Characteristic momentum of Hydro+
&
a Bound on the sound speed enhancement near the QCD critical point

CPOD2022 - Workshop on Critical Point and Onset of Deconfinement

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In collaboration with **Matthias Kaminski**
University of Alabama

Outline

1. Hydrodynamics

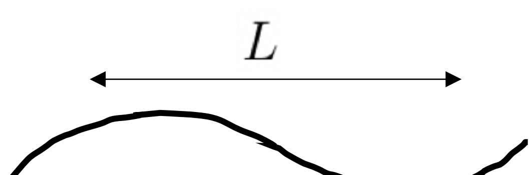
2. Correlation functions

3. A simple application to QCD critical point

Hydrodynamics

Perturbing a thermal equilibrium state:

- When $\ell_{\mathrm{mic}} \ll L$, only conserved quantities are relevant



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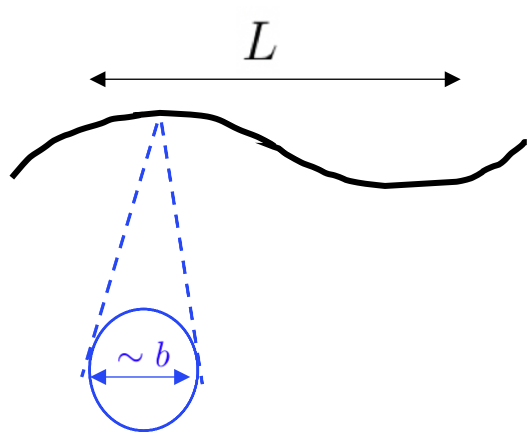

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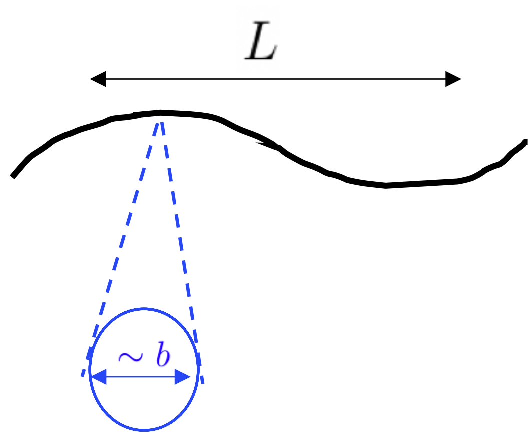

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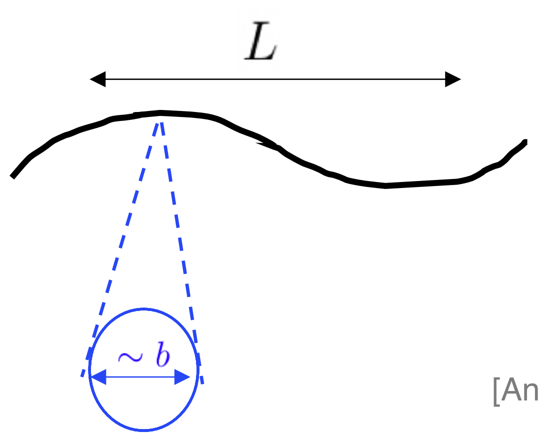

[An, Basar, Stephanov, Yee 1902.09517]

What about 2pt correlators?

Indeed, $\langle \phi_A(t, \boldsymbol{x}_1) \phi_B(t, \boldsymbol{x}_2) \rangle$ must be considered as well! $(\phi = \hat{\psi} - \psi)$

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$$\langle \delta\varrho(\mathbf{r}_1)\, \delta\varrho(\mathbf{r}_2) \rangle = \varrho T (\partial\varrho/\partial P)_T\, \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (88.2)$$

$$\langle \delta P(\mathbf{r}_1)\, \delta P(\mathbf{r}_2) \rangle = \varrho T (\partial P/\partial \varrho)_S\, \delta(\mathbf{r}_1 - \mathbf{r}_2) = \varrho T u^2 \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (88.3)$$

$$\langle \delta s(\mathbf{r}_1)\, \delta s(\mathbf{r}_2) \rangle = (c_p/\varrho)\, \delta(\mathbf{r}_1 - \mathbf{r}_2), \quad (88.4)$$

[Landau Lifshitz Vol9]

Hydro 2pt correlators

Let us define: $\langle \phi_A(t, \boldsymbol{x} + \boldsymbol{y}/2) \phi_B(t, \boldsymbol{x} - \boldsymbol{y}/2) \rangle \equiv G_{AB}(x, \boldsymbol{y})$

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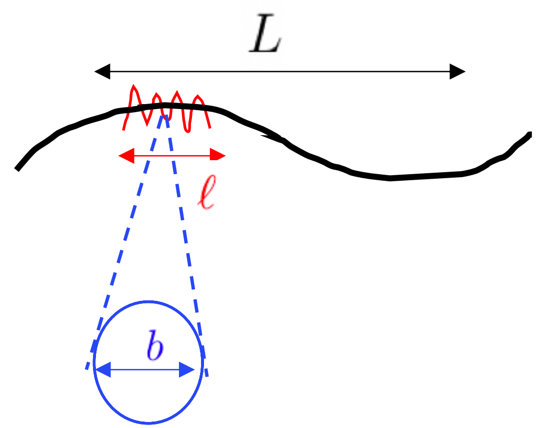

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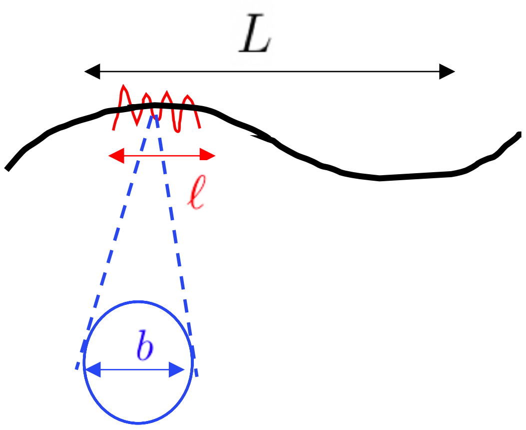

  - The separation of scales $\ell_{\rm mic} \ll b \ll \ell \ll L$
    or equivalently $q \ll Q < \Lambda \ll \ell_{\rm mic}^{-1}$ $(\Lambda \equiv b^{-1})$

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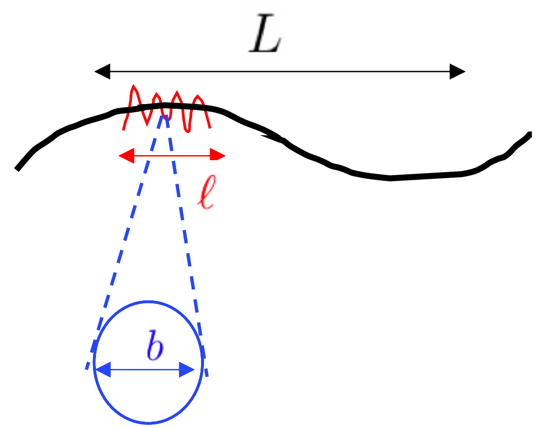

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Suggest to work with

$$G_{\boldsymbol{Q}}(x) = \int_{\boldsymbol{Q}} G(x,\boldsymbol{y})e^{-i\boldsymbol{Q}\cdot\boldsymbol{y}}$$

Hydro fluctuations in static fluid

- $G_{\boldsymbol{Q}}(t, \boldsymbol{x})$ does not depend on $\boldsymbol{x}$.
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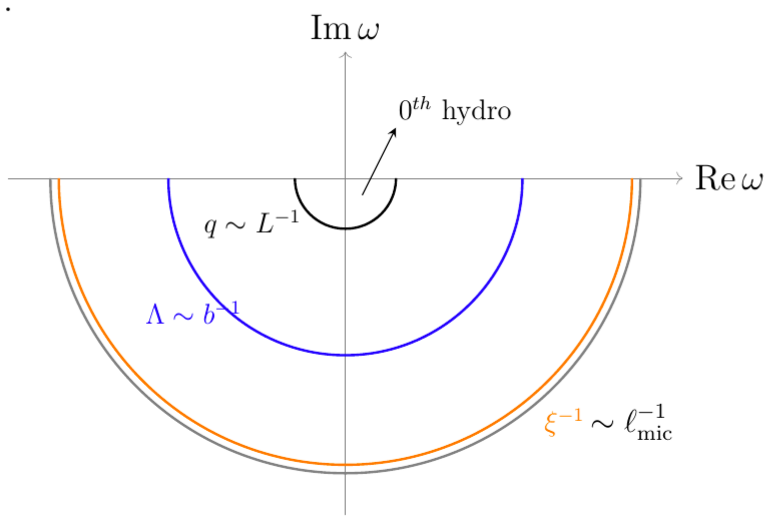

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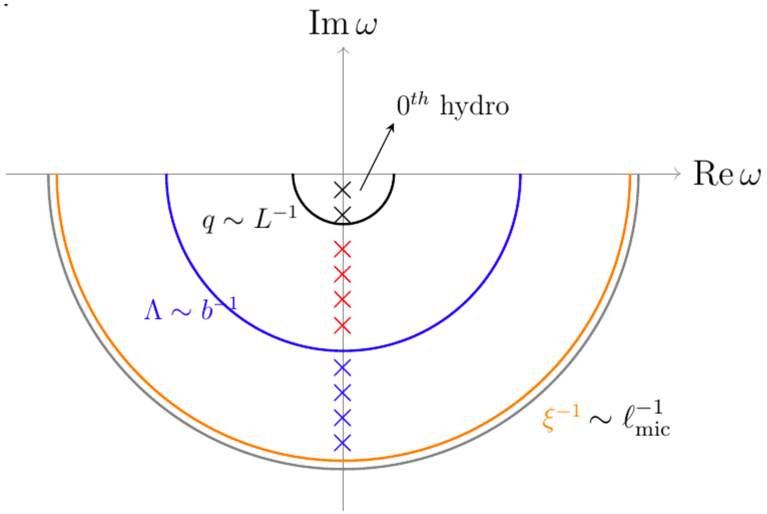

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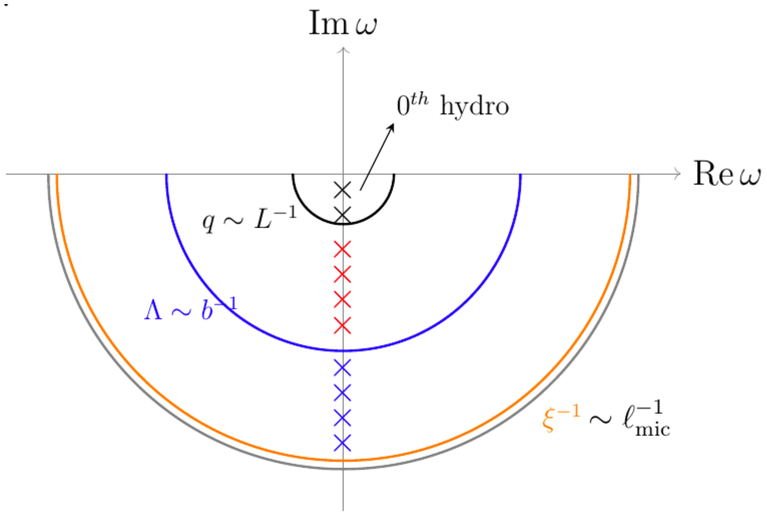

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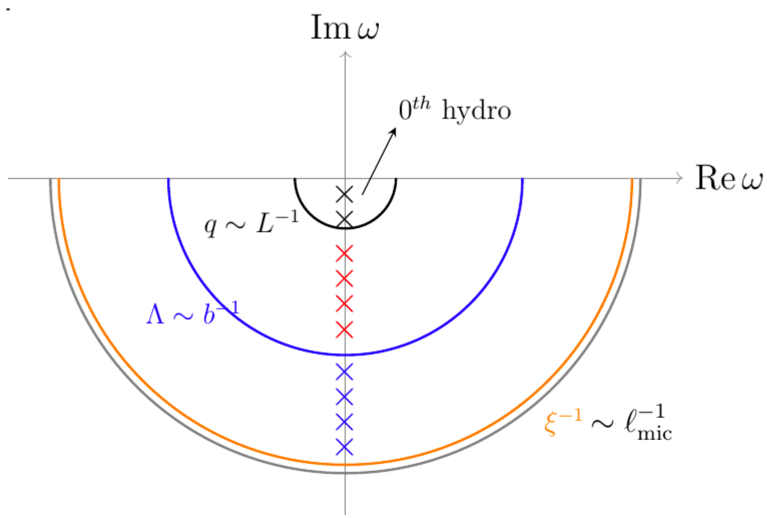

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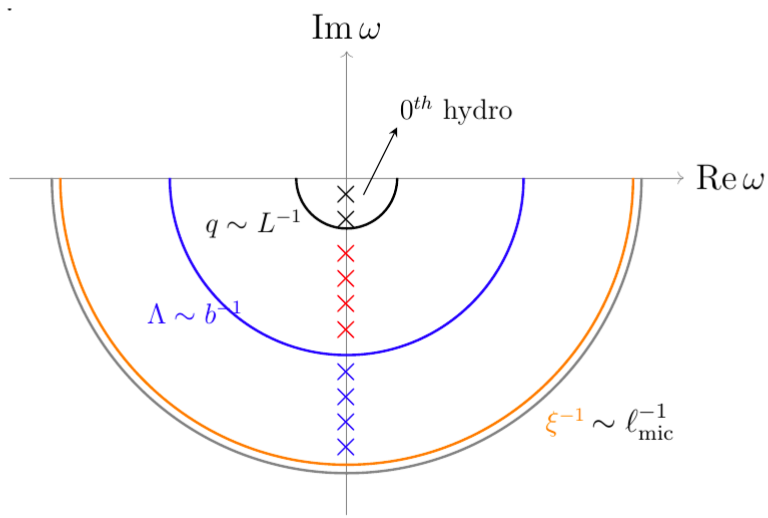

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[Akamatsu, Mazeliauskas, Teaney 1606.07742]

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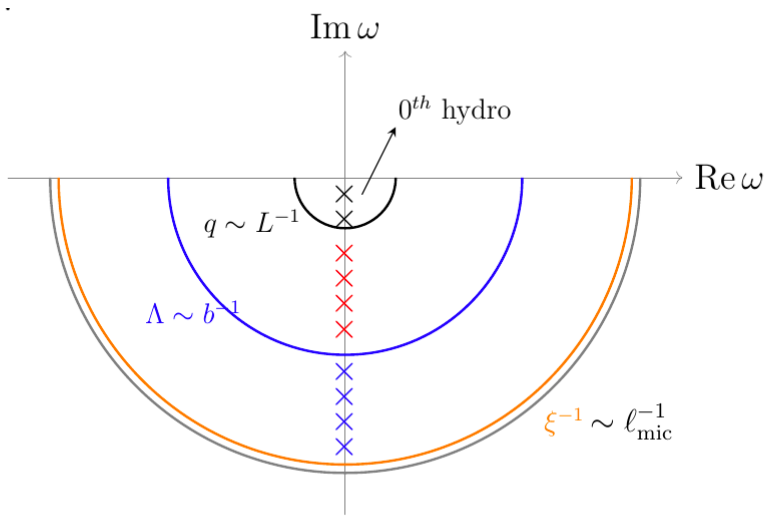

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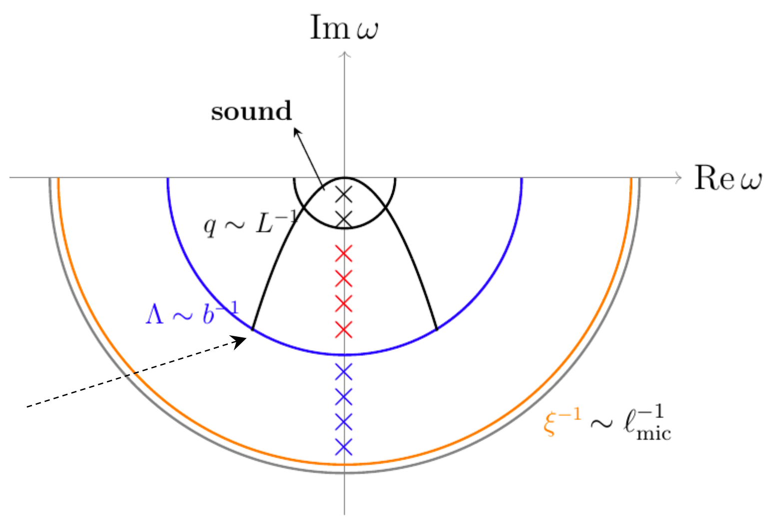

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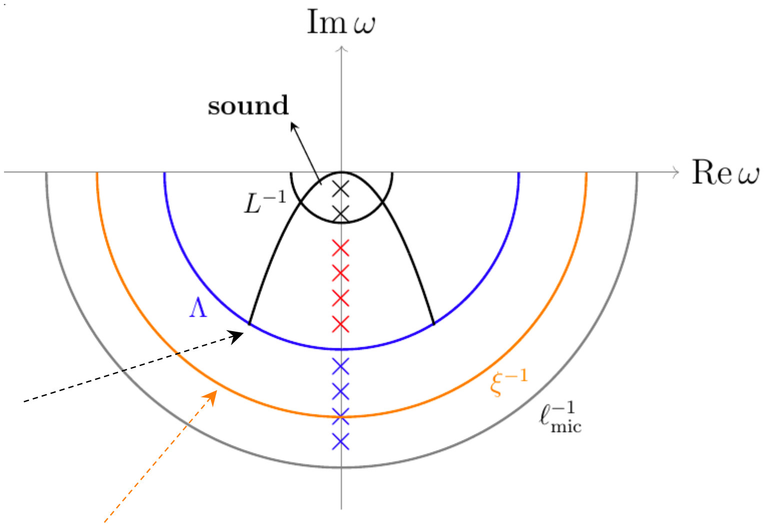

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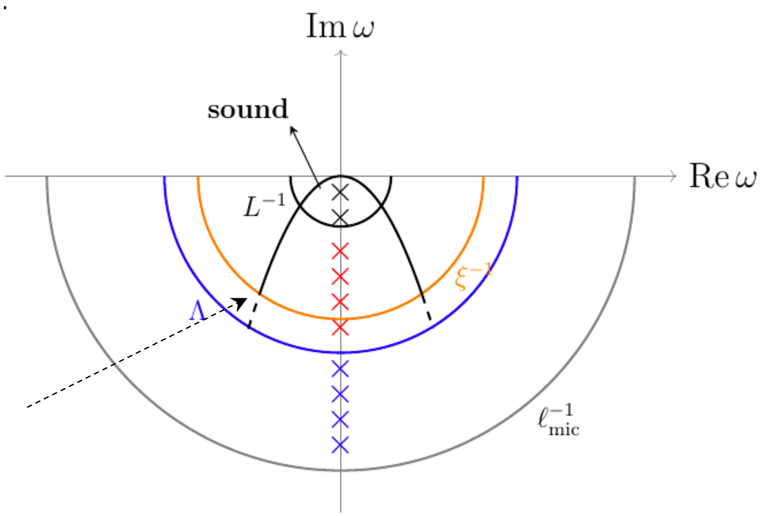

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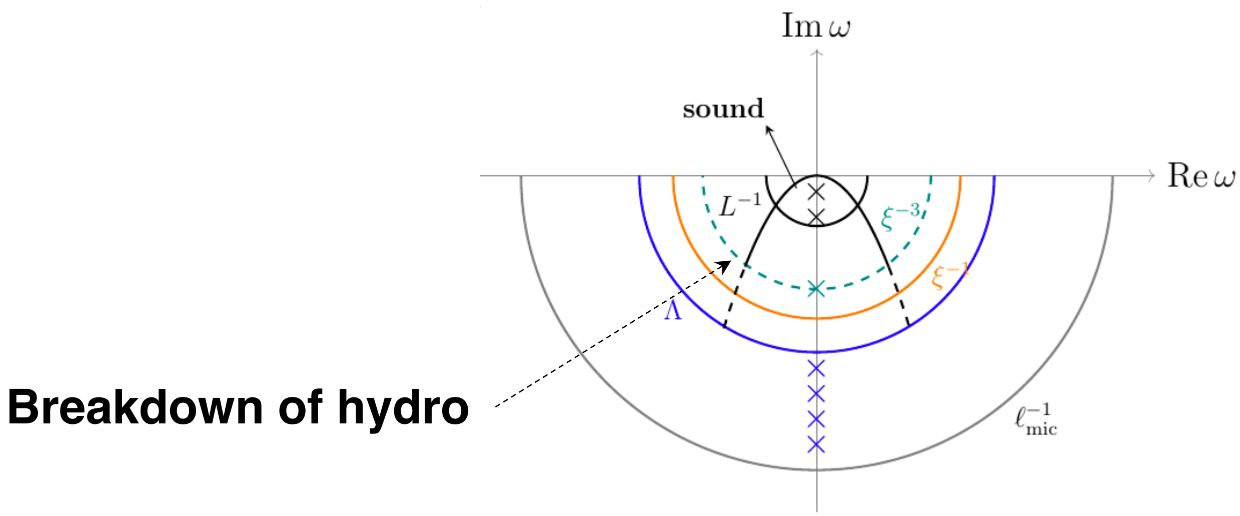

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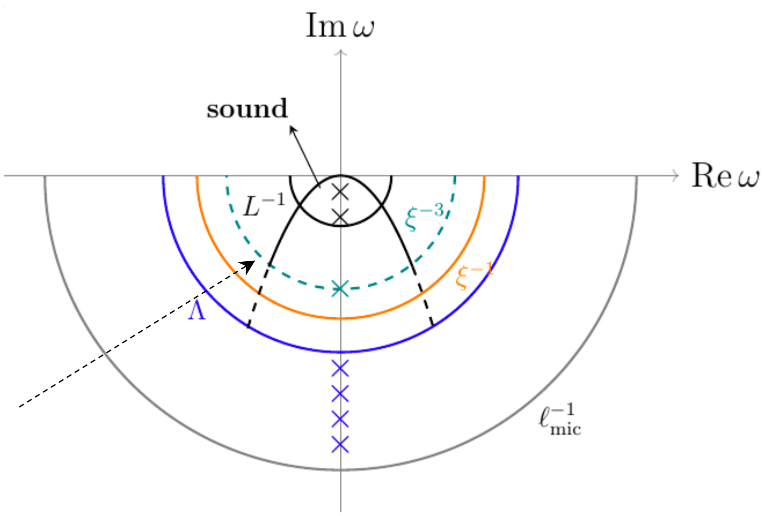

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Near a CP, hydro breaks down at $q \sim \xi^{-3}$.

(Critical) Slowing down & Hydro+

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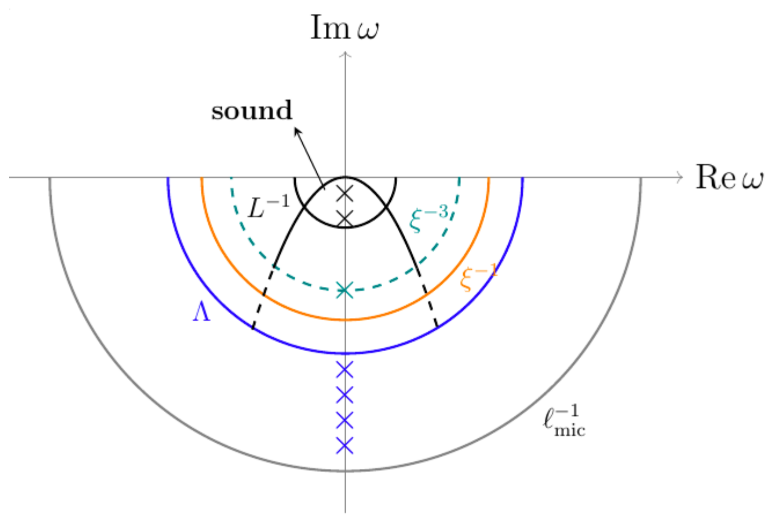

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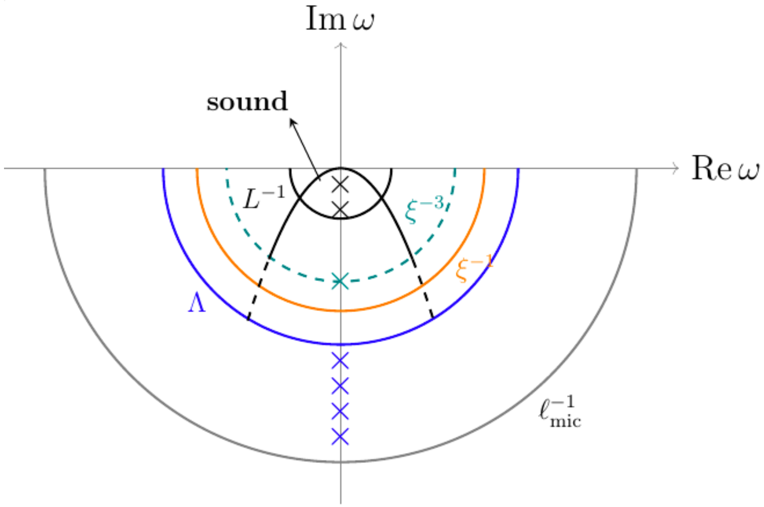
- Relaxation time of the slow mode diverges: **"critical slowing down"**

[Berdnikov, Rajagopal 992274]



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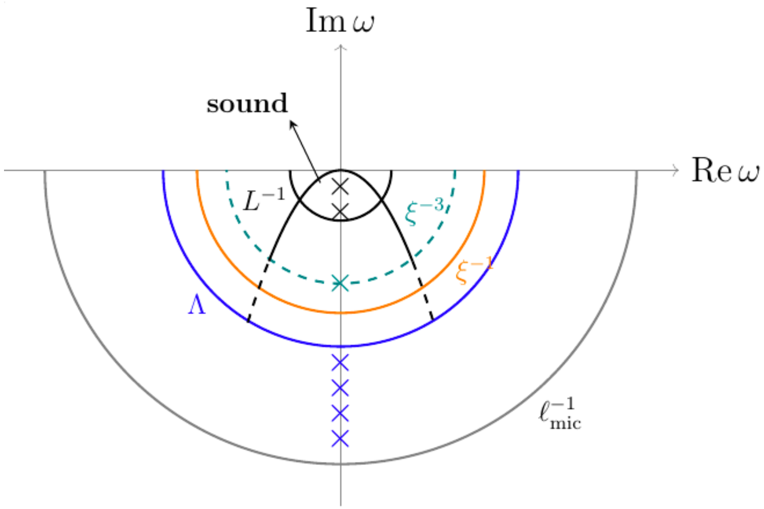

- To extend the hydro regime back to the expected range ,

We must include them in the hydro picture

This is the philosophy of the Hydro+. [Stephanov, Yin 1712.10305]

Hydro+

The simplest setup:

"single-mode hydro+"

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Hydro+ eqs. = hydro eqs. **+** relaxation eq. of the slow mode

$$
\begin{aligned}
D\epsilon &= -w_{(+)}\theta - \partial_{(\mu}u_{\nu)}\,\Pi^{\mu\nu}, \\
Dn &= -n\theta - \partial\cdot\Delta J, \\
w_{(+)}Du^{\nu} &= -\partial_{\perp}^{\nu}p - \delta_{\perp\lambda}^{\nu}\,\partial_{\mu}\Pi^{\mu\lambda}, \\
D\phi &= -\gamma_{\pi}\pi - A_{\phi}\theta + \cdots
\end{aligned}
$$

(+) indicates that thermo functions are now functions of $\epsilon, \ n$ and ϕ

$$
ds_{(+)} = \beta_{(+)}\,d\varepsilon - \alpha_{(+)}\,dn - \pi d\,\phi
$$

Enhancement in c_s^2

- Linearizing equations gives:

$$F(\omega, q^2) = \omega^2 - q^2 \left(c_s^2 + \frac{\omega}{\omega + i\Gamma_\pi} \frac{\beta p_\pi^2}{\phi_\pi w} \right) = 0$$

Due to the slow mode:

$$\boxed{\Delta c_s^2 = \frac{\omega^2}{\omega^2 + \Gamma_\pi^2} \frac{\beta p_\pi^2}{\phi_\pi w}}$$

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- In terms of dimensionless quantities $\mathfrak{w} = \frac{\omega}{\Gamma_\pi}, \quad \mathfrak{q} = \frac{c_s\, q}{\Gamma_\pi}, \quad \alpha = \frac{\Delta c_s^2(\infty)}{c_s^2}$

$$F(\mathfrak{w}, \mathfrak{q}^2) = \mathfrak{w}^2 - \mathfrak{q}^2 \frac{i + \mathfrak{w} + \alpha\, \mathfrak{w}}{i + \mathfrak{w}} = 0$$

[NA, Kaminski 2112.14747]

Spectrum of excitations

- There are three modes:

$$\mathfrak{w}_1(\mathfrak{q}) = -\frac{i}{12}\left(4 + \frac{2^{7/3}(-1+3(1+\alpha)\mathfrak{q}^2)}{3\,\mathcal{D}(\mathfrak{q})} - \frac{2^{2/3}}{3}\mathcal{D}(\mathfrak{q})\right)\,,$$

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Interestingly:

$$\mathcal{D}(\mathfrak{q}) = \left(2i + 9i(2-\alpha)\mathfrak{q}^2 + 3\sqrt{3}\sqrt{-4-4\mathfrak{q}^4(1+\alpha^3)+\mathfrak{q}^2(-8+20\alpha+\alpha^2)}\right)^{1/3}$$

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- 4 branch points:

$$(\mathfrak{q}_1^*)^2 = \frac{\alpha^2 + 20\alpha - 8 + \sqrt{\alpha - 8}\,(\alpha^{3/2} - 8\alpha^{1/2})}{8(1+\alpha)^3}$$
$$(\mathfrak{q}_2^*)^2 = \frac{\alpha^2 + 20\alpha - 8 - \sqrt{\alpha - 8}\,(\alpha^{3/2} - 8\alpha^{1/2})}{8(1+\alpha)^3}$$

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$$\mathfrak{w}_2(\mathfrak{q}) = -\frac{i}{12}\left(4 + \frac{2^{4/3}(-i+\sqrt{3})(-1+3(1+\alpha)\mathfrak{q}^2)}{\mathcal{D}(\mathfrak{q})} - 2^{2/3}(i+\sqrt{3})\mathcal{D}(\mathfrak{q})\right)$$

$$\mathfrak{w}_3(\mathfrak{q}) = -\frac{i}{12}\left(4 + \frac{2^{4/3}(-i-\sqrt{3})(-1+3(1+\alpha)\mathfrak{q}^2)}{\mathcal{D}(\mathfrak{q})} - 2^{2/3}(1+i\sqrt{3})\mathcal{D}(\mathfrak{q})\right)$$

Interestingly:

$$\mathcal{D}(\mathfrak{q}) = \left(2i + 9i(2-\alpha)\mathfrak{q}^2 + 3\sqrt{3}\sqrt{-4-4\mathfrak{q}^4(1+\alpha^3)+\mathfrak{q}^2(-8+20\alpha+\alpha^2)}\right)^{1/3}$$

- 4 branch points:

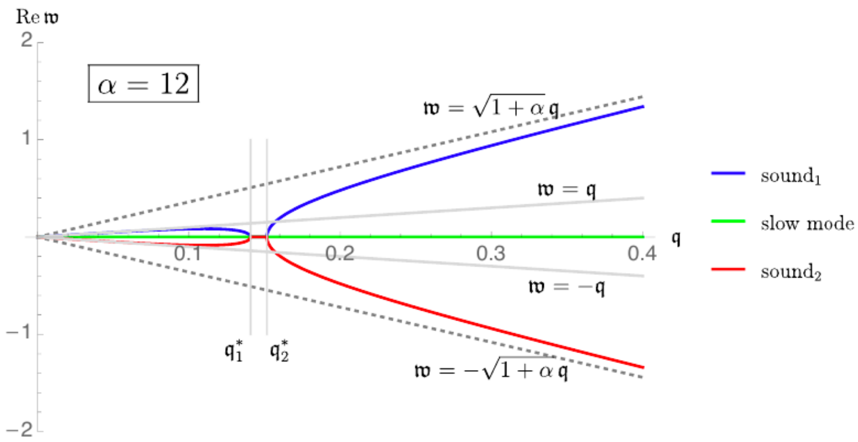
$$(\mathfrak{q}_1^*)^2 = \frac{\alpha^2+20\alpha-8+\sqrt{\alpha-8}\,(\alpha^{3/2}-8\alpha^{1/2})}{8(1+\alpha)^3}$$

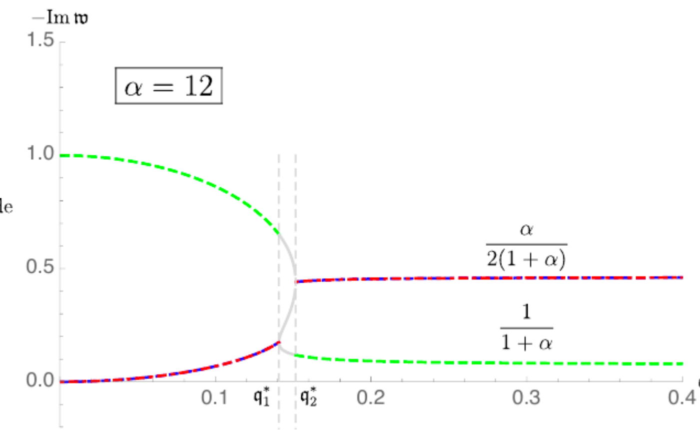
$$(\mathfrak{q}_2^*)^2 = \frac{\alpha^2+20\alpha-8-\sqrt{\alpha-8}\,(\alpha^{3/2}-8\alpha^{1/2})}{8(1+\alpha)^3}$$

[NA, Kaminski 2112.14747]

The story begins…

Characteristic momentum of Hydro+





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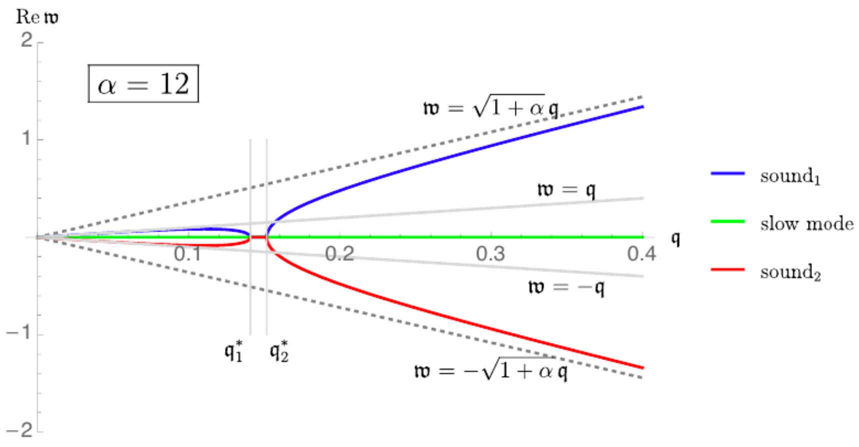

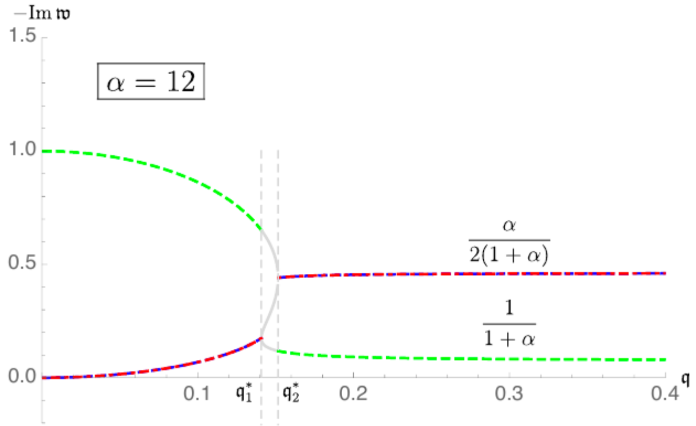

- At $q < q_1$ the slowest modes are the two sound modes
- At $q > q_2$ the slow mode is the slowest mode

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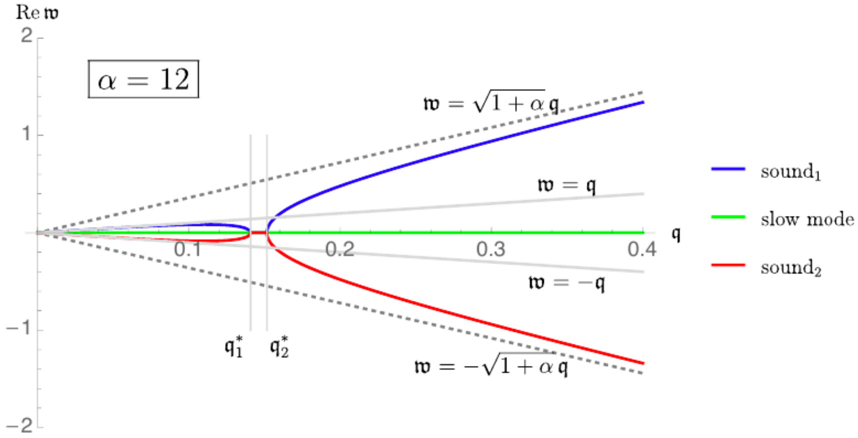

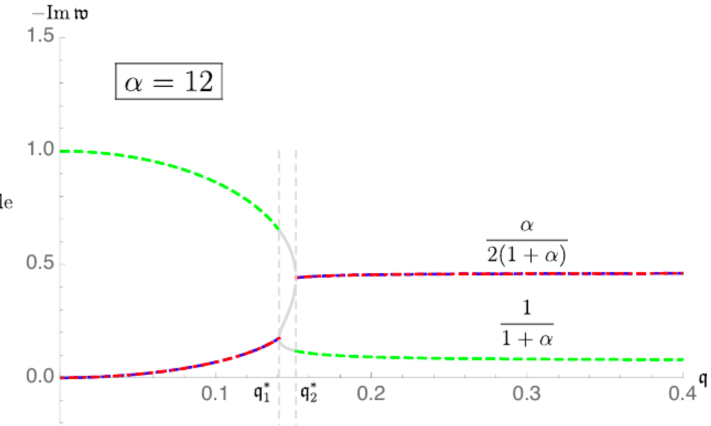

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We refer to $\mathfrak{q}_c = \min\{|\mathfrak{q}_1^*|, |\mathfrak{q}_2^*|\}$ as the

characteristic momentum of Hydro+

beyond which, the standard hydrodynamics breaks down.

[NA, Kaminski 2112.14747]

Hydro+ near the QCD critical point

- The slowest mode is the $G_{\boldsymbol{Q}}(x)$ corresponding to **order parameter field.**

  The slow mode of Hydro+,

  it is called $\phi_{\boldsymbol{Q}}(t,\boldsymbol{x})$: $D\phi_{\boldsymbol{Q}} = -\Gamma_{\boldsymbol{Q}}\left(\phi_{\boldsymbol{Q}} - \bar{\phi}_{\boldsymbol{Q}}\right)$

$$\bar{\phi}_{\boldsymbol{Q}} \approx \frac{c_M \, \xi^2}{1+(Q\xi)^2}$$

[Rajagopal, Ridgway, Weller, Yin 1908.08539]

$$\Gamma_{\boldsymbol{Q}} = \frac{2D_0\xi_0}{\xi^3}K(Q\xi)$$

[Kawasaki 1970]

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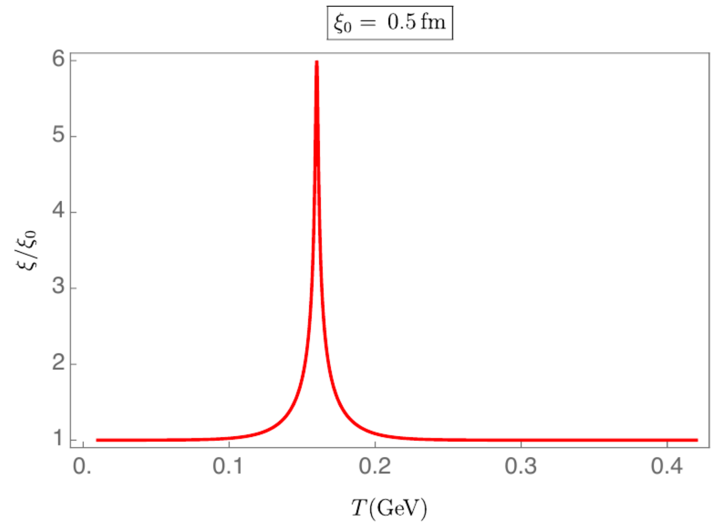
- We also parametrize ξ, as

[Rajagopal, Ridgway, Weller, Yin 1908.08539]

$$\left(\frac{\xi}{\xi_0}\right)^{-2} = \sqrt{\tanh^2\left(\frac{T-T_c}{\Delta T}\right)\left(1-\left(\frac{\xi_{\max}}{\xi_0}\right)^{-4}\right)+\left(\frac{\xi_{\max}}{\xi_0}\right)^{-4}}$$

$$\Delta T = 0.2 T_c$$



Question that we want to address:

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What we need to know:

1. The equation of state near the CP. . [Rajagopal, Ridgway, Weller, Yin 1908.08539]
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Here is the idea:

the characteristic momentum of single-mode Hydro+ puts constrain on the limits of the above integral.

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The purpose is to illustrate the effect of qc, not a precise quantitative analysis

Connection to single mode Hydro+

The contribution of any mode is given by

$$\alpha_{\boldsymbol{Q}}(\omega \gg \Gamma_{\boldsymbol{Q}}) = \frac{\Delta c^2_{s,\boldsymbol{Q}}(\infty)}{c_s^2} = \frac{c_s^2}{2s}\,\frac{Q^2\,\Delta Q}{2\pi^2}\,[f_2(Q\xi)]^2 \left(\frac{\xi}{\xi_0}\right)^4 \left(T\frac{\partial}{\partial T}\left(\frac{\xi}{\xi_0}\right)^{-2}\right)^2$$

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- Then the above picture can be applied to any point (μ, T) near the critical point (in the phase space). $q_c \equiv q_c(Q, \mu, T)$

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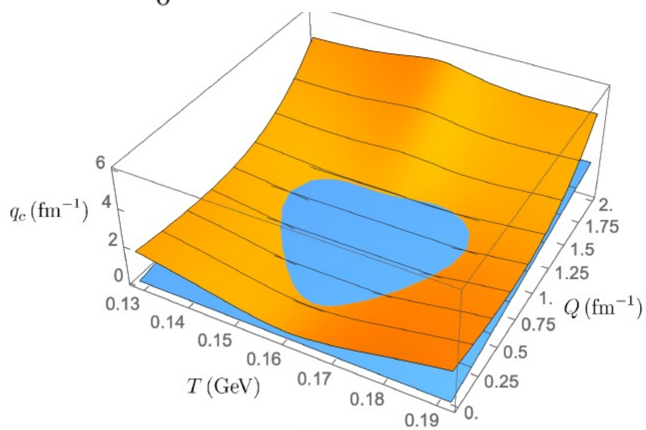
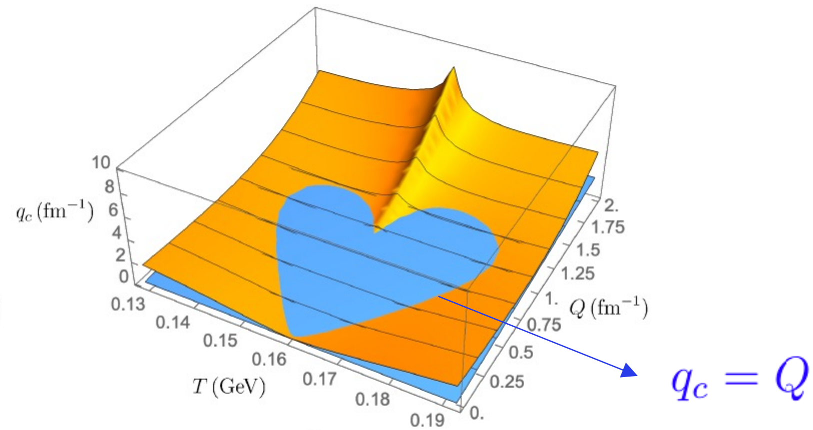
- We limit our study to small μ region; $q_c \equiv q_c(Q, T)$
- Only $\phi_{\boldsymbol{Q}}$ modes satisfying $q_c(Q,T) \ll Q$ contribute.

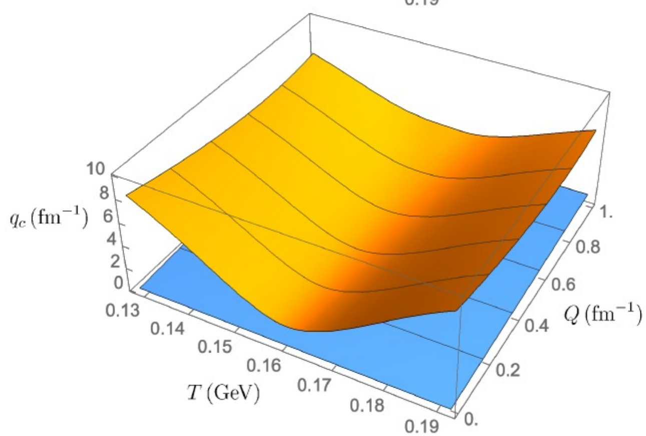
Characteristic momentum near CP

$\xi_{\max} = 1\,\mathrm{fm}$ and $D_0 = 0.1\,\mathrm{fm}$

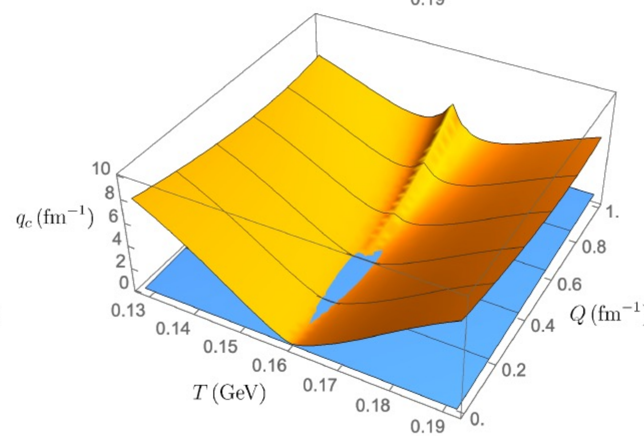


$\xi_{\max} = 3\,\mathrm{fm}$, $D_0 = 0.1\,\mathrm{fm}$





$\xi_{\max} = 1\,\mathrm{fm}$ and $D_0 = 0.5\,\mathrm{fm}$



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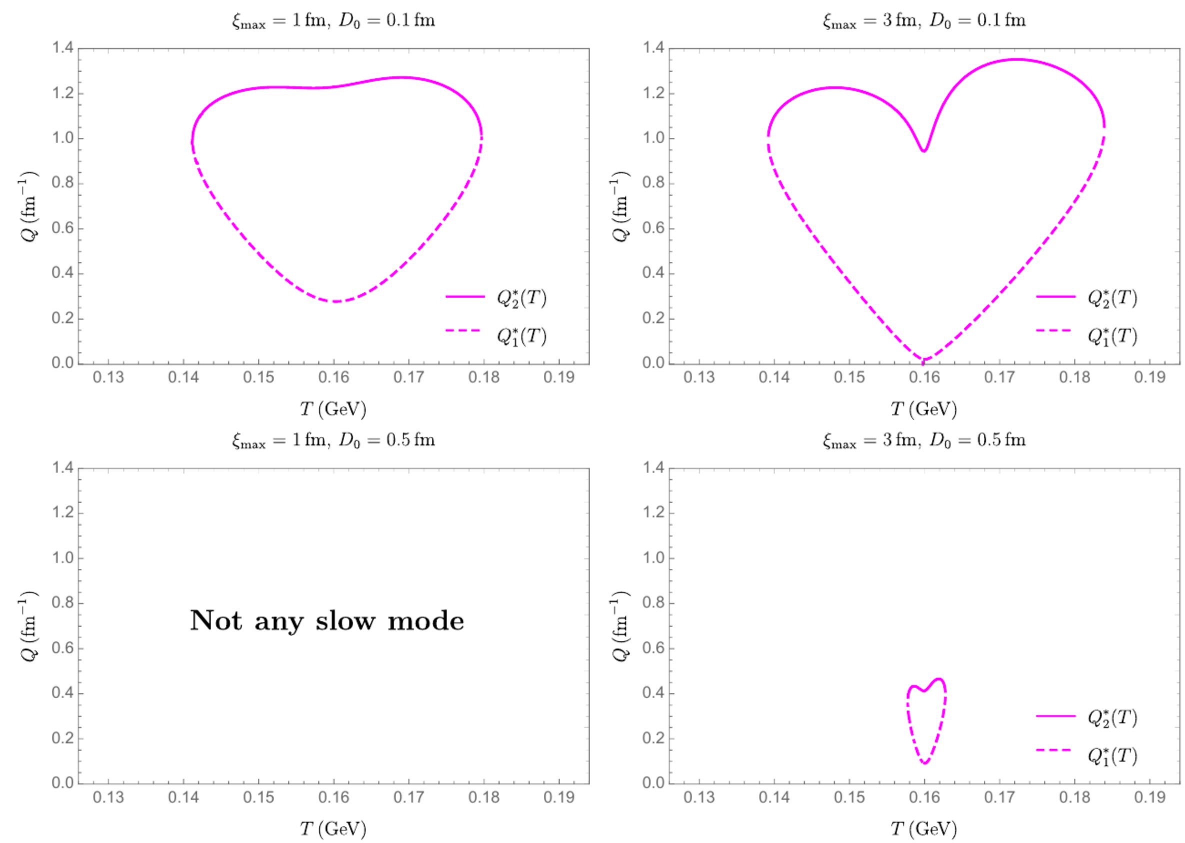
Contributing modes

$$\Delta c_s^2(\omega) = \frac{c_s^4}{2s} \int \frac{d^3\boldsymbol{Q}}{(2\pi)^3} \, [f_2(Q\xi)]^2 \left(\frac{\xi}{\xi_0}\right)^4 \left(T\frac{\partial}{\partial T}\left(\frac{\xi}{\xi_0}\right)^{-2}\right)^2 \frac{\omega^2}{\omega^2 + \Gamma_{\boldsymbol{Q}}^2}$$

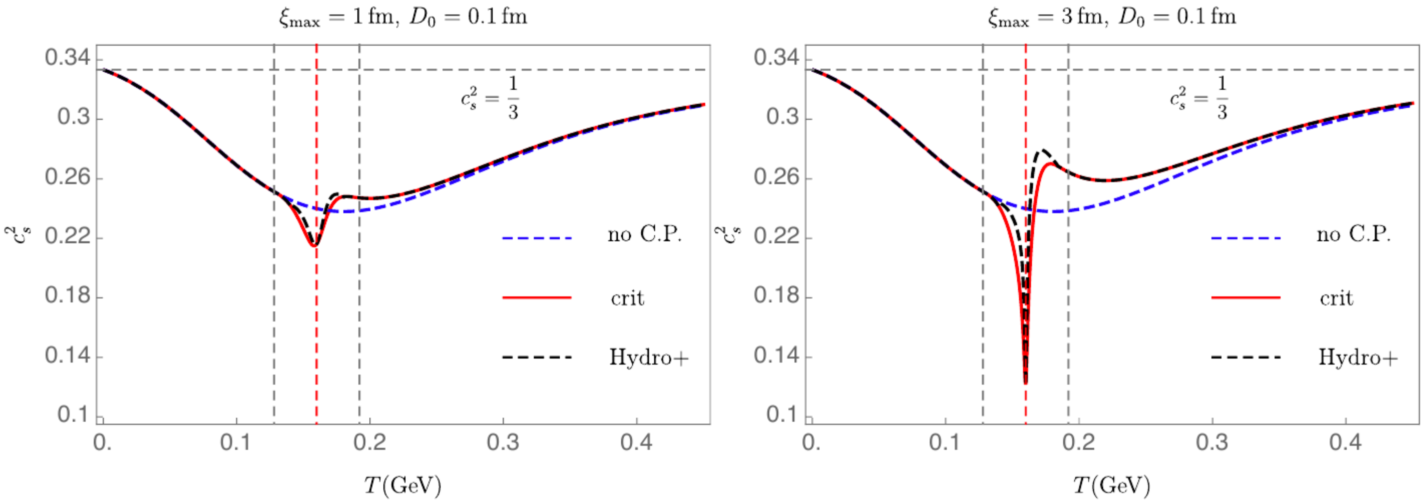


At any T, only modes

with $Q_1^*(T) \ll Q \ll Q_2^*(T)$.

contribute to Δc_s^2.

c_s^2 near the critical point



[NA, Kaminski 2112.14747]

- The larger $\xi_{\rm max}$, the more enhancement in the speed of sound.
- The enhancement of the speed of sound in any case.

(similar to bulk viscosity enhancement [Martinez, Schafer, Skokov 1906.11306])

Where does this idea come from?

- Holography:

Branch point* singularities of spectral function of linear excitations

= **Radius of convergence** of the hydrodynamic derivative expansion

[Withers 1803.08058] [Grozdanov, Kovtun, Starinets, Tadić 1904.01018] [NA, Tahery 2007.10024]

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In a (1+1)d SYK chain, [NA 2112.12751]

the spectrum of excitations

in the presence of longtime tails:

$$\mathfrak{w} \equiv \mathfrak{w}(\mathfrak{q}^2) \quad \rightarrow \quad \mathfrak{w} \equiv \mathfrak{w}(|\mathfrak{q}|)$$

The series is convergent.

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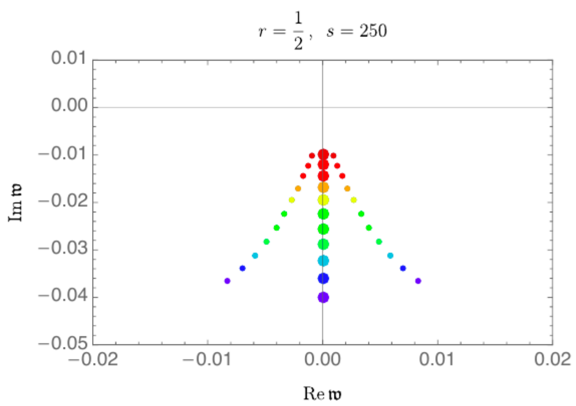

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Outlook

In an upcoming work: [NA Kaminski Tavkol To Appear]

We show that:

Collison between (modified)hydro and

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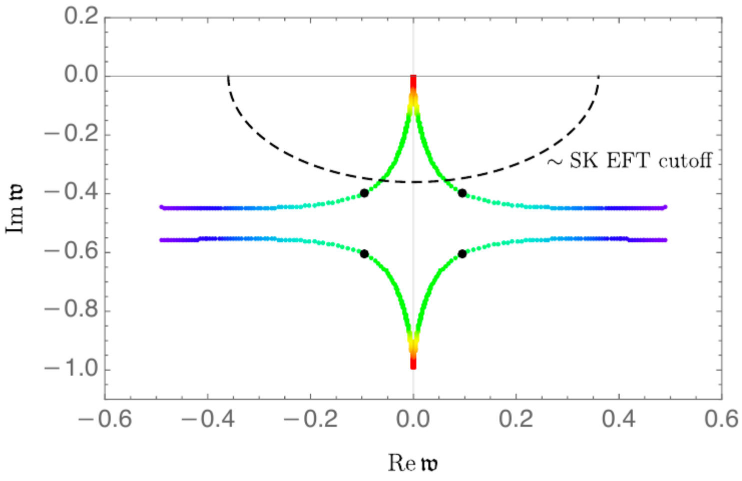
Equivalently, due to some BP singularity.



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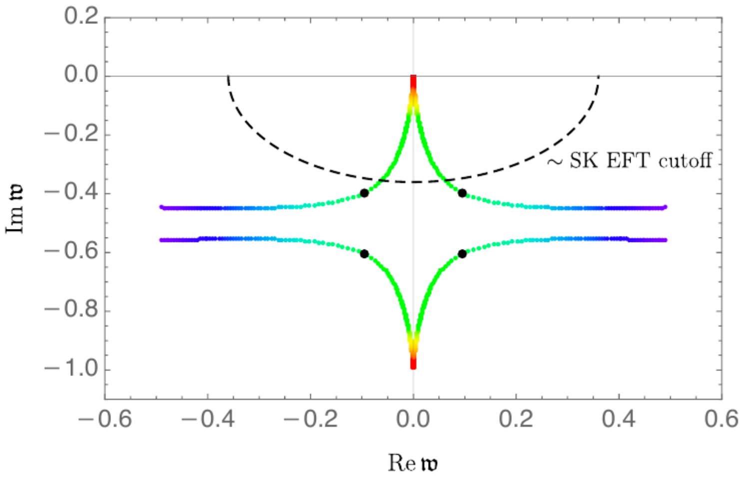

- Hydro+ near the critical point is able to give the longtime tails.

[Stephanov, Yin 1712.10305]

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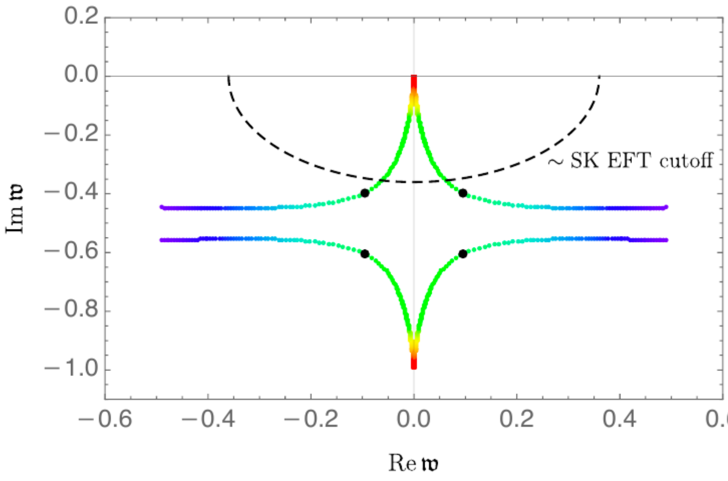

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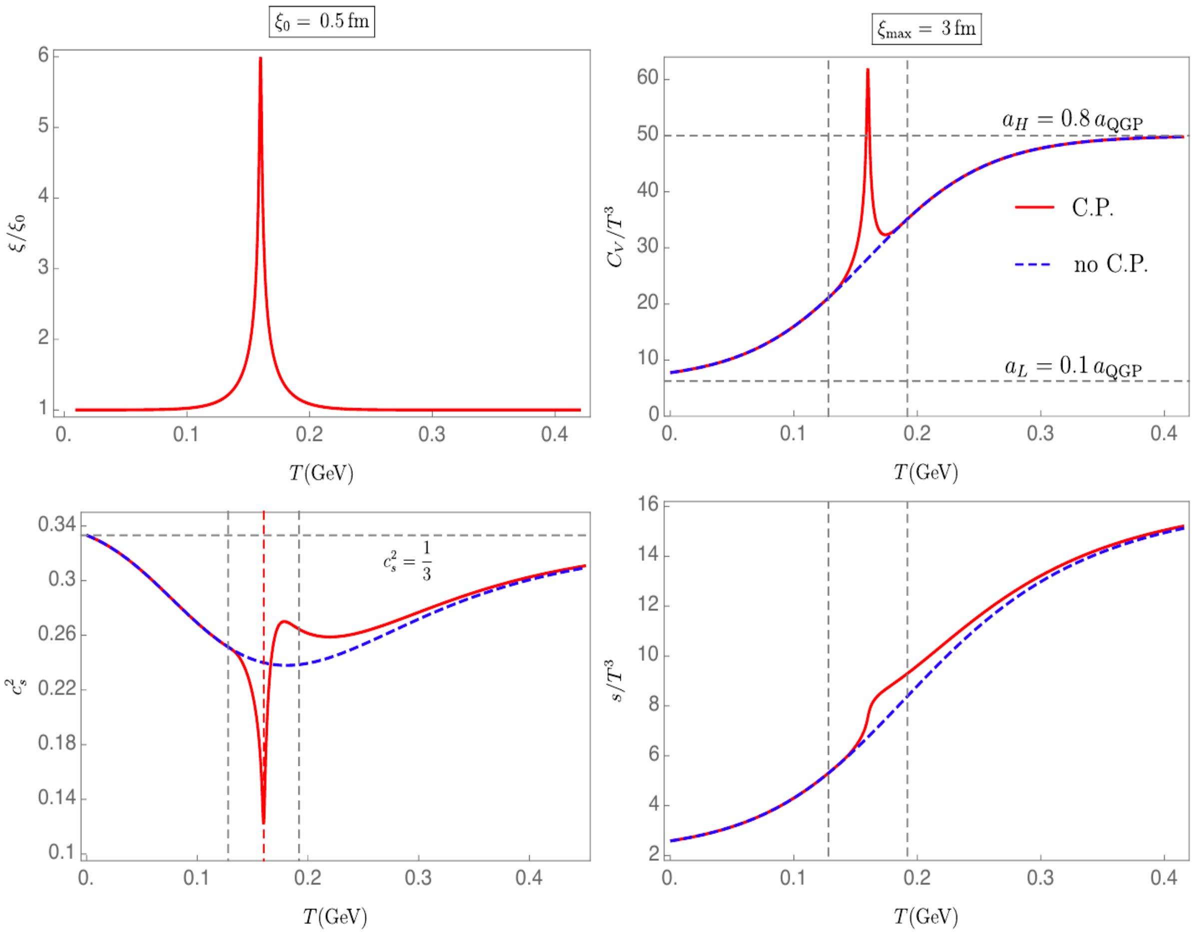
- It would be interesting to go beyond assumptions in this work and extract a modified dispersion relation near the critical point.

Thank you for your attention

EOS

$$\frac{c_V^{\text{no C.P.}}}{T^3} = \left[\left(\frac{a_H + a_L}{2}\right) + \left(\frac{a_H - a_L}{2}\right)\tanh\left(\frac{T - T_{\text{C.O.}}}{\Delta T_{\text{C.O.}}}\right)\right]$$



$T_{\text{C.O.}} = T_c$ and $\Delta T_{\text{C.O.}} = 0.6\,T_c$

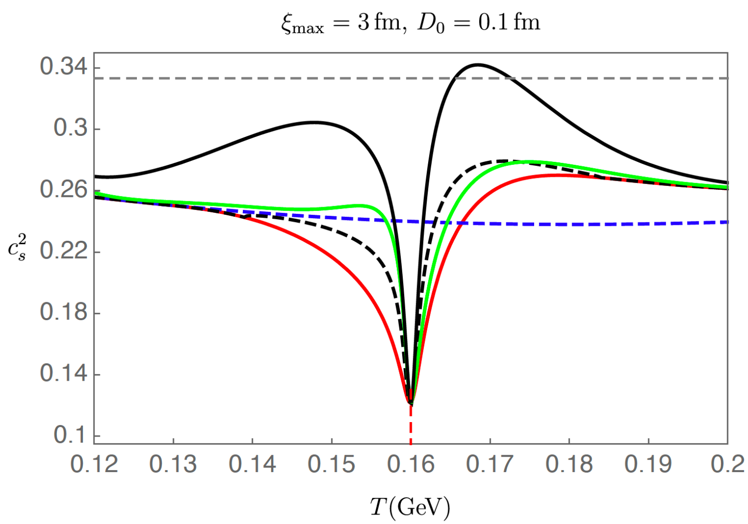
$a_L = 0.1\,a_{\text{QGP}}, \quad a_H = 0.8\,a_{\text{QGP}}$

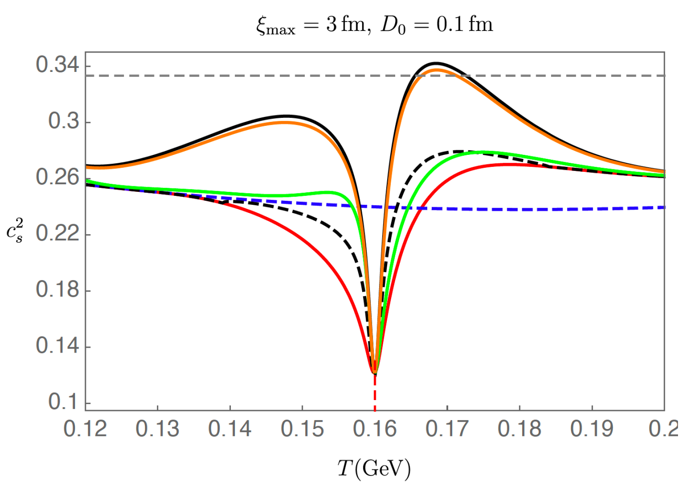
$$a_{\text{QGP}} = \frac{4\pi^2(N_c^2 - 1) + 21\pi^2 N_f}{15}$$

$$s(T) = \int_0^T dT' \frac{c_V(T')}{T'},$$

$$c_s^2 = \frac{s}{c_V}.$$

Bound





$\int_0^\infty$

$\int_{Q_c(Q)<Q}$

$\int_0^{\xi^{-1}}$

$\int_{Q_c<Q}$

Needs improvement …