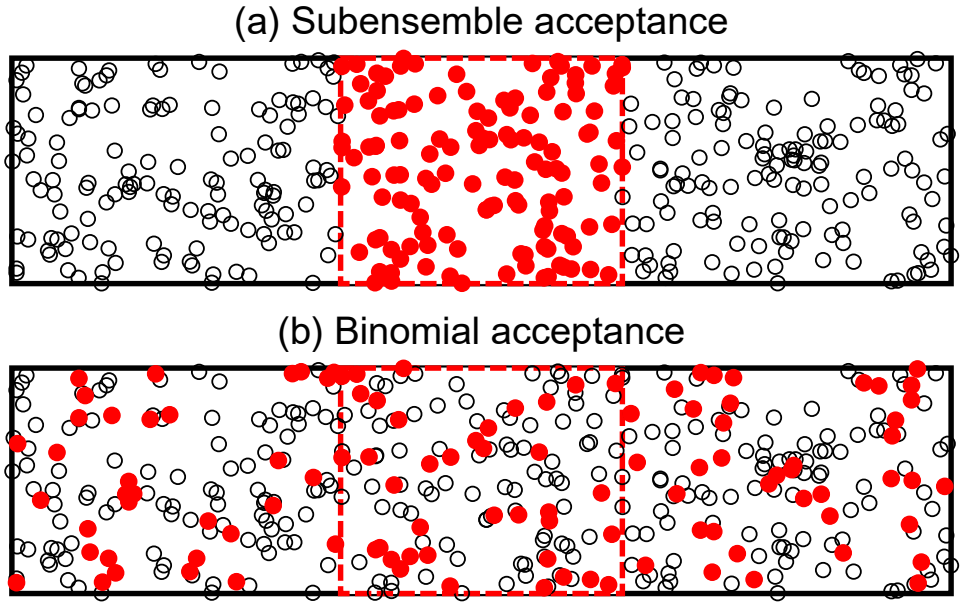
$p(\bar{p})$ - probability that a (anti)baryon is inside the acceptance and measured as (anti)proton,
$\langle N_b\rangle(\langle\bar{N}_b\rangle)$ - mean (anti)baryon number w/o baryon conservation,
$\langle N_b\rangle_c(\langle\bar{N}_b\rangle_c)$ - mean (anti)baryon number with baryon conservation,
$z = \sqrt{\langle N_b\rangle\langle\bar{N}_b\rangle}$, $z_c = \sqrt{\langle N_b\rangle_c\langle\bar{N}_b\rangle_c}$,
$\langle N\rangle_c = \langle N_b\rangle_c + \langle\bar{N}_b\rangle_c$,
$\Delta = z_c^2 - z^2$,
$\gamma = z_c^2 + \Delta\langle N\rangle_c$,
$\beta = \gamma(\langle N\rangle_c + 2) + 2\Delta^2$.

Net-baryon cumulants from baryon conservation and short-range correlations

V. Vovchenko, O. Savchuk, R.V. Poberezhnyuk, M.I. Gorenstein, V. Koch,

Phys. Lett. B **811**, 135868 (2020)

- Subensemble acceptance method (short-range correlations).
- Statistical mechanics, thermodynamic limit.
- They derived net-baryon cumulants with baryon number conservation and short-range correlations in terms of cumulants without baryon conservation.



Problem formulation and method

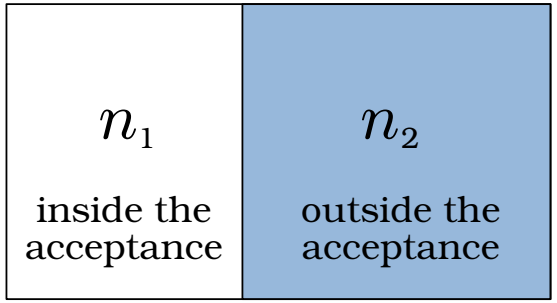
Goal:
calculate the baryon number cumulants
inside the acceptance
with global baryon number conservation
and short-range correlations

MB and A. Bzdak, Phys. Rev. C 106, no. 2, 024904 (2022)

MB and A. Bzdak, [arXiv:2210.15394 [hep-ph]]

Assumptions

- The system is divided into two subsystems (inside and outside the acceptance).



- Only baryons, no antibaryons

  (applies to low energies where the critical point is expected).
- Probability distribution: $P_B(n_1, n_2) = A\ P_1(n_1) P_2(n_2)\ \underbrace{\delta_{n_1+n_2,B}}_{\text{B conservation}}$,

  n_1 baryons in the first subsystem (inside the acceptance),
  n_2 baryons in the second one (outside the acceptance),
  A - normalization const.
- P_1, P_2 include only short-range correlations: $\hat{C}_k^{(i)} = \langle n_i \rangle \alpha_k$, $i = 1, 2$,
  α_k - k-particle short-range correlation strength
- Inside the acceptance: $P_B(n_1) = \sum_{n_2} P_B(n_1, n_2)$.

Calculation

- The factorial cumulant generating function in the 1st subsystem with baryon number conservation:
  $G_{(1,B)}(z) = \ln\left[\sum_{n_1} P_B(n_1) z^{n_1}\right]$,
  $G_{(1,B)}(z) = \ln\left[\frac{A}{B!}\,\frac{d^B}{dx^B}\exp\left(\sum_{k=1}^{\infty}\frac{(xz-1)^k\hat{C}_k^{(1)}+(x-1)^k\hat{C}_k^{(2)}}{k!}\right)\Bigg|_{x=0}\right]$.
- The factorial cumulants in the acceptance can be calculated:

$$\hat{C}_k^{(1,B)} = \frac{d^k}{dz^k}G_{(1,B)}(z)\Bigg|_{z=1}.$$

- Cumulants are calculated from factorial cumulants.
- Details of calculations in our papers.

Results

Cumulants with baryon conservation and short-range correlations can be calculated from the cumulants without baryon conservation.

Cumulants as the power series in terms of B

$$\kappa_n^{(1,B)} \approx \underbrace{\kappa_n^{(1,B,\mathrm{LO})}}_{\substack{\propto B^1 \\ \text{thermodynamic limit}}} + \underbrace{\kappa_n^{(1,B,\mathrm{NLO})}}_{\propto B^0} + \underbrace{\cdots}_{O(B^{-1})}$$

$$\kappa_1^{(1,B)} = fB = f\kappa_1^{(G)}$$

$$\kappa_2^{(1,B,\mathrm{LO})} = \bar{f} f \kappa_2^{(G)}$$

$$\kappa_2^{(1,B,\mathrm{NLO})} = \frac{1}{2}\bar{f} f \frac{(\kappa_3^{(G)})^2 - \kappa_2^{(G)}\kappa_4^{(G)}}{(\kappa_2^{(G)})^2}$$

$$\kappa_3^{(1,B,\mathrm{LO})} = \bar{f} f (1-2f)\kappa_3^{(G)}$$

$$\kappa_3^{(1,B,\mathrm{NLO})} = \frac{1}{2} f \bar{f} (1-2f) \frac{\kappa_3^{(G)}\kappa_4^{(G)} - \kappa_2^{(G)}\kappa_5^{(G)}}{(\kappa_2^{(G)})^2}$$

- LO reproduces net-baryon cumulants from

  V. Vovchenko, O. Savchuk, R.V. Poberezhnyuk, M.I. Gorenstein, V. Koch, PLB **811**, 135868 (2020)

- NLO is new.

$\kappa_n^{(1,B)}$ - cumulants in the subsystem with the baryon conservation and short-range correlations

$\kappa_n^{(G)}$ - short-range cumulants in the whole system without baryon conservation

f - a fraction of particles in the acceptance, $\bar{f} = 1 - f$

Cumulants as the power series in terms of B

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$$\kappa_4^{(1,B,\mathrm{LO})} = f\bar{f}\left[\kappa_4^{(G)} - 3f\bar{f}\left(\kappa_4^{(G)} + (\kappa_3^{(G)})^2/\kappa_2^{(G)}\right)\right]$$

$$\kappa_4^{(1,B,\mathrm{NLO})} = \frac{1}{2}f\bar{f}\left\{\frac{\kappa_3^{(G)}\kappa_5^{(G)} - \kappa_2^{(G)}\kappa_6^{(G)}}{(\kappa_2^{(G)})^2} + 3f\bar{f}\left[\frac{2(\kappa_3^{(G)})^4 - 5\kappa_2^{(G)}(\kappa_3^{(G)})^2\kappa_4^{(G)} + (\kappa_2^{(G)})^2\kappa_3^{(G)}\kappa_5^{(G)}}{(\kappa_2^{(G)})^4} + \frac{(\kappa_4^{(G)})^2 + \kappa_2^{(G)}\kappa_6^{(G)}}{(\kappa_2^{(G)})^2}\right]\right\}$$

$\kappa_n^{(1,B)}$ - cumulants in the subsystem with the baryon conservation and short-range correlations,
$\kappa_n^{(G)}$ - short-range cumulants in the whole system without baryon conservation,

f - a fraction of particles in the acceptance, $\bar{f} = 1 - f$.

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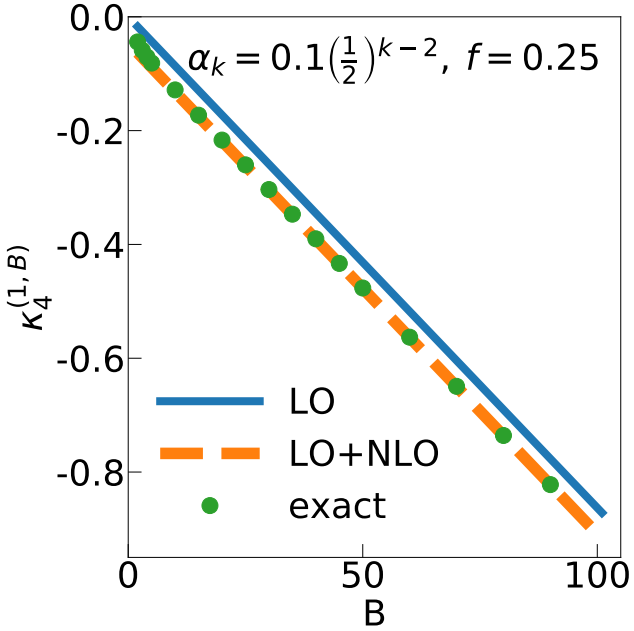
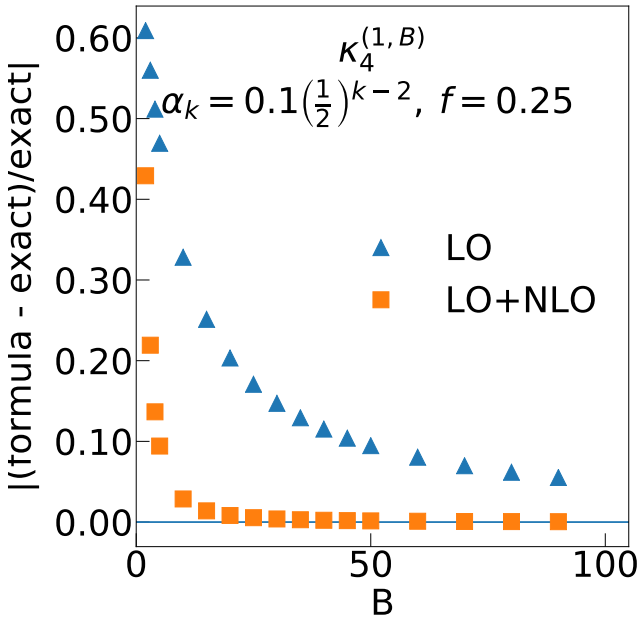
Example





- exact - a straightforward differentiation of the factorial cumulant gen. func.,
- α_k - k-particle short-range correlation strength, $\alpha_k = 0.1\left(\frac{1}{2}\right)^{k-2}$, $k = 2...6$, $\alpha_1 = 1$,
- f - a fraction of particles in the acceptance.

- NLO improves the results.

Remark: α_k's cannot be arbitrary

- α_k - k-particle short-range correlation strength.
- For any probability distribution, $0 \leq P(m) \leq 1$.
  Multiplicity distribution can be written as
  $P(m) = \frac{1}{m!} \frac{d^m}{dz^m} \left[\exp\left(\sum_{k=1}^{\infty} \frac{(z-1)^k}{k!} \hat{C}_k \right) \right] \Big|_{z=0}$,
  $\hat{C}_k$ - kth factorial cumulant of P,
  $\hat{C}_k = \alpha_k \langle N \rangle$ for short-range correlations.
- The even central moments are ≥ 0,
  e.g., variance $\sigma^2 = \langle n^2 \rangle - \langle n \rangle^2 \geq 0$
  $\hat{C}_2 = \frac{d^2 G(z)}{dz^2} \Big|_{z=1} = -\langle n \rangle^2 + \langle n(n-1) \rangle = \alpha_2 \langle n \rangle$
  $\Rightarrow \alpha_2 \geq -1$

Summary

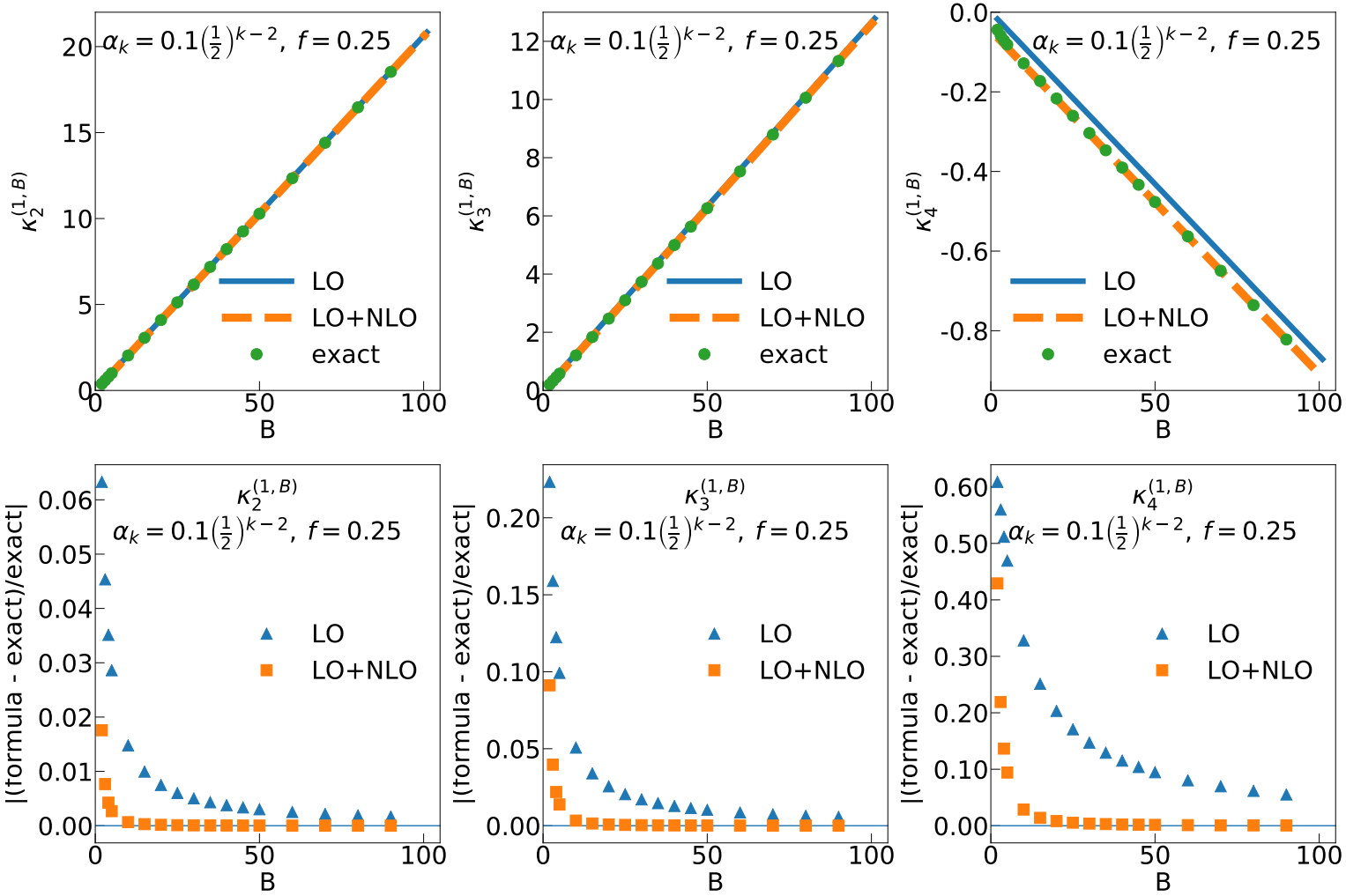
- The mathematical model for calculating cumulants from global baryon conservation and short-range correlations was presented.
- Cumulants with baryon conservation and short-range correlations can be calculated having the cumulants without baryon conservation.
- The resulting cumulants were shown (LO and NLO terms).
- NLO correction improves the results.
- Possible extension: take also antibaryons and long-range correlations into account.

Thank you

Backup

Example

- exact - a straightforward differentiation of the factorial cumulant gen. func.
- α_k - k-particle short-range correlation strength ($\alpha_k = 0.1\left(\frac{1}{2}\right)^{k-2}$, $k = 2...6$, $\alpha_1 = 1$)
- f - a fraction of particles in the acceptance



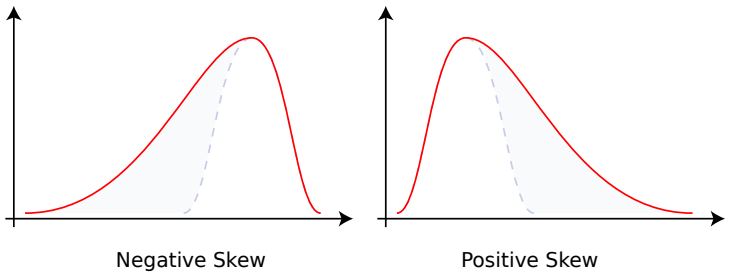
- NLO improves the results.

Quantify fluctuations: cumulants

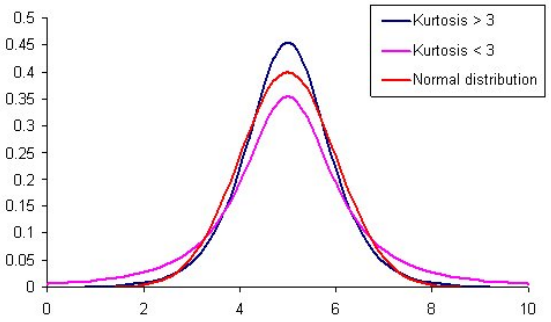
- Cumulant generating function for random variable n:

$$K(t) = \ln\left(\sum_{n=0}^{\infty} P(n) e^{tn}\right)$$

- Cumulants: $\kappa_i = \left.\frac{d^i K}{dt^i}\right|_{t=0}$
- $\kappa_1 = \langle n \rangle$ (mean)
  $\kappa_2 = \mu_2 = \sigma^2$ (variance)
  $\kappa_3 = \mu_3 = S\sigma^3$ (scaled skewness)
  $\kappa_4 = \mu_4 - 3\mu_2^2 = K\sigma^4$ (scaled kurtosis), $\mu_k = \langle (n - \langle n \rangle)^k \rangle$
- Cumulants are broadly studied in heavy ion collisions.



R. Hermans, en.wikipedia.org



vosesoftware.com/riskwiki/Kurtosis(K).php

Quantify fluctuations: factorial cumulants

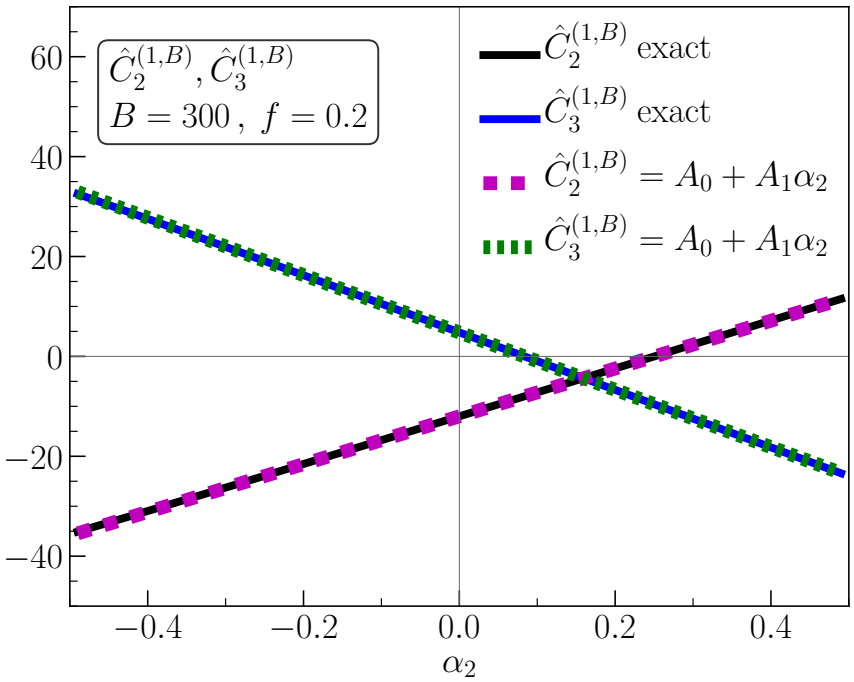
- Factorial cumulant generating function:

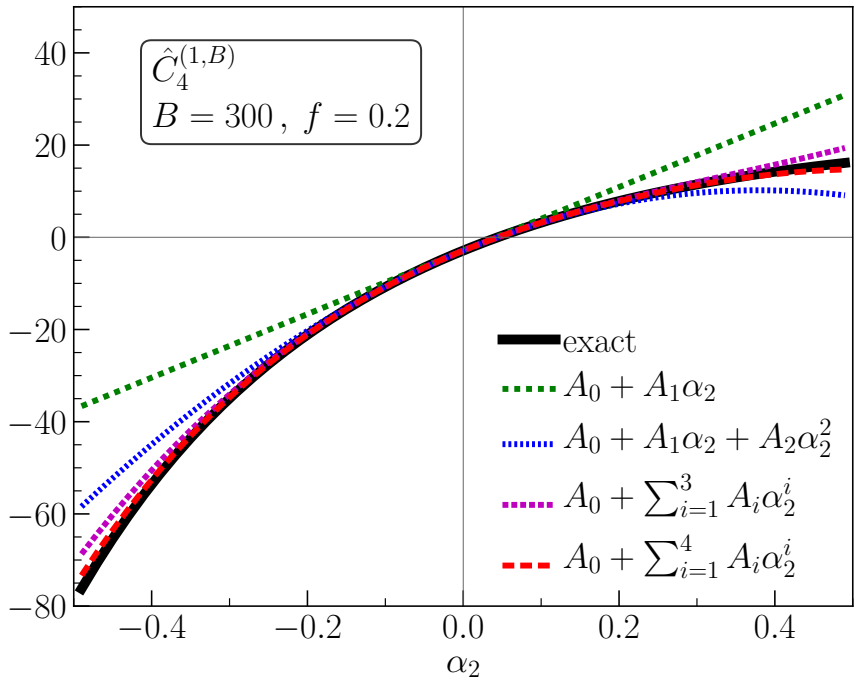
$$G(z) = \ln\left(\sum_{n=0}^{\infty} P(n) z^n\right)$$

- Factorial cumulants: $\hat{C}_k = \left.\frac{d^k G}{dz^k}\right|_{z=1}$
- $C_2(y_1, y_2) = \varrho_2(y_1, y_2) - \varrho(y_1)\varrho(y_2)$ - two-particle density correlation function
- $\hat{C}_2 = \int dy_1 dy_2 C_2(y_1, y_2)$
- No correlations: $\hat{C}_k = 0$ $(k > 1)$

Example

- $B = 300$, $f = 0.2$, $\alpha_k = 0$ for $k > 2$
- $f = \langle n_1 \rangle / \langle N \rangle$ - fraction of particles in the 1st subsystem
- Factorial cumulants in the subsystem assuming global baryon conservation and only 2-particle short-range correlations ($\hat{C}_k^{(i)} = \langle n_i \rangle \alpha_k$, $i = 1, 2$ with $\alpha_k = 0$ for $k > 2$, always $\alpha_1 = 1$).
- Exact vs approximate results.





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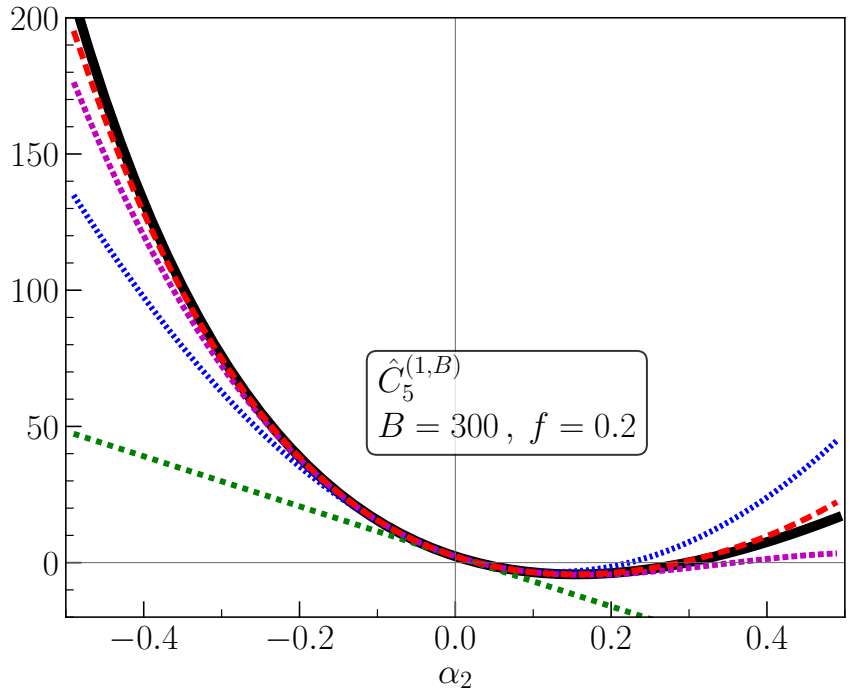

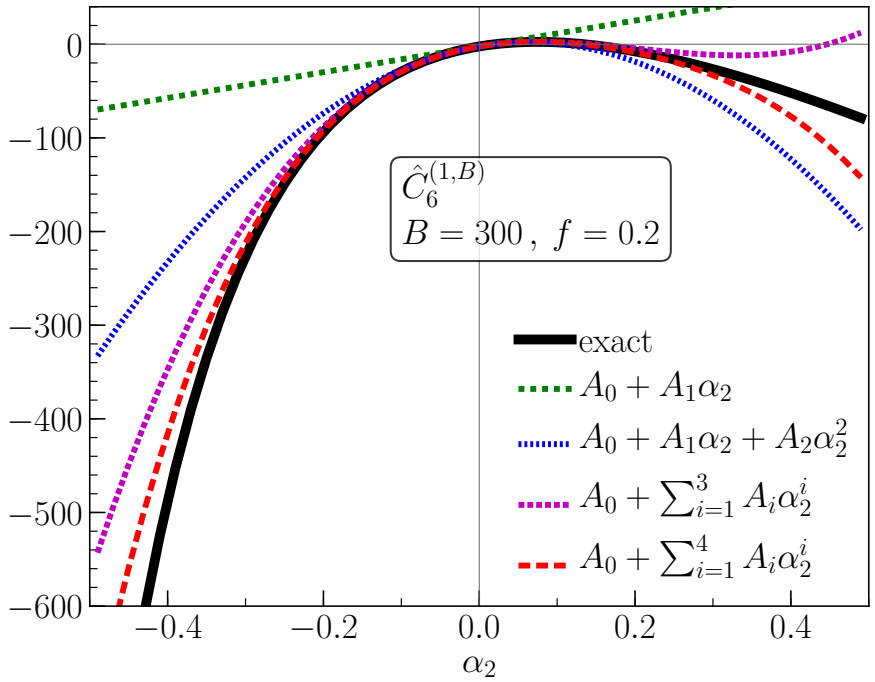

Cumulants vs statistical mechanics

$$\frac{\partial^n \left(P/T^4\right)}{\partial \left(\mu_B/T\right)^n} = \frac{\kappa_n}{VT^3} \,,$$

κ_n - grand canonical cumulant (without baryon number conservation)
μ_B - baryochemical potential

Cumulant with baryon number conservation - follows canonical ensemble.