Correlation of fluctuation with parametric slow modes: a signature of the QCD critical point?

Based on
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Outline

- **Introduction & motivation**
- **The dynamic structure factor**
- **Why parametric slow mode?**
- **Framework: Second-order hydrodynamics**
- **Results**
- **Summary & conclusion**

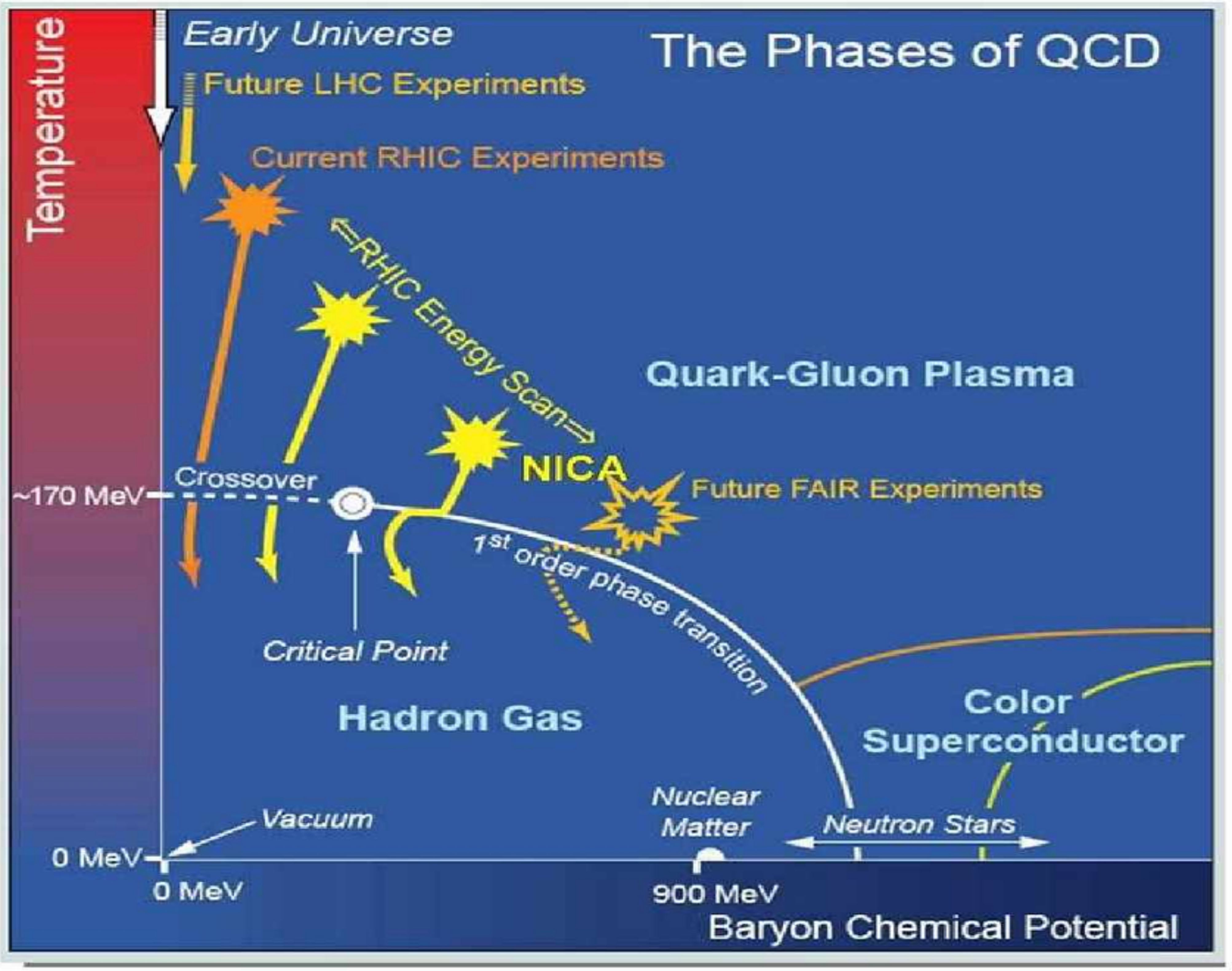
Introduction & Motivation

- Thermodynamics: Critical End-Point (CEP) at the end of the first order phase transition line.

- QCD based effective models: The existence of the CEP based on the parameters of the models used.

- Lattice calculations, not valid $\mu_B \neq 0$: sign problem, exact location of the CEP is still not conclusive.

- Lack of conclusiveness of the precise location of the CEP demands for more theoretical as well as experimental studies.



- Beam Energy Scan (BES) at BNL is particularly dedicated for finding the CEP by tuning the $\sqrt{s}$.

The density fluctuation

- **The propagation of the perturbations: unveil the thermodynamic state of a fluid.**

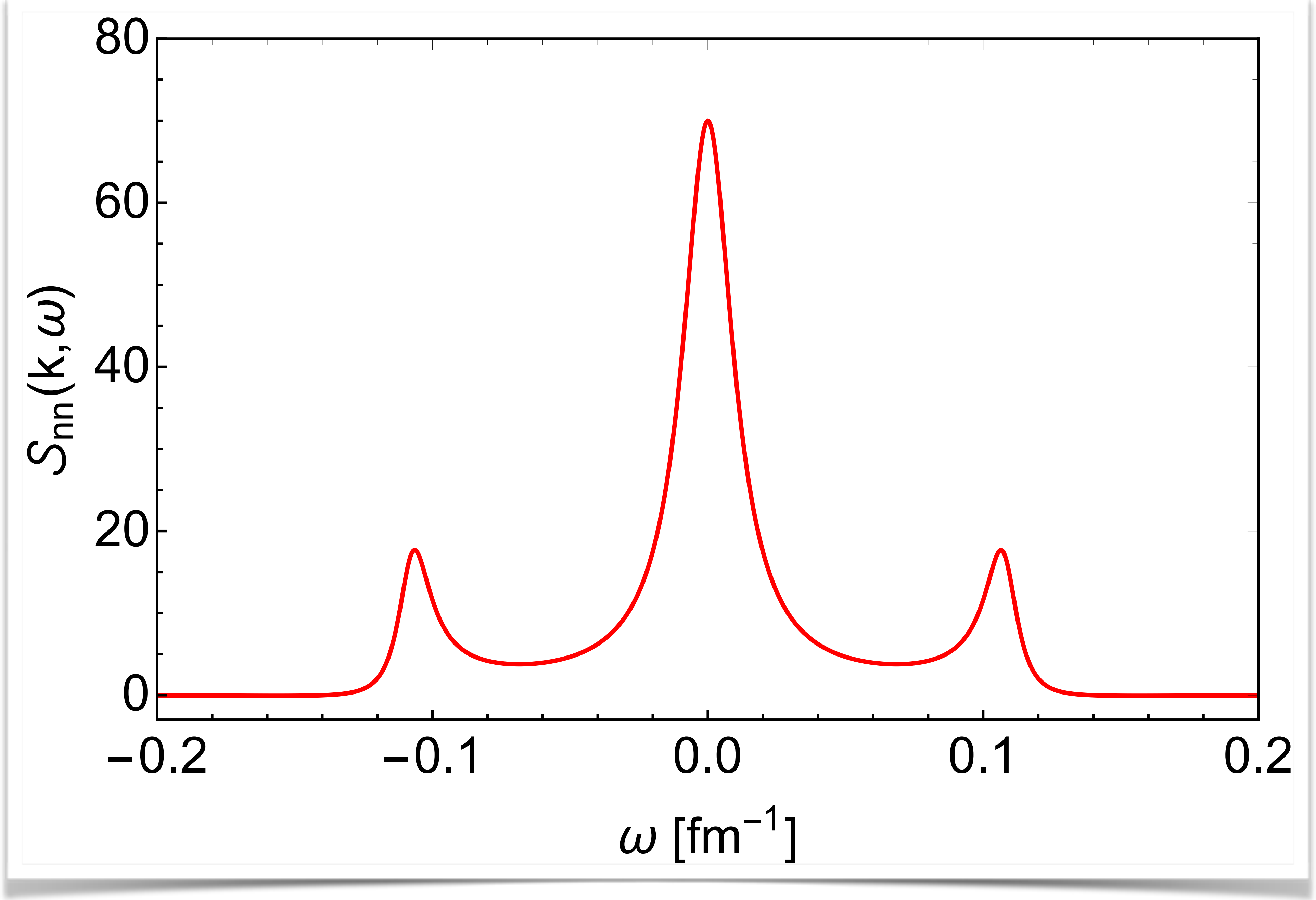
- **The hydrodynamic response to the perturbation: imprinted on fluid and thus translated into the particle spectra.**

- **Consequences of the CEP on the hydrodynamic evolution if an isentropic trajectory passes through the critical region.**

- **Density fluctuation (dynamical) $\delta n(\vec{r}, t)$ and its correlation can play the key role in heavy-ion collision to be understood in event-by-event analysis.**

The dynamic structure factor

MH et. al., EPJA 57:283 (2021)



- **The structure factor is calculated as:**
  $\mathcal{S}_{nn}(\overrightarrow{k},\omega) = < \delta n(\overrightarrow{k},\omega)\ \delta n(\overrightarrow{k},0) >$.

- **The $\mathcal{S}_{nn}(\overrightarrow{k},\omega)$ generally contains Rayleigh peak (thermal fluctuations) and Brillouin peaks (pressure fluctuation).**

- **The width $D_R \sim \kappa/\rho C_P$, and $D_B \sim \zeta + (4/3)\eta$.**

- **The intensities $I_R/I_B = C_P/C_V - 1$, is called the Landau-Placzek ratio.**

- **The position of the Brillouin peaks help to determine the speed of sound, $c_s = \frac{\omega_B}{k}$.**

The non-hydrodynamic modes, ϕ

- The hydrodynamics is applied for the slowly evolving modes, the faster non-hydrodynamic modes are set by the collision dynamics.
- The separation of timescale is necessary to apply hydro.
- Approaching the CEP, the separation of the timescale disappears, the applicability of hydro becomes invalid.
- Hydro can still be applied for systems far away from equilibrium by including the higher-order gradients of hydrodynamic fields (not suitable for systems with large fluctuations near the CEP where the out-of-equilibrium modes relax slowly).

P. Romatschke, PRL 120, 012301 (2018)

Heller et. al., PRL 115, 072501 (2015)

or

- The validity of the hydro can be extended by introducing a slowly evolving scalar non-hydrodynamic field, ϕ, incorporated in the definition of entropy → **Hydro+**

Stephanov et. al., PRD 98, 036006 (2018)

Framework

- **Hydro equations:** $\partial_\mu T^{\mu\nu} = 0; \partial_\mu j^\mu = 0.$

$u^\mu u_\mu = -1$

where, $T^{\mu\nu} = \epsilon u^\mu u^\nu + (P + \Pi)\Delta^{\mu\nu} + h^\mu u^\nu + u^\nu h^\mu + \pi^{\mu\nu}, \quad j^\mu = n u^\mu + \nu^\mu$

- **In hydro+ formalism, the entropy density with ϕ becomes**

Stephanov et. al., PRD 98, 036006 (2018)

$ds_+ = \beta_+ d\epsilon - \alpha_+ dn - \pi d\phi$ **where,** $\beta_+ = 1/T, \alpha_+ = \mu_+/T,$ **π is the energy cost for ϕ**

- **The relaxation equation for ϕ**

$$D\phi = -F_\phi + A_\phi \theta \qquad D = u^\mu \partial_\mu, \theta = \partial_\mu u^\mu$$

- **F_ϕ and A_ϕ can be found out by imposing the second law of thermodynamics**

$$\partial_\mu s^\mu \geq 0\,; \; \partial_\mu s^\mu = (F_\phi - n\partial_\mu q^\mu)\pi - q^\mu \partial_\mu(\beta + b\pi) - \beta \partial_\mu u_\nu \Delta T^{\mu\nu} + \partial_\mu(\Delta s^\mu + \beta q^\mu + b\pi q^\mu) + S_n \theta$$

$s^\mu = s u^\mu + \Delta s^\mu$

b represents the coupling strength between the slow mode and the heat flux

Framework

- **The expression,** $S_n = s - \beta(\epsilon + P) + \beta\mu n + \pi A_\phi$

$\partial_\mu s^\mu \geq 0$ **is satisfied by taking** $S_n = 0$ **along with**

$$\Delta s^\mu = -\beta q^\mu - b\pi q^\mu$$

$$q^\mu = -\kappa T\left[Du^\mu - \frac{1}{\beta}\Delta^{\mu\nu}\partial_\nu(\beta + b\pi)\right],$$

$$F_\phi = \gamma\pi - b\partial_\mu\left[\kappa T D u^\mu - \frac{\kappa}{\beta}\Delta^{\mu\nu}\partial_\nu(\beta + b\pi)\right],$$

γ is relaxation rate of the slow modes

- **The thermodynamic extensivity condition allows** → $A_\phi = \phi$

- **The dissipative fluxes in IS theory becomes**

$$\Pi = -\frac{1}{3}\zeta[\partial_\mu u^\mu + \beta_0 D\Pi - \tilde{\alpha}_0\partial_\mu q^\mu],$$

$$q^\mu = -\kappa T\Delta^{\mu\nu}[\beta\partial_\nu(T + b\pi) + Du_\nu + \beta_1 Dq_\nu - \tilde{\alpha}_0\partial_\nu\Pi - \tilde{\alpha}_1\partial_\lambda\pi^\lambda_\nu],$$

$$\pi^{\mu\nu} = -2\eta[\Delta^{\mu\nu\rho\lambda}\partial_\rho u_\lambda + \beta_2 D\pi^{\mu\nu} - \tilde{\alpha}_1\Delta^{\mu\nu\rho\lambda}\partial_\rho q_\lambda],$$

Framework

The hydro equations in linearized form

$$
\begin{aligned}
0 &= -\frac{\partial \delta\epsilon}{\partial t} - (\epsilon_0 + P_0) + \nabla \cdot \delta \boldsymbol{u} - \nabla \cdot \delta \boldsymbol{q}\,, \\
0 &= -(\epsilon_0 + P_0)\frac{\partial}{\partial t}\delta u^i - \partial^i(\delta P + \delta\Pi) + \frac{\partial}{\partial t}\delta q^i - \partial_j \Pi^{ij}\,, \\
0 &= -\frac{\partial}{\partial t}\delta n - n_0 \nabla \cdot \delta \boldsymbol{u}\,, \\
0 &= \delta\Pi + \frac{1}{3}\zeta[\nabla \cdot \delta \boldsymbol{u} + \beta_0 \frac{\partial}{\partial t}\delta\Pi - \tilde{\alpha}_0 \nabla \cdot \delta \boldsymbol{q}]\,, \\
0 &= \delta q^i + \kappa T_0 \nabla^i \delta T + \kappa T_0 \frac{\partial}{\partial t}\delta u^i + \kappa T_0 \beta_1 \frac{\partial}{\partial t}\delta q^i - \kappa T_0 \tilde{\alpha}_0 \nabla^i \delta\Pi - \kappa T_0 \tilde{\alpha}_1 \nabla_j \pi^{ij} \\
0 &= \delta\pi^{ij} + 2\eta\delta^{ijlm}(\partial_l \delta u_m - \tilde{\alpha}_1 \partial_l \delta q_m) + 2\eta\beta_2 \frac{\partial}{\partial t}\delta\pi^{ij}\,, \\
0 &= -\frac{\partial}{\partial t}\delta\phi - (\gamma + T_0^2 \frac{K_{q\pi}^2}{\kappa}\nabla^2)C_{\phi\pi}\delta\phi - \left[\gamma + (T_0^2 \frac{K_{q\pi}^2}{\kappa} - \frac{K_{q\pi}}{C_{T\pi}})\nabla^2\right]C_{T\pi}\delta T \\
&\quad -(\gamma + T_0^2 \frac{K_{q\pi}}{\kappa}\nabla^2)C_{\pi n}\delta n - T_0 K_{q\pi}\frac{\partial}{\partial t}(\nabla_i \delta u^i) + \tilde{\phi}\nabla_i \delta u^i,
\end{aligned}
$$

where, $K_{q\pi} = b\kappa$ and

$$C_{A\pi} = \frac{\partial \pi}{\partial A}, \quad \text{with } A \equiv (T, n, \phi)\,.$$

Framework

By taking the Fourier-Laplace transformation, the equations in $\omega - k$ space, take the form

$$\mathscr{M}\,\delta\mathcal{Q} = \mathscr{A}$$

$$\delta\mathcal{Q} = \begin{pmatrix} \delta n \\ \delta T \\ \delta\phi \\ \delta u_{||} \\ \delta q_{||} \\ \delta\pi \\ \delta\pi_{||||} \\ \delta\pi_{\perp\perp} \\ \delta u_{\perp} \\ \delta q_{\perp} \\ \delta\pi_{||\perp} \end{pmatrix}, \qquad \mathcal{A} = \begin{pmatrix} -\delta n(\boldsymbol{k}, t=0) \\ -\epsilon_n \delta n(\boldsymbol{k}, t=0) - \epsilon_\phi \delta\phi(\boldsymbol{k}, t=0) - e_T \delta T(\boldsymbol{k}, t=0) \\ -\delta\phi(\boldsymbol{k}, t=0) - ikT_0 \delta u_{||}(\boldsymbol{k}, t=0)\kappa_{q\pi} \\ (\epsilon_0 + P_0)\delta u_{||}(\boldsymbol{k}, t=0) - \delta q_{||}(\boldsymbol{k}, t=0) \\ -\chi\beta_1 T_0 \delta q_{||}(\boldsymbol{k}, t=0) - T_0\chi\delta u_{||}(\boldsymbol{k}, t=0) \\ -\frac{1}{3}\beta_0\zeta\delta\pi(\boldsymbol{k}, t=0) \\ -2\beta_2\eta\delta\pi_{||||}(\boldsymbol{k}, t=0) \\ -2\beta_2\eta\delta\pi_{\perp\perp}(\boldsymbol{k}, t=0) \\ (\epsilon_0 + P_0)\delta u_{\perp}(\boldsymbol{k}, t=0) - \delta q_{\perp}(\boldsymbol{k}, t=0) \\ -\chi\beta_1 T_0 \delta q_{\perp}(\boldsymbol{k}, t=0) - T_0\chi\delta u_{\perp}(\boldsymbol{k}, t=0) \\ -2\beta_2\eta\delta\pi_{||}\perp(\boldsymbol{k}, t=0) \end{pmatrix}$$

$$\mathcal{L}_m = \begin{pmatrix} i\omega & 0 & 0 & ikn_0 & 0 & 0 & 0 & 0 \\ i\omega\epsilon_n & i\omega\epsilon_T & i\omega\epsilon_\phi & ikw_0 & -ik & 0 & 0 & 0 \\ C_{\mathrm{n}\pi}B(k) & C_{\mathrm{T}\pi}B(k) + k^2\kappa_{\mathrm{q}\pi} & C_{\phi\pi}B(k) + i\omega & ikD_\phi(\omega) & 0 & 0 & 0 & 0 \\ -ikP_n & -ikP_T & -ikP_\phi & -i\omega w_0 & i\omega & -ik & -ik & 0 \\ -ikT_0^2 C_{\mathrm{n}\pi}\kappa_{\mathrm{q}\pi} & ik(\chi - T_0^2 C_{\mathrm{T}\pi}\kappa_{\mathrm{q}\pi}) & -ikT_0^2 C_{\phi\pi}\kappa_{\mathrm{q}\pi} & iT_0\chi\omega & 1 + i\beta_1 T_0\chi\omega & -\chi_T\tilde{\alpha}_0 & -\chi_T\tilde{\alpha}_1 & 0 \\ 0 & 0 & 0 & \frac{i\zeta k}{3} & -\frac{1}{3}i\zeta k\tilde{\alpha}_0 & 1 + \frac{1}{3}i\beta_0\zeta\omega & 0 & 0 \\ 0 & 0 & 0 & \frac{4i\eta k}{3} & -\frac{4}{3}i\eta k\tilde{\alpha}_1 & 0 & K(\omega) & 0 \\ 0 & 0 & 0 & -\frac{4}{3}i\eta k & \frac{4}{3}i\eta k\tilde{\alpha}_1 & 0 & 0 & K(\omega) \end{pmatrix},$$

$$\mathbb{M} = \begin{pmatrix} \mathcal{L}_m & \mathbf{0} \\ \mathbf{0} & \mathcal{T}_m \end{pmatrix}, \qquad \mathcal{T}_m = \begin{pmatrix} -i\omega(\epsilon_0 + P_0) & i\omega & -ik \\ iT_0\chi\omega & 1 + i\beta_1 T_0\chi\omega & -ikT_0\chi\tilde{\alpha}_1 \\ i\eta k & -i\eta k\tilde{\alpha}_1 & 1 + 2i\beta_2\eta\omega \end{pmatrix}$$

$$\delta\mathcal{Q} = \mathscr{M}^{-1}\mathscr{A}$$

Framework

- **The density fluctuation is expressed as:**

$$
\begin{aligned}
\delta n(\boldsymbol{k},\omega) = {} & (-\epsilon_n \mathbb{M}^{-1}_{12} - \mathbb{M}^{-1}_{11})\delta n(\boldsymbol{k}, t=0) + \epsilon_T(-\mathbb{M}^{-1}_{12})\delta T(\boldsymbol{k}, t=0) + (-\epsilon_\phi \mathbb{M}^{-1}_{12} - \mathbb{M}^{-1}_{13})\delta\phi(\boldsymbol{k}, t=0) \\
& + (\epsilon_0 \mathbb{M}^{-1}_{14} - ikT_0 \mathbb{M}^{-1}_{13}\kappa_{q\pi} + \mathbb{M}^{-1}_{14} P_0 - T_0\chi \mathbb{M}^{-1}_{15})\delta u_{||}(\boldsymbol{k}, t=0) \\
& + (-\beta_1 T_0 \chi \mathbb{M}^{-1}_{15} - \mathbb{M}^{-1}_{14})\delta q_{||}(\boldsymbol{k}, t=0) + \frac{1}{3}\zeta \mathbb{M}^{-1}_{16}\beta_0 \delta\pi(\boldsymbol{k}, t=0) - 2\eta \mathbb{M}^{-1}_{17}\beta_2 \delta\pi_{||||}(\boldsymbol{k}, t=0) \\
& - 2\eta \mathbb{M}^{-1}_{18}\beta_2 \pi_{\perp\perp}(\boldsymbol{k}, t=0) + (\epsilon_0 \mathbb{M}^{-1}_{19} + \mathbb{M}^{-1}_{19} P_0 - T_0\chi \mathbb{M}^{-1}_{110})\delta u_\perp(\boldsymbol{k}, t=0) \\
& + (-\beta_1 T_0 \chi \mathbb{M}^{-1}_{110} - \mathbb{M}^{-1}_{19})\delta q_\perp(\boldsymbol{k}, t=0) - 2\eta \mathbb{M}^{-1}_{111}\beta_2 \delta\pi_{||\perp}(\boldsymbol{k}, t=0),
\end{aligned}
$$

- **We are concerned with the correlation of density Fluctuation**

$$\mathcal{S}'_{nn}(\boldsymbol{k},\omega) = \langle \delta n(\boldsymbol{k},\omega)\delta n(\boldsymbol{k},0)\rangle$$

- **The correlations between independent variables must vanish**

$$\langle \delta\mathcal{Q}_i(\boldsymbol{k},\omega)\delta\mathcal{Q}_j(\boldsymbol{k}, t=0)\rangle = 0, \quad i \neq j.$$

- **Finally we are able to write**

$$
\begin{aligned}
\mathcal{S}_{nn}(\boldsymbol{k},\omega) &= \frac{\mathcal{S}'_{nn}(\boldsymbol{k},\omega)}{\langle \delta n(\boldsymbol{k}, t=0)\delta n(\boldsymbol{k}, t=0)\rangle} \\
&= -\left[\left(\frac{\partial\epsilon}{\partial n}\right)\mathbb{M}^{-1}_{12} - \mathbb{M}^{-1}_{11}\right].
\end{aligned}
$$

Transport coefficients and thermodynamic response functions

$$\kappa_T = \kappa_T^0 |r|^{-\gamma'}, C_V = C_0|r|^{-\alpha}, C_P = \frac{\kappa_0 T_0}{n_0}\left(\frac{\partial p}{\partial T}\right)_n^2 |r|^{-\gamma'},$$

$$c_s^2 = \frac{T_0}{n_0 h_0 C_0}\left(\frac{\partial p}{\partial T}\right)_n^2 |r|^{\alpha}, \alpha_p = \kappa_0\left(\frac{\partial p}{\partial T}\right)_n |r|^{-\gamma'}$$

$$\eta = \eta_0 |r|^{1+a_\kappa/2-\gamma'}, \zeta = \zeta_0|r|^{-\alpha_\zeta}, \kappa = \kappa_0|r|^{-a_\kappa}$$

Guida & Zinn-Justin, Nucl.Phys. B489, 626 (1997)
Rajagopal & Wilczek Nucl.Phys. B399, 395 (1993)

Equation of Sate (EOS)

- **Prime objectives is to see the critical effects on $\mathcal{S}_{nn}(\vec{k}, \omega)$; the effects enters into the equation via the EOS with the CEP**

Nonaka & Asakawa, PRC(2005), MH et al, PRC (2020)

- **Constructed on the basis of universality hypothesis: CEP of the QCD belongs to the same universality class of 3D Ising model.**

- **The universality hypothesis permits linear mapping between the CEP of the Ising model and the CEP of the QCD phase diagram as:**

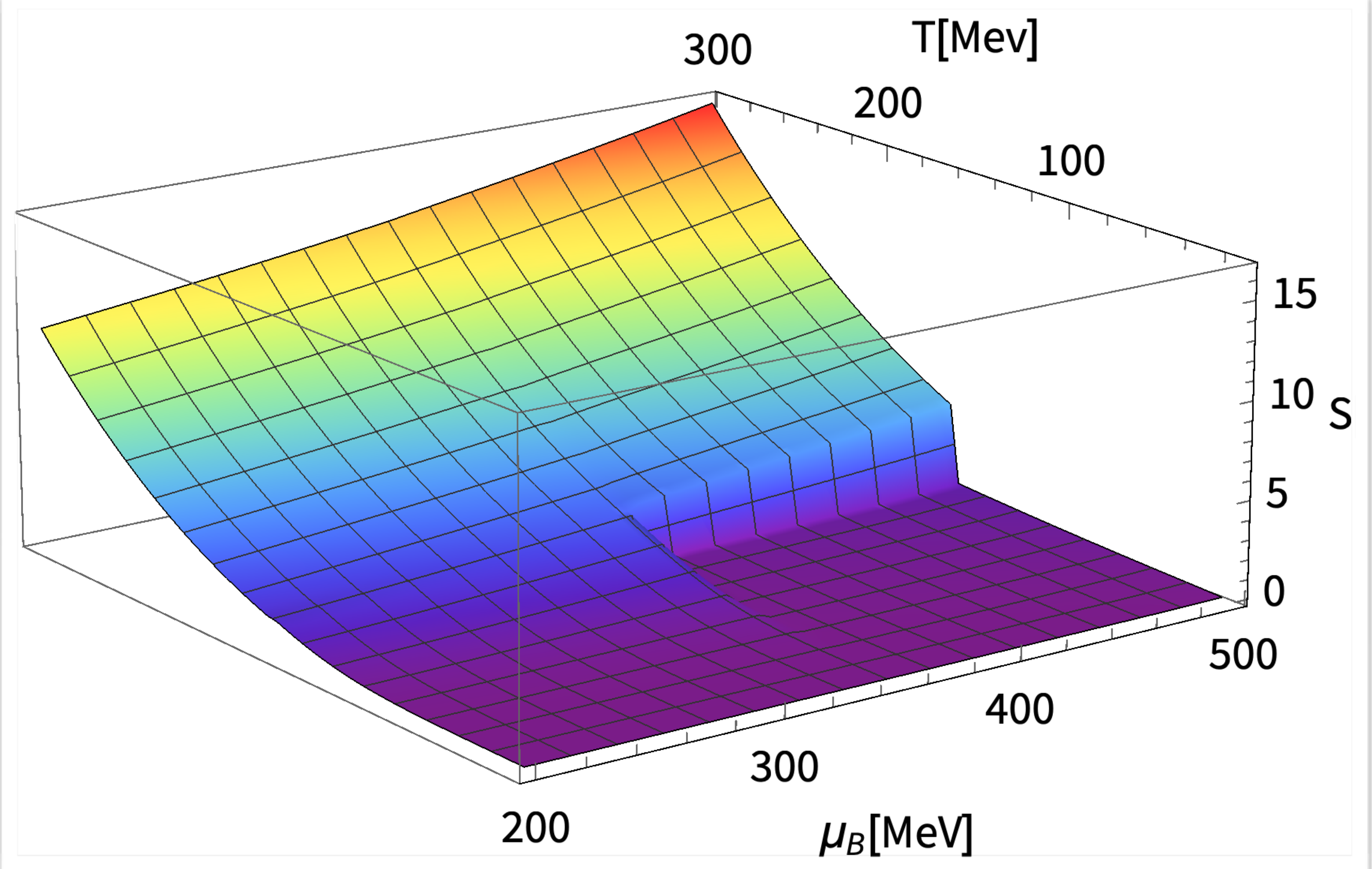
$$r = \frac{\mu - \mu_c}{\Delta\mu_c}, \; h = \frac{T - T_c}{\Delta T_c}$$

- **Magnetization, $M(r, h)$ (order parameter) in Ising model is mapped as critical entropy density $s_c(T, \mu)$ in QCD.**

$$s_c(T, \mu) = \frac{M(r, h)}{\Delta T_c} = M\left(\frac{\mu - \mu_c}{\Delta\mu_c}, \frac{T - T_c}{\Delta T_c}\right)\frac{1}{\Delta T_c}$$

Equation of Sate (EOS)

$(T_c, \mu_c) = (154{,}367) MeV$



- **A dimensionless entropy density is calculated as:**

$$S_c = \left[D\sqrt{\Delta T_c^2 + \Delta \mu_c^2} \right] s_c$$

- **S_c as switching function connects the entropy densities s_Q and s_H :**

$$s(T,\mu) = \frac{1}{2}[1 - \tanh S_c(T,\mu)] s_Q(T,\mu) + \frac{1}{2}[1 + \tanh S_c(T,\mu)] s_H(T,\mu)$$

- **Once entropy density is known, other thermodynamic variables can be calculated as:**

$$n(T,\mu) = \int_0^T \frac{\partial s(T',\mu)}{\partial \mu} dT'$$

$$p(T,\mu) = \int_0^T s(T',\mu) dT'$$

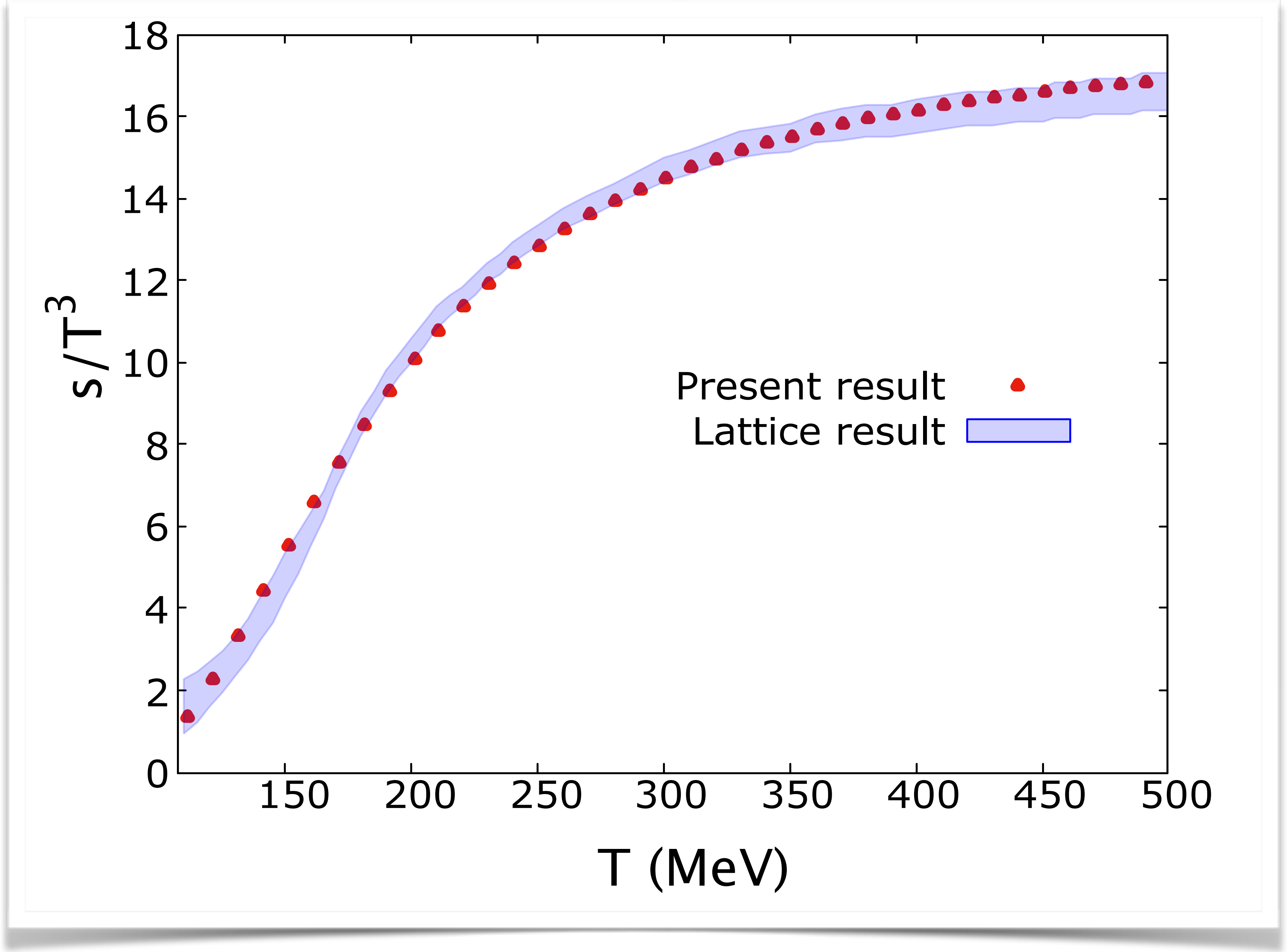
$$\epsilon(T,\mu) = Ts(T,\mu) - p(T,\mu) + \mu n$$

Equation of Sate (EOS)

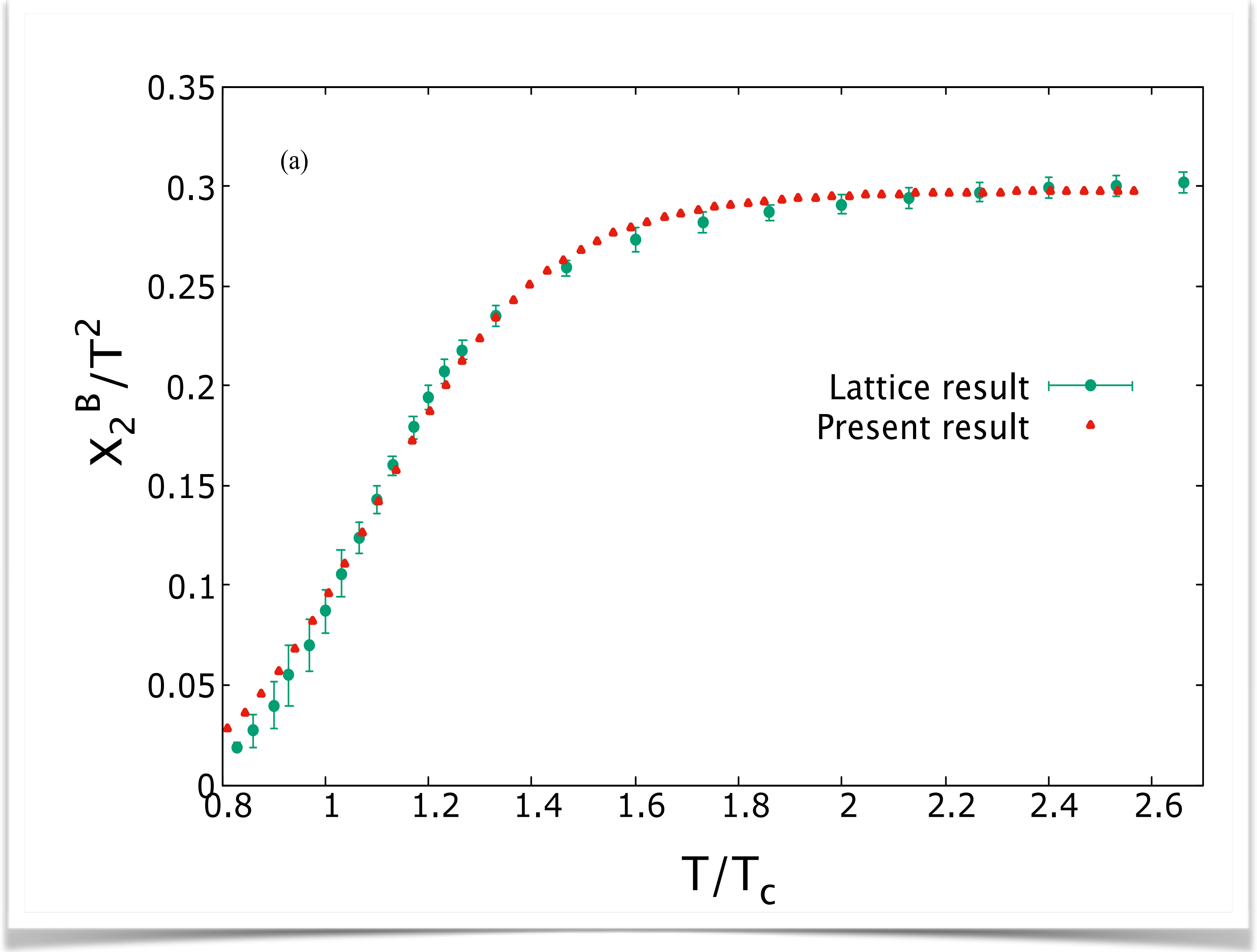
S. Borsanyi et al, JHEP (2012), PLB (2014)

- To check the reliability of the EOS, we compare our results with the available lattice results

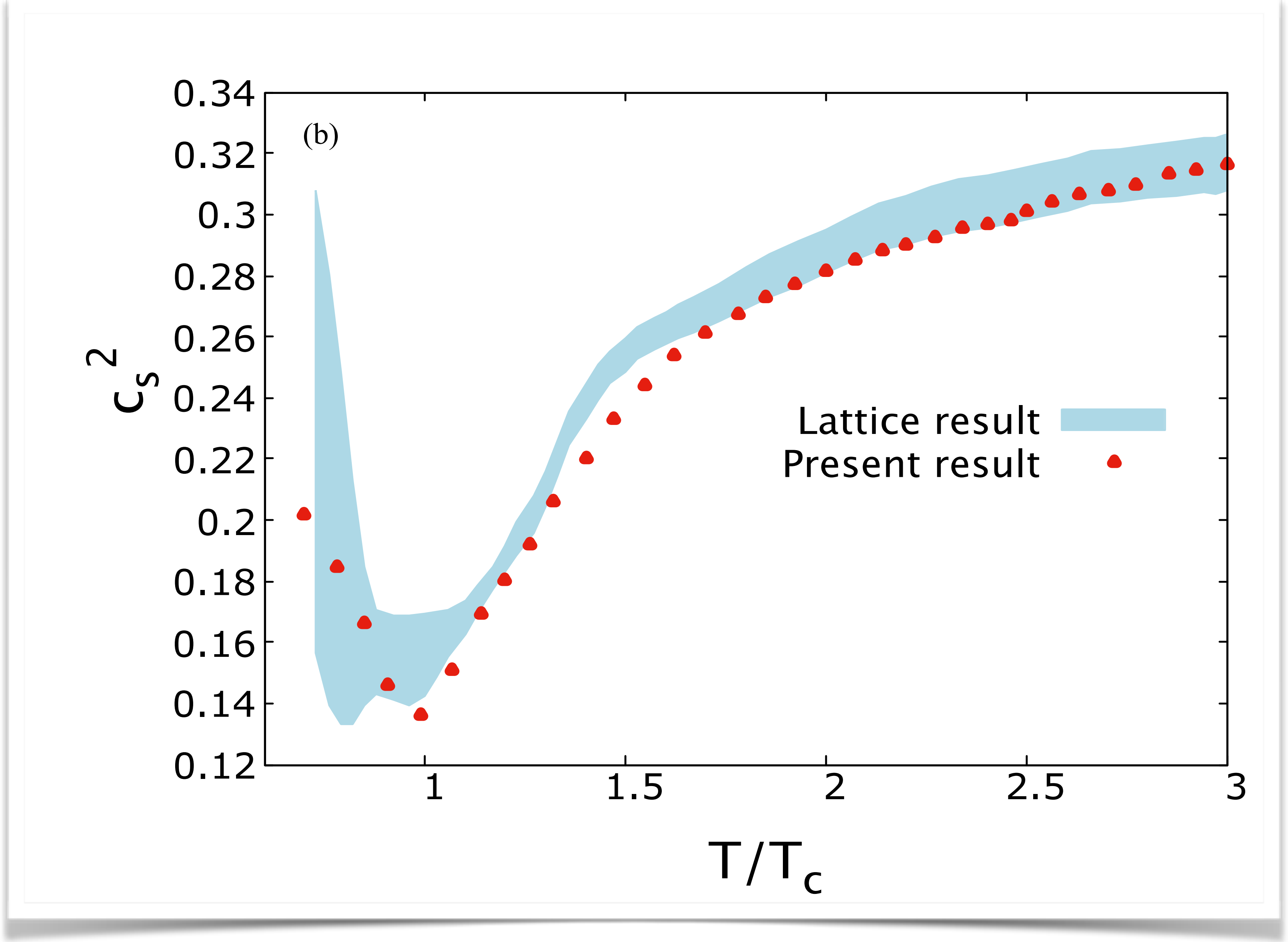
Entropy density



Baryon number susceptibility ~ $\left(\frac{\partial^2 P}{\partial \mu^2}\right)$

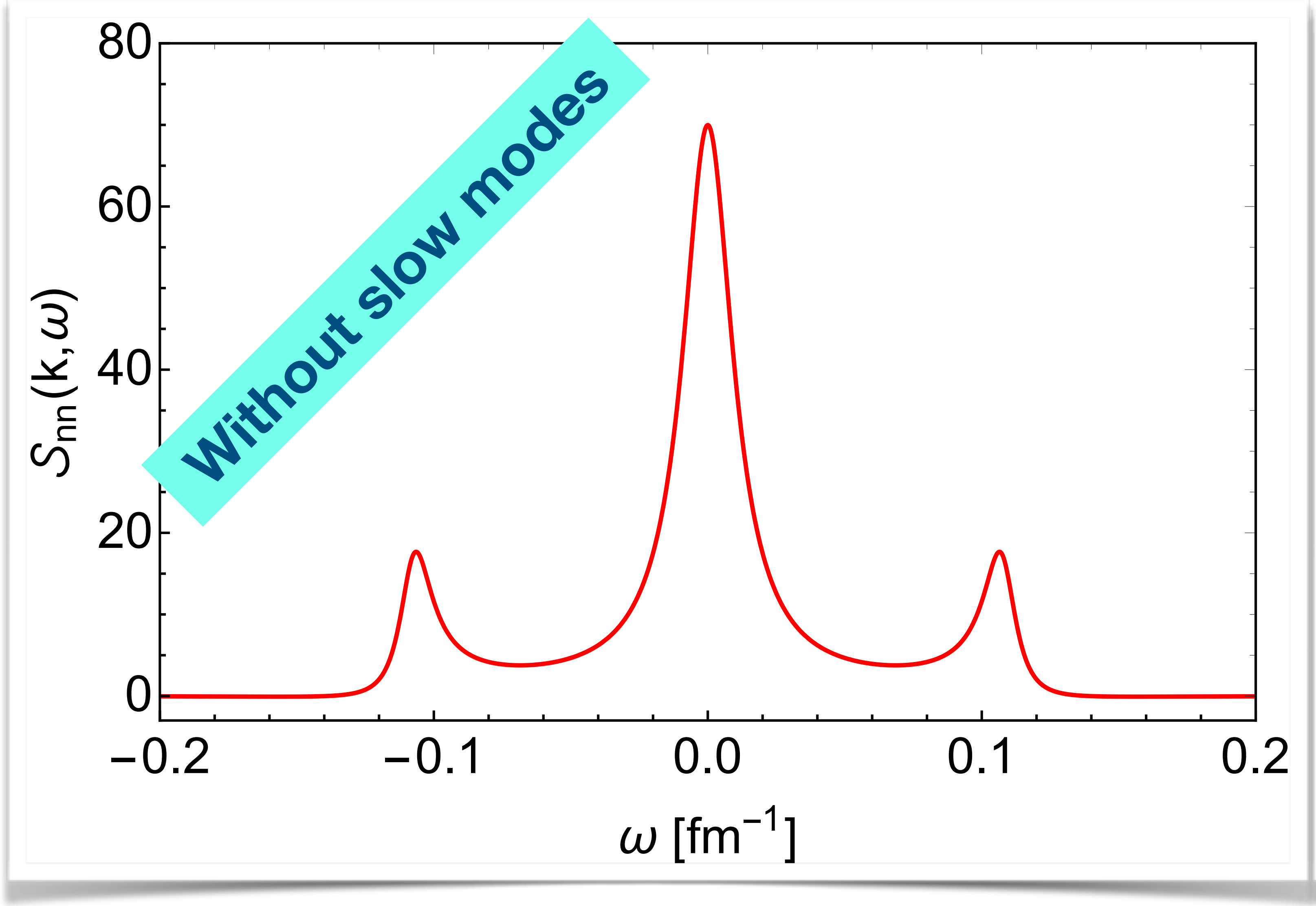


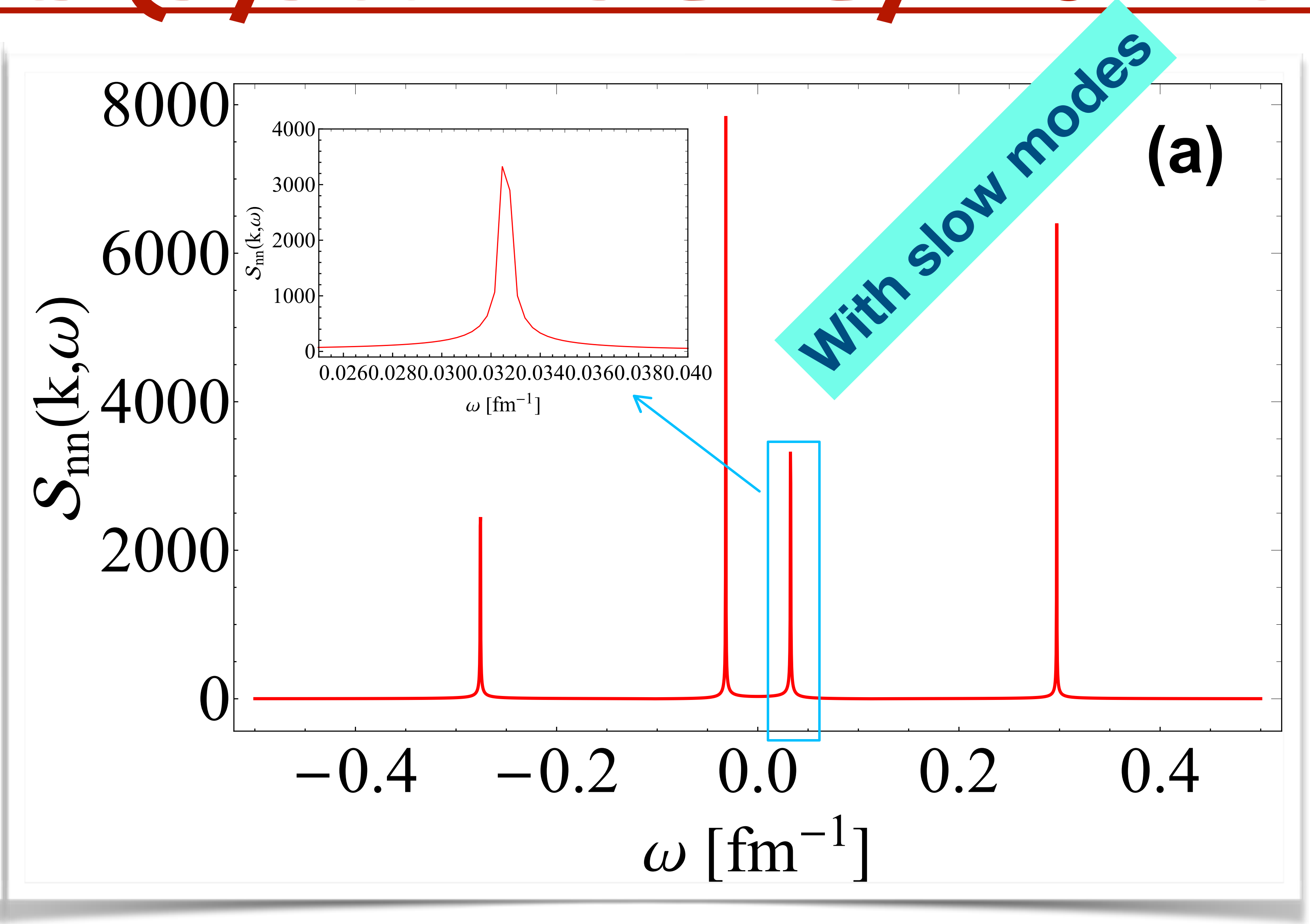
Speed of sound~ $\left(\frac{\partial P}{\partial \epsilon}\right)$

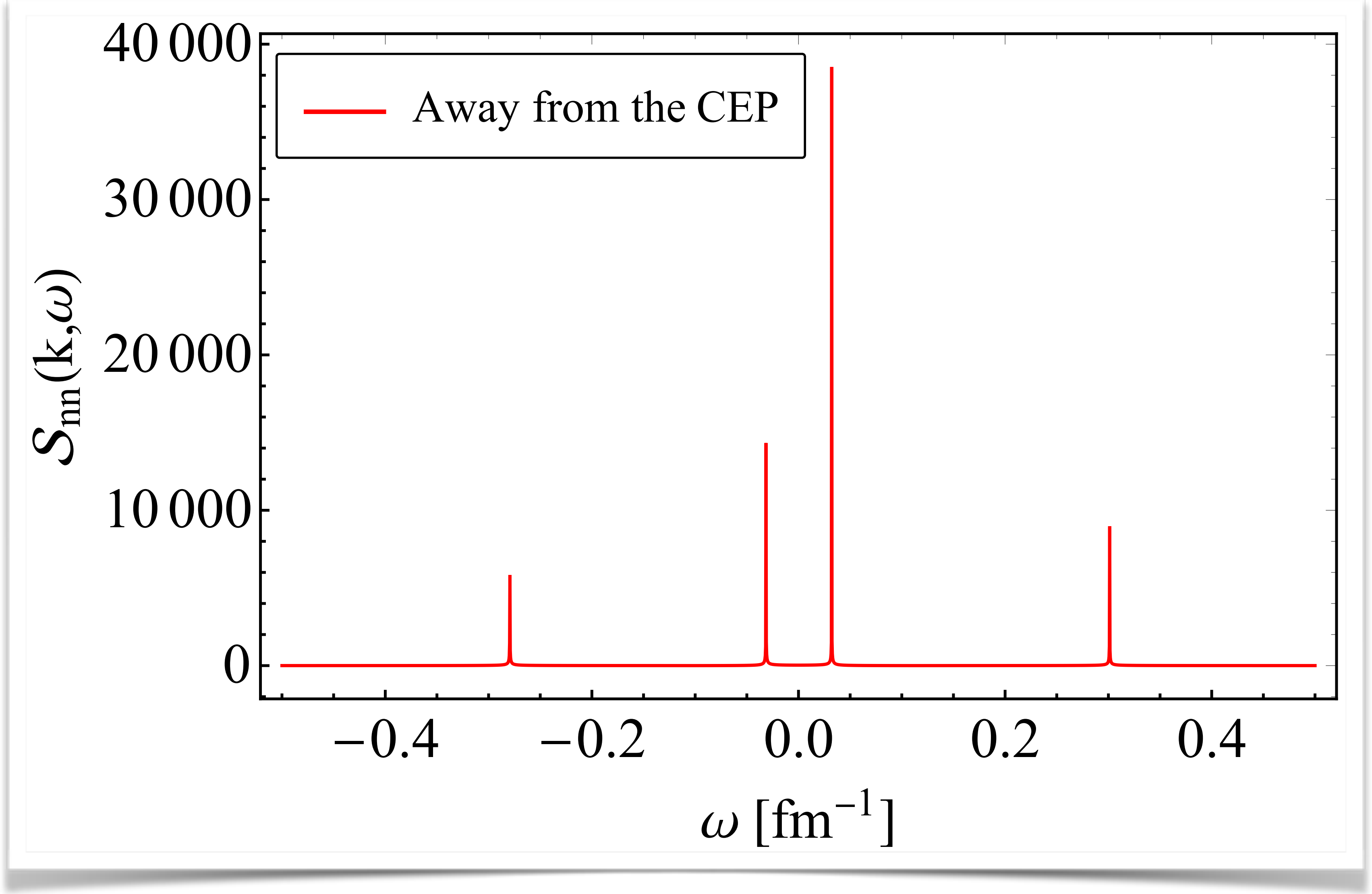


Agrees with the lattice results

Results (system is away from the CEP)







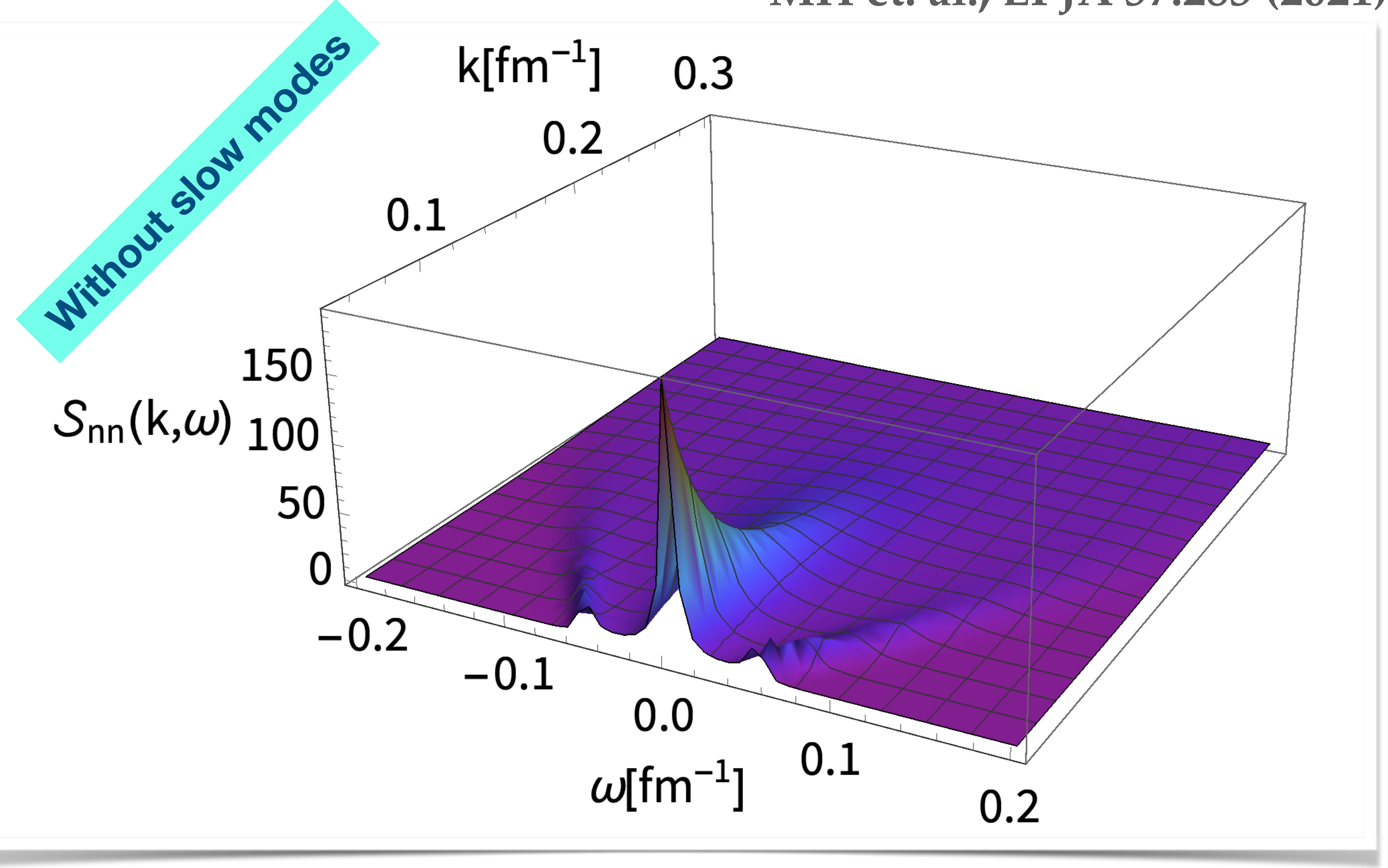
- One Rayleigh peak, two Brillouin peaks
- Rayleigh peak is at $\omega = 0$.
- Brillouin peaks are symmetrically situated about origin with even magnitudes.
- Finite widths of the peaks.

- One Rayleigh peak, two Brillouin peaks, one extra peak due to coupling of hydro modes with ϕ.
- Rayleigh peak is not at $\omega = 0$.
- Brillouin peaks are asymmetrically situated about $\omega = 0$, with uneven magnitudes.
- Narrow widths of the peaks: critical slowing down

- Increasing ϕ induces fluctuation in the system: increases the magnitudes of the peaks.

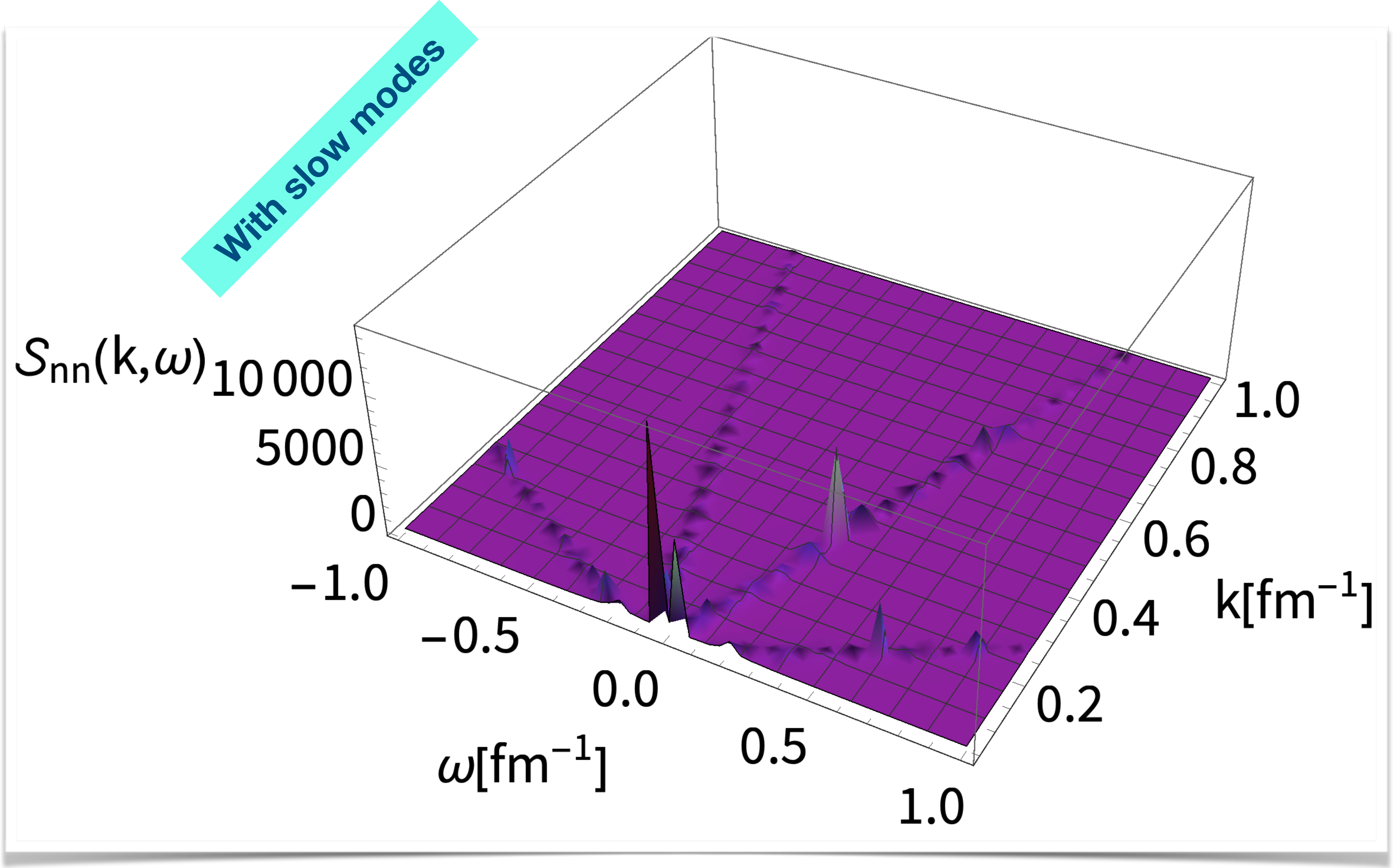
Results (system is away from the CEP)

MH et. al., EPJA 57:283 (2021)

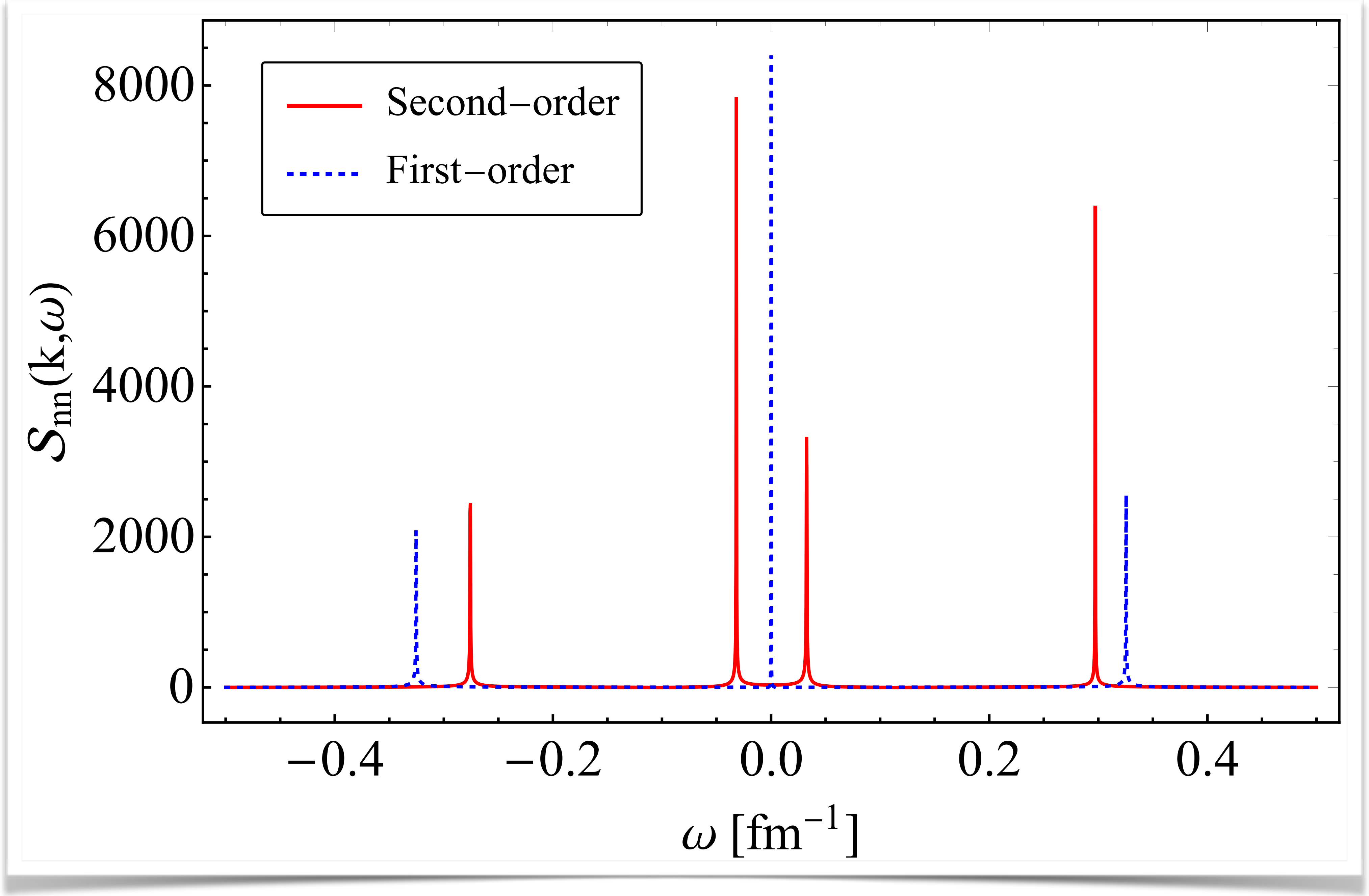


- Brillouin peaks shifts away from the Rayleigh peak.
- Larger the k-values, smaller is the magnitudes of the peaks.

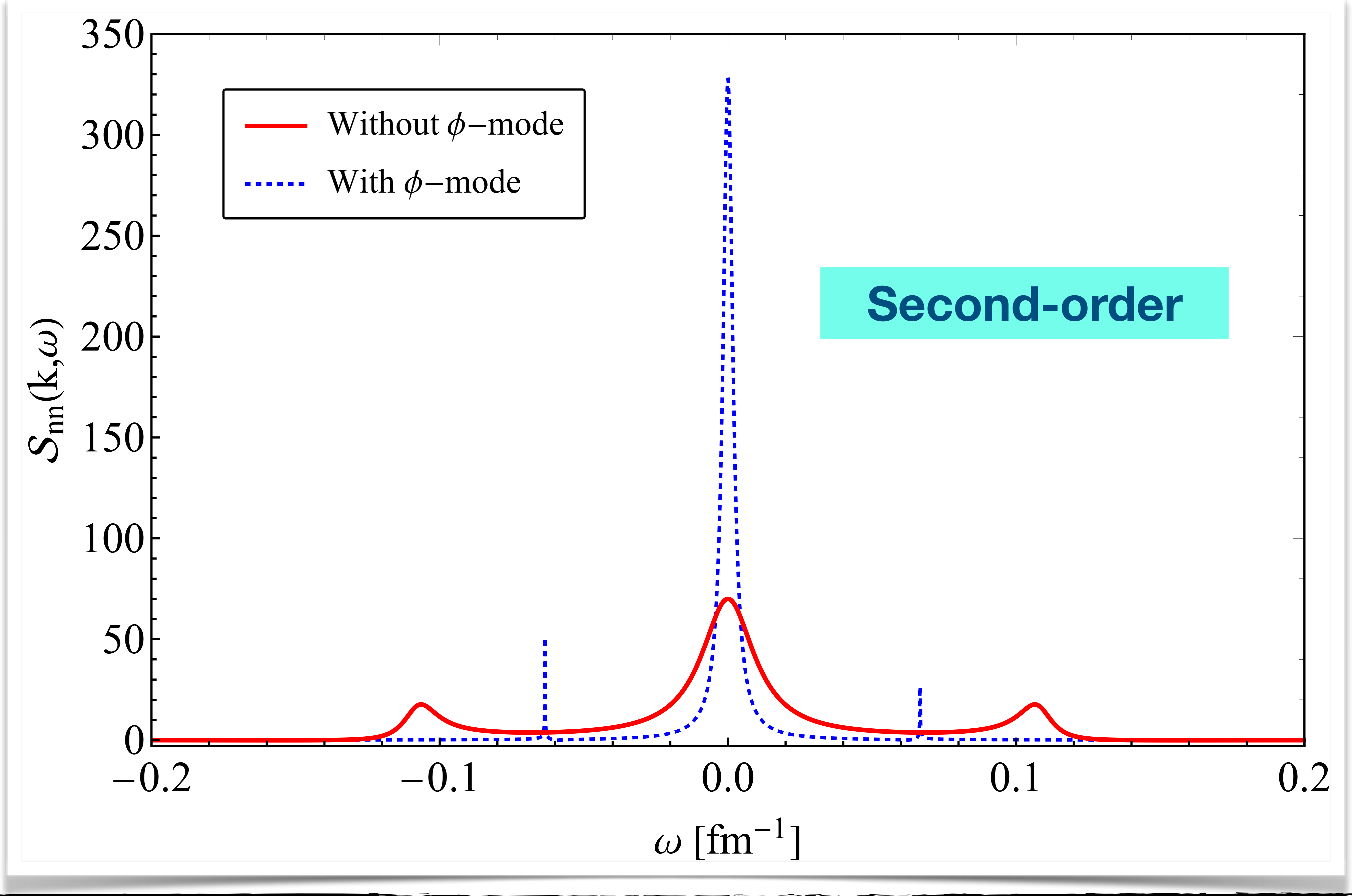
- Brillouin peaks shifts away from the Rayleigh peak.
- Due to the coupling strength, the peaks may reappear at larger k-values.



Results (system is near the CEP)



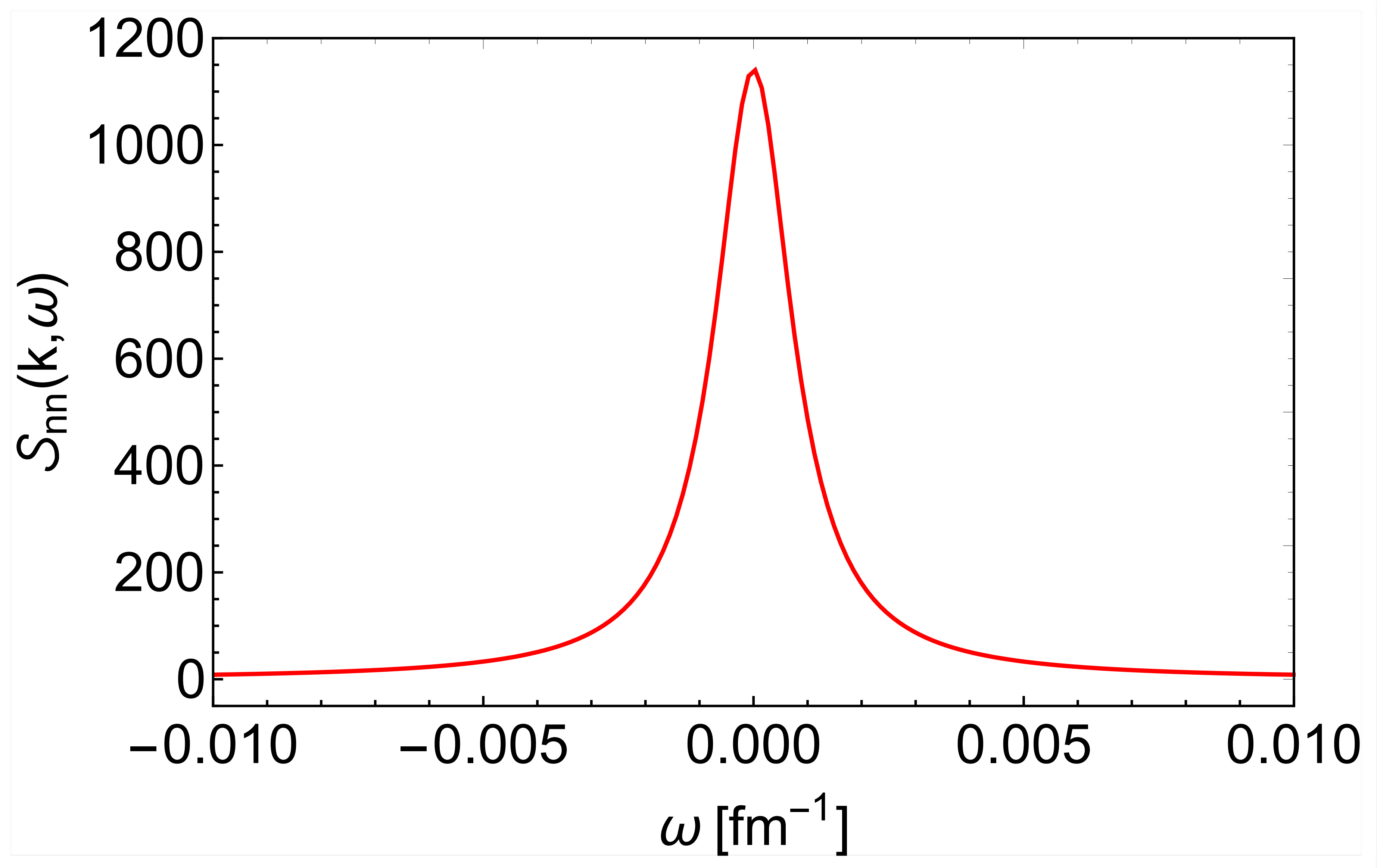
- Only longitudinal modes.
- Without ϕ, finite width of the peaks, symmetric Brillouin peaks.
- With ϕ, no extra peak, asymmetry in Brillouin peaks.



- No extra peak is seen in First-order theory (relaxation & coupling coefficients are not present).
- Effects of ϕ remains (narrower widths and asymmetry of the Brillouin peaks).

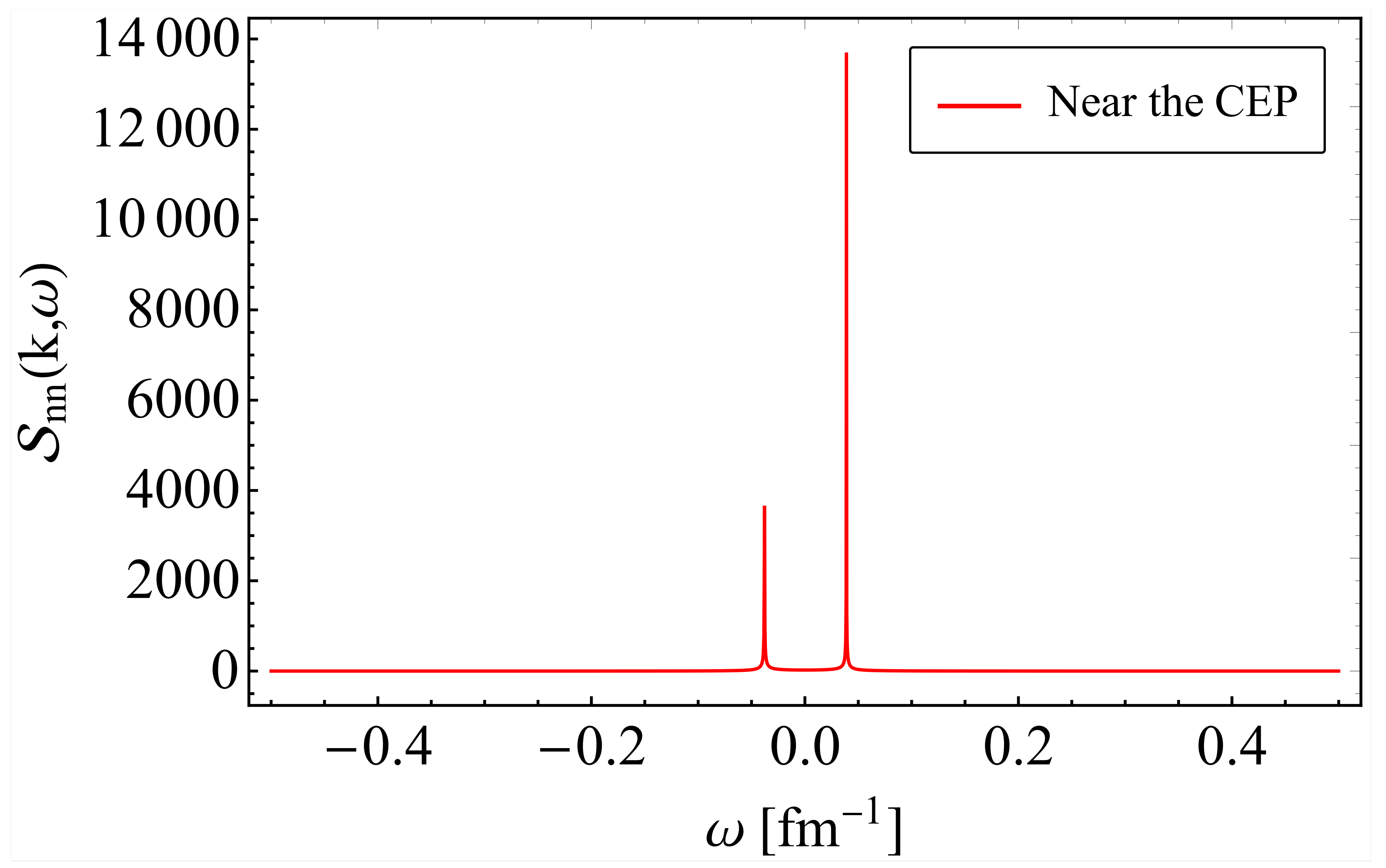
Extra peak appears only when the transverse modes are present in the second-order theory

Results (system is near the CEP)



- No distinct Brillouin peaks found, they merge with the Rayleigh peak gives rise to enhanced Rayleigh peak.
- Width of Rayleigh peak becomes very narrow.

- Brillouin peaks merge with the Rayleigh peak, enhancing the magnitude of the Rayleigh peak.
- The extra peak remains distinctly visible.



- Brillouin peaks merge with the Rayleigh peak due to the absorption of sound.
- Flow harmonics gets suppressed.
- Mach cone formation is prevented.
- Effect of Mach cone in particle correlation in away side jet vanishes.

Minami et al, Prog. Theor. Phys (2009)
MH et al, EPJA (2021)
MH et al, PRC (2020)
Sarwar et al, PLB (2021)

Summary & conclusion

- The behaviour of the $\mathcal{S}_{nn}(\overrightarrow{k}, \omega)$ is studied within the scope of Mueller-Israel-Stewart hydrodynamics, whose validity can be extended near the CEP by introducing an extra slow degree of freedom, ϕ.
- Away from the CEP, the $\mathcal{S}_{nn}(\overrightarrow{k}, \omega)$ shows four peaks of Lorentzian distribution, identified as a Rayleigh peak, two Brillouin peaks, and a peak due to coupling of hydro with slow modes.
- The extra peak appears due to the presence of the transverse modes of hydrodynamic fields in second-order hydrodynamics.
- Near the CEP, Brillouin peaks merge with the Rayleigh peak due to the absorption of sound.
- As a consequence Mach cone formation is prevented, could be reflected in the particle correlation of away side jet.

Thank You