Equilibrium expectations for non-Gaussian fluctuations near a QCD critical point

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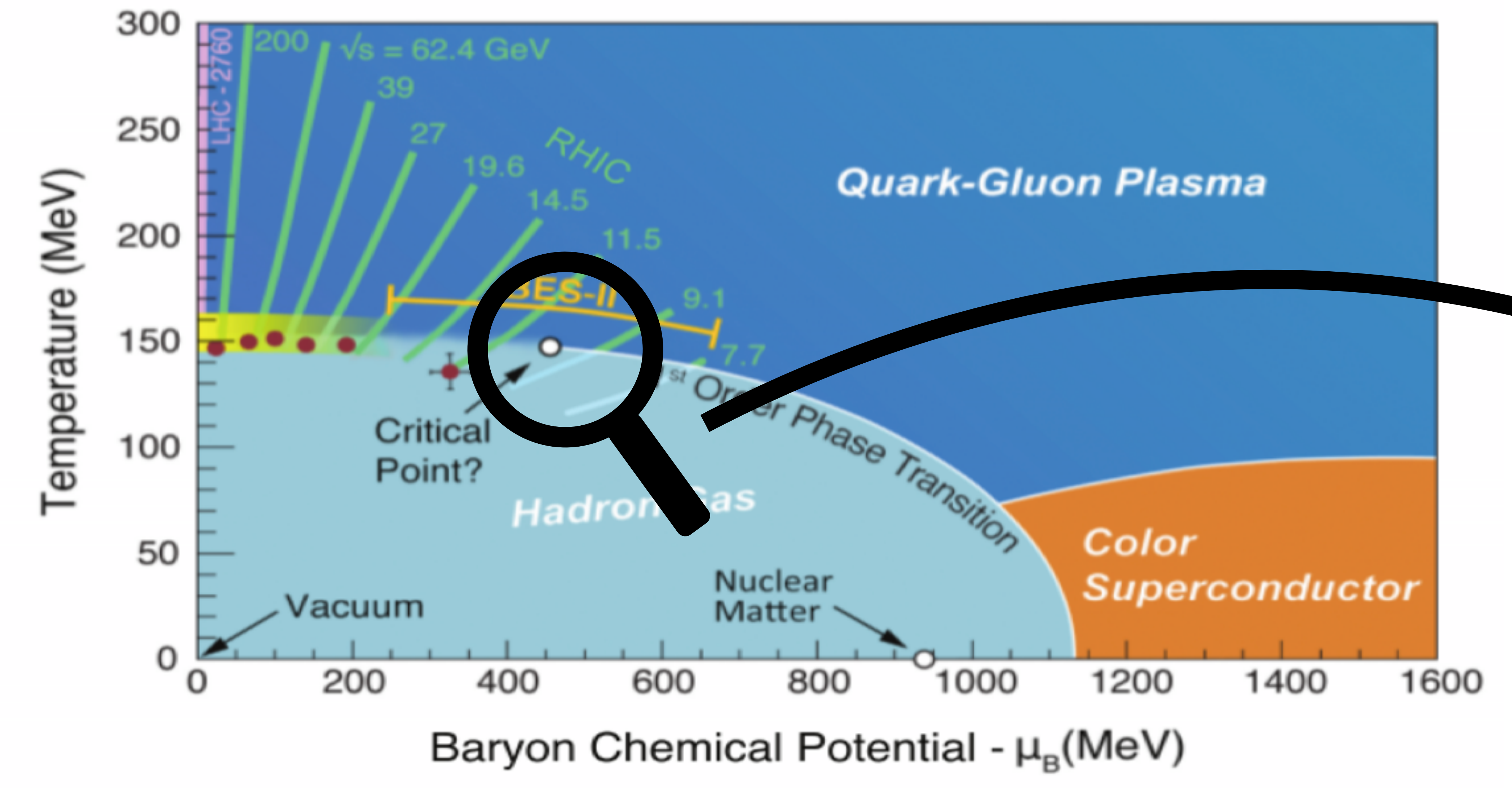
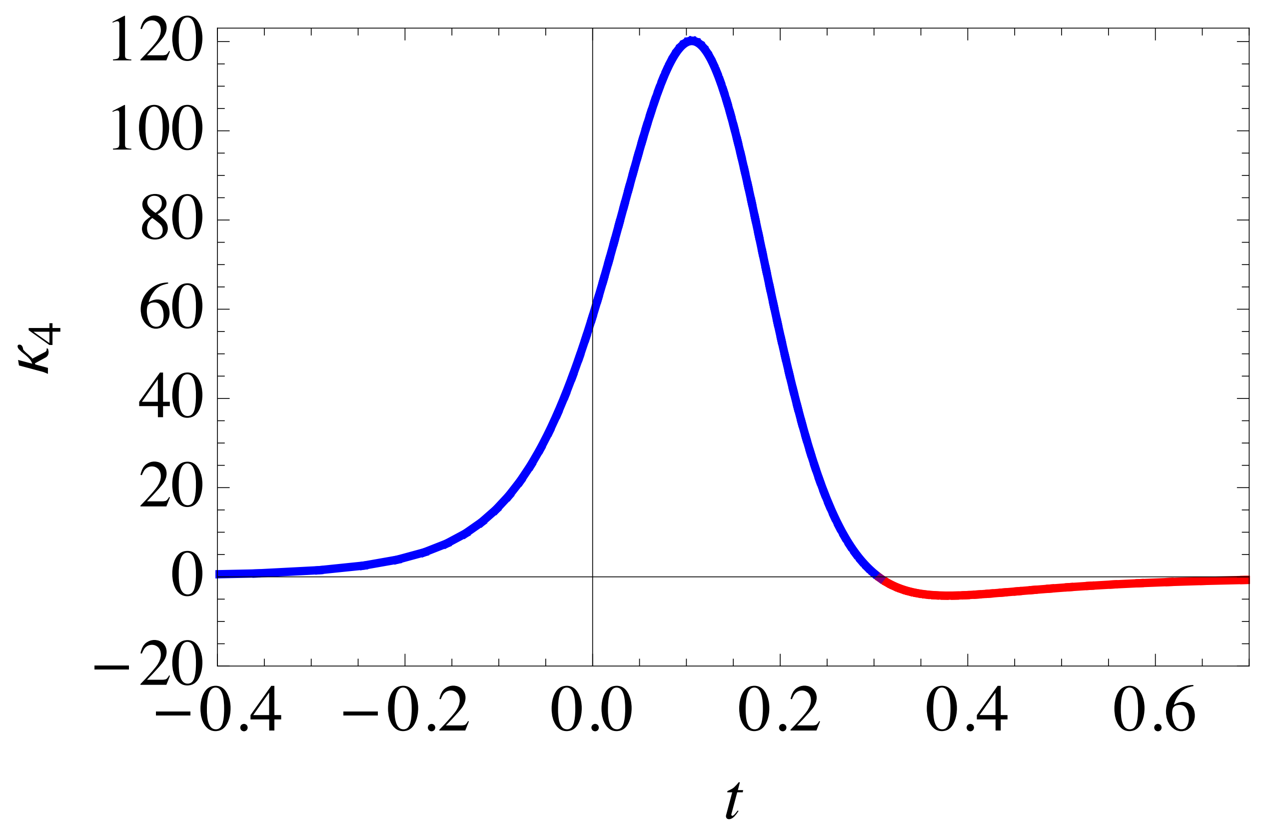
Collaborators: Maneesha Pradeep, Misha Stephanov,
Krishna Rajagopal, and Yi Yin

Search for Criticality

- Ongoing search for critical point requires support from theory community to provide candidates for criticality-carrying observables





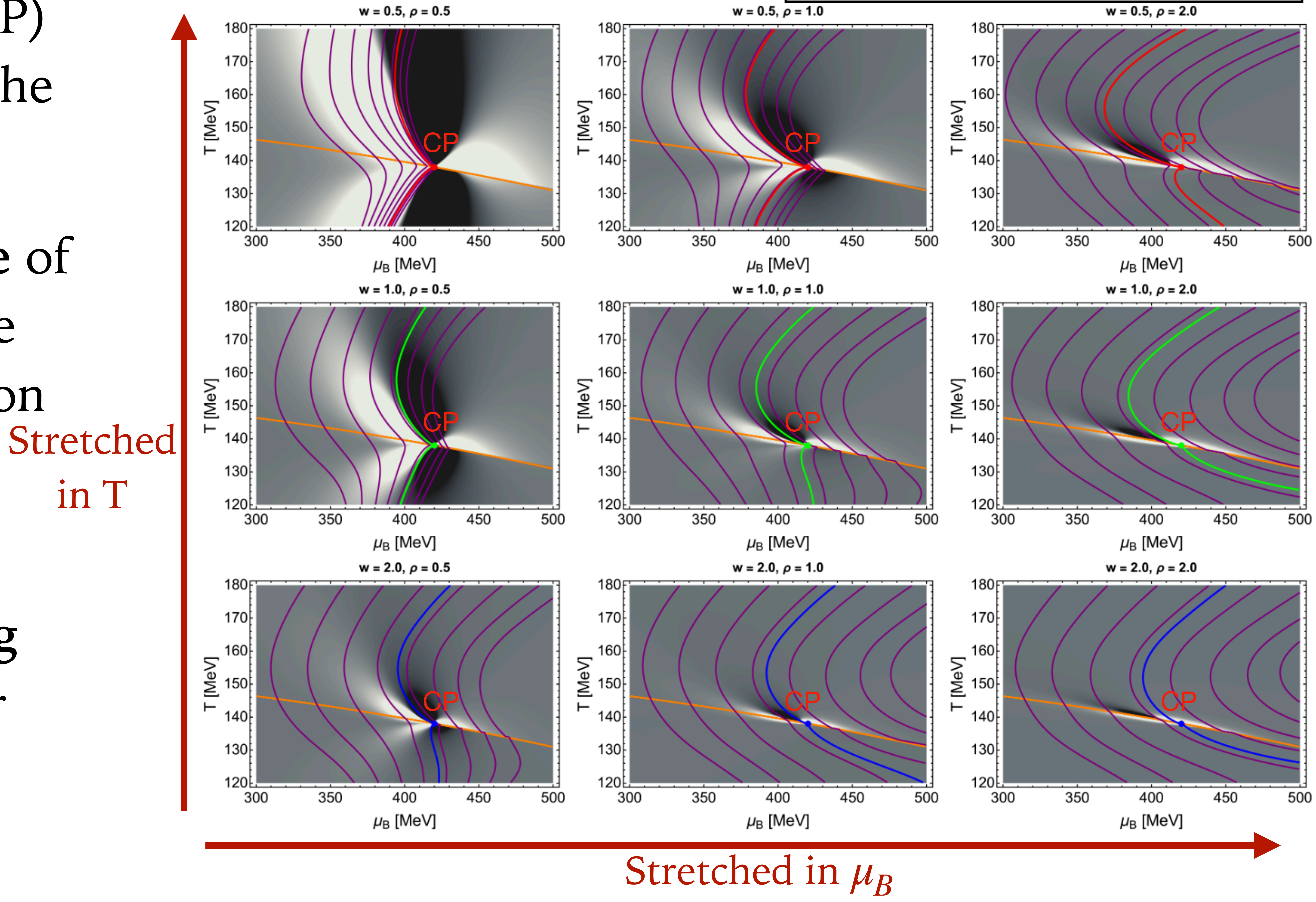
- Higher order susceptibilities diverge with higher power of the correlation length, $\kappa_4 \propto \xi^7$ $\quad \chi_n^B \equiv \frac{\partial^n (p/T^4)}{\partial (\mu_B/T)^n}$
- Related to moments of the net-proton distribution: can be measured experimentally $\quad \kappa_4 \sigma^2 = \chi_4^B / \chi_2^B$

NSAC 2015 Long Range Plan for Nuclear Physics
M. Stephanov, K. Rajagopal and E. Shuryak, PRD (1999)
M. Stephanov, PRL (2011)

Kurtosis and Critical Lensing in Equilibrium

See talk by T. Dore on Thursday

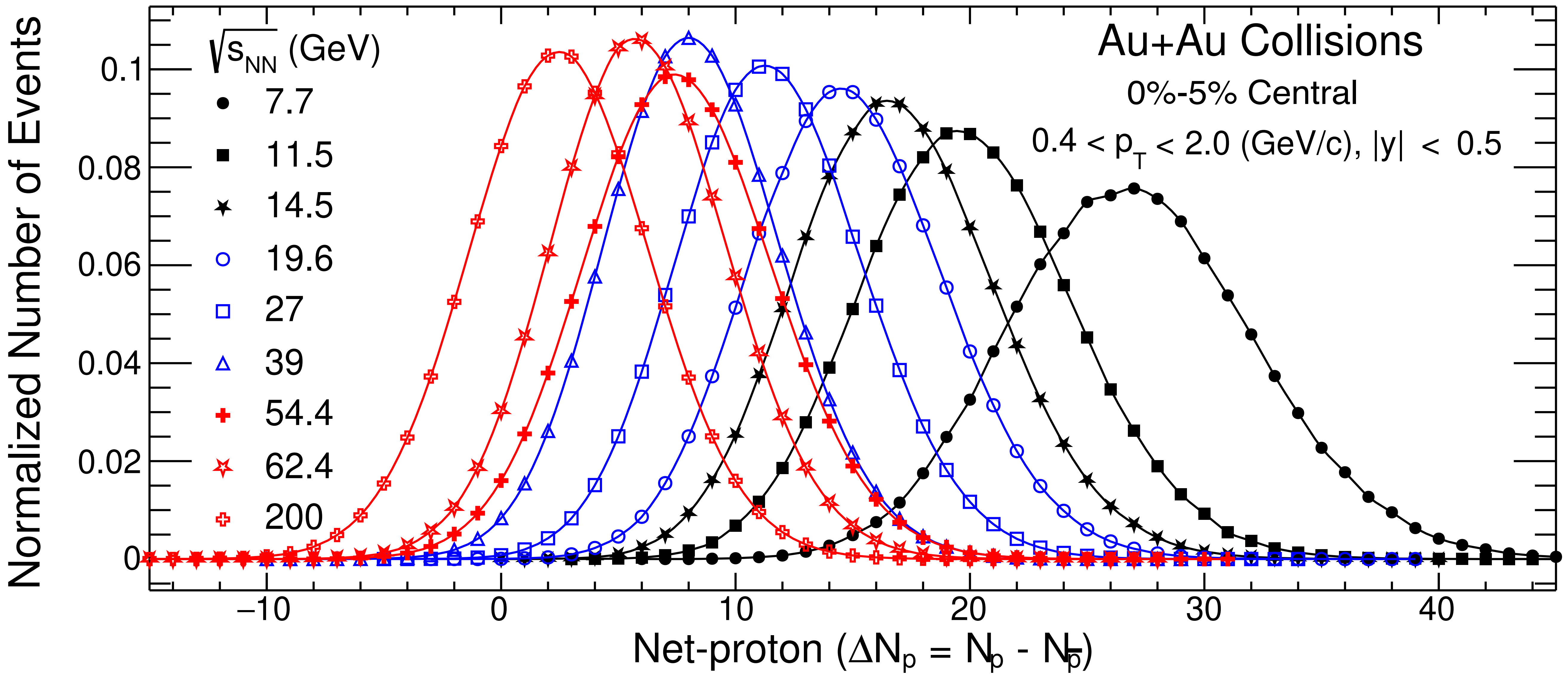
- <u>Critical lensing</u>: critical point (CP) is an attractor of trajectories in the QCD phase diagram
  - Study how the **size** and **shape** of the critical region affects these trajectories within the Equation of State with a critical point from the BEST COLLABORATION
- Critical regions **extending along the T-direction** show a stronger lensing effect

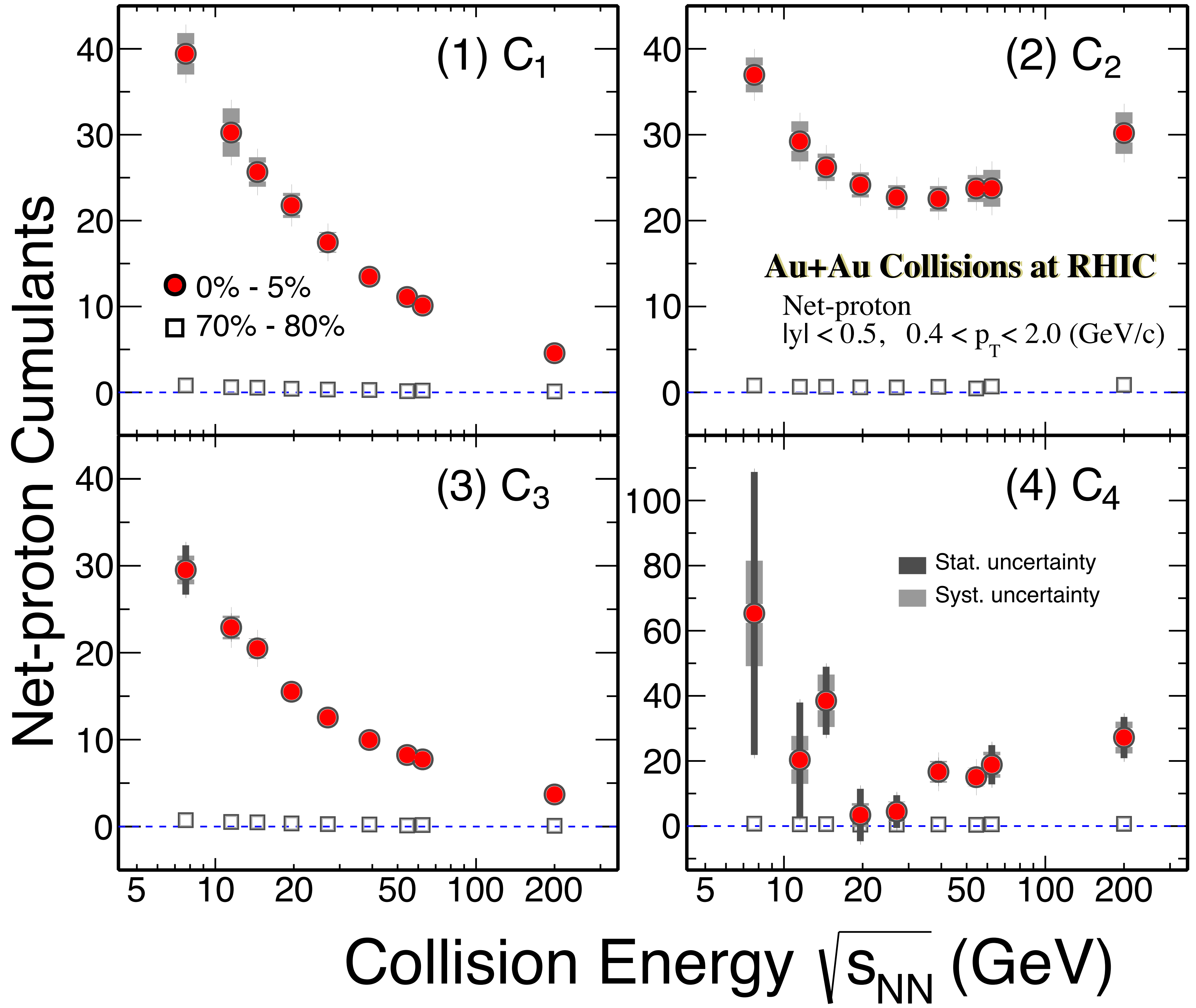


*T. Dore, **J.M. Karthein**, D. Mroczek et al, PRD (2022)*

Experimental Particle Distributions and Moments

➤ Event-by-event fluctuations are available for several particle species, including updated results from the BES program on net-proton fluctuations in search of the QCD critical point

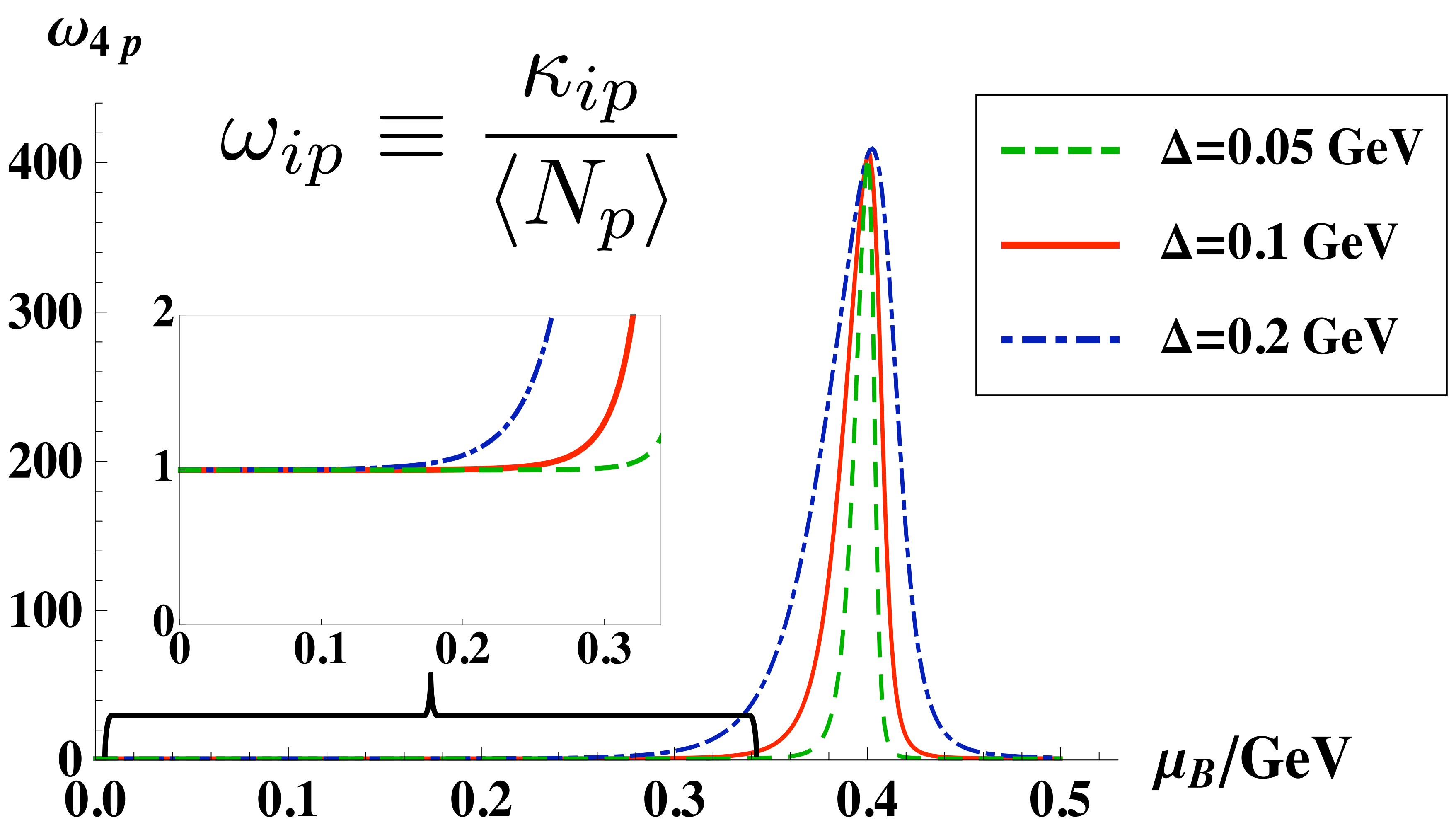




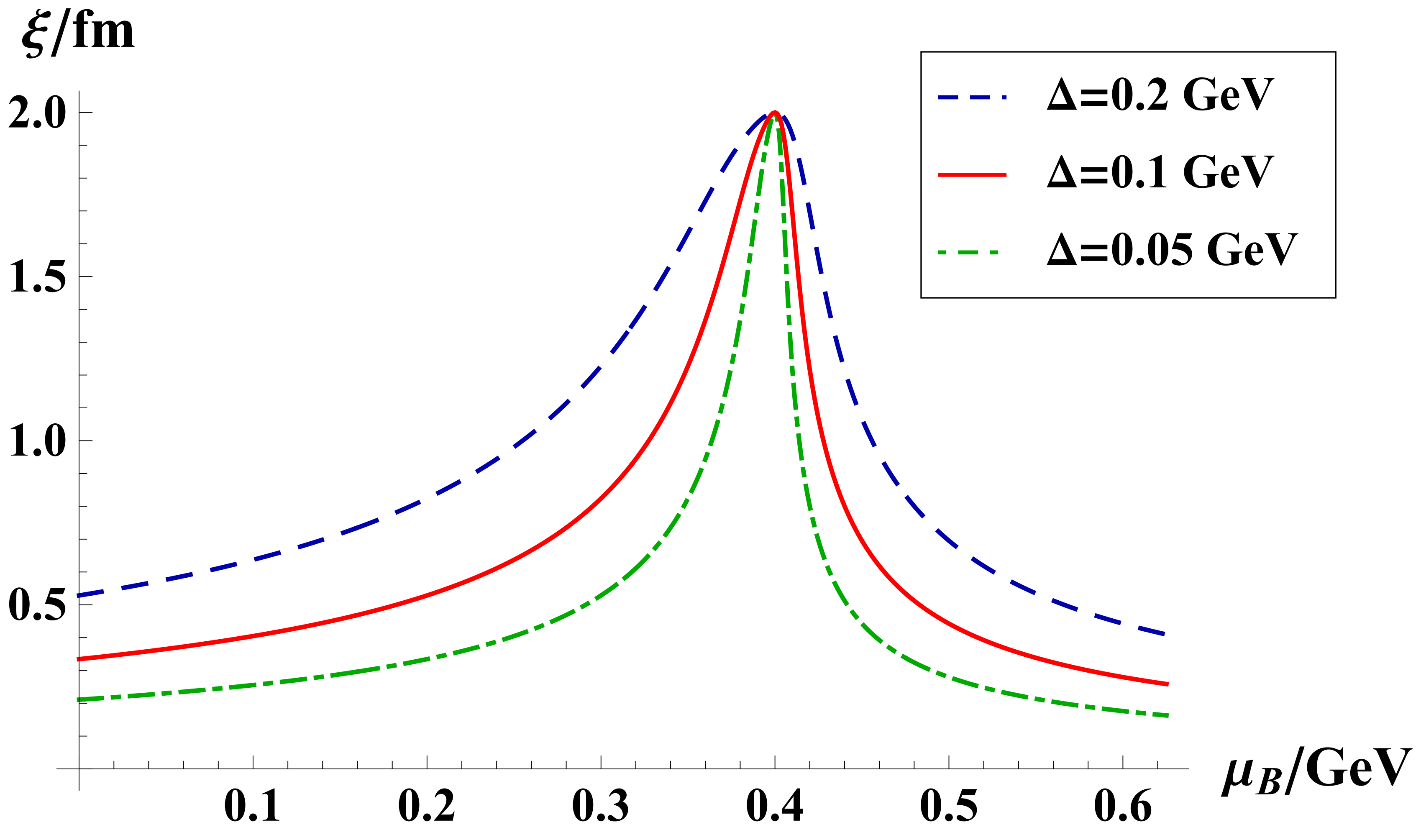
STAR Collaboration (L. Adamczyk et al.), PRL (2014), PRL (2021)

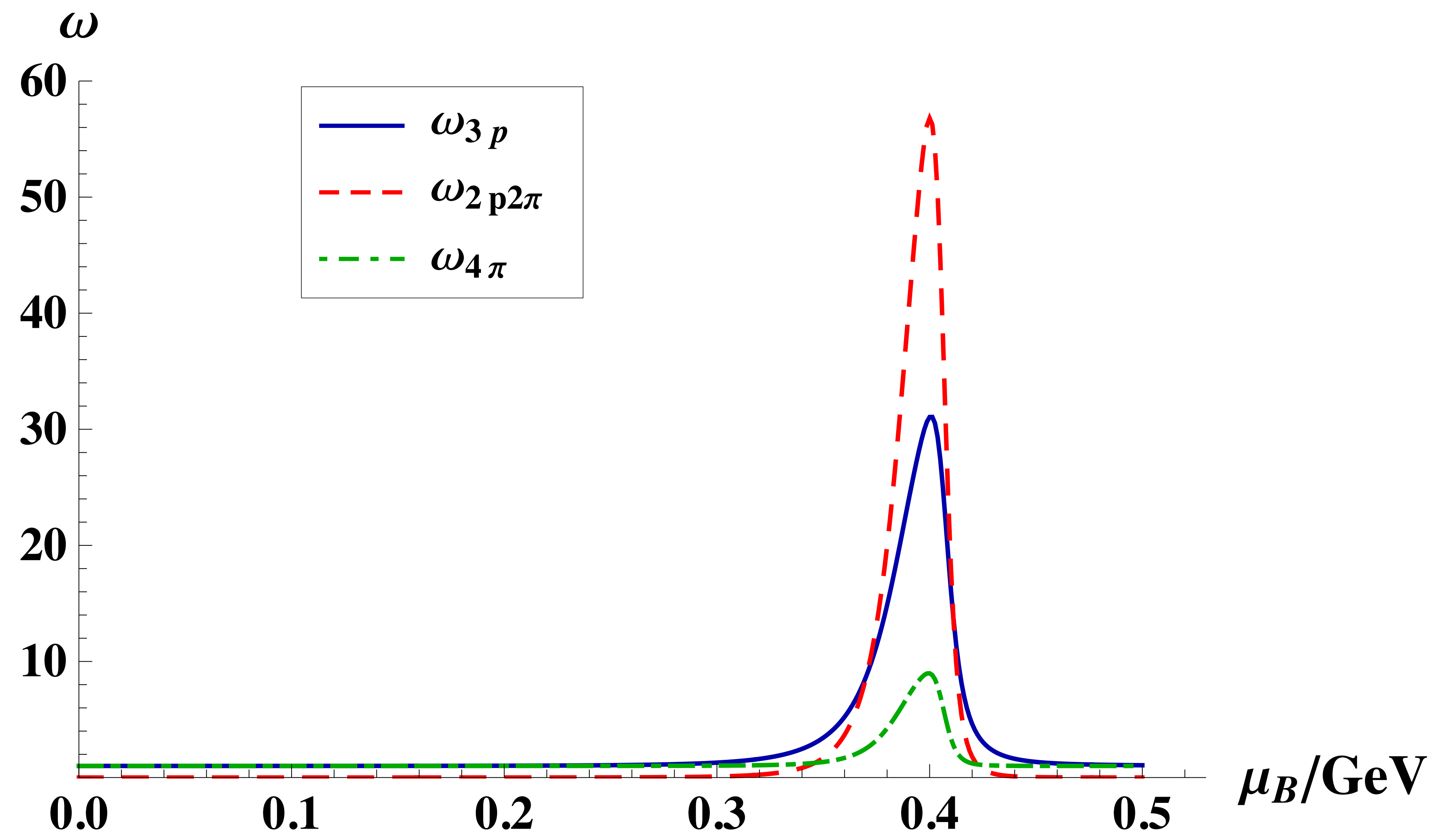
Early Estimates of Equilibrium Fluctuations

- Order-of-magnitude predictions of volume-independent normalized cumulants from 2010 relied on ansätze
  - Original estimates used parametrized correlation length with width Δ



$$\xi(\mu_B) = \frac{\xi_{\max}}{\left[1 + \frac{(\mu_B - \mu_B^c)^2}{W(\mu_B)^2}\right]^{1/3}}$$





C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)

Effective Field Theory for Critical Fluctuations

See talk by M. Pradeep next

- Fluctuations near the critical point are driven by coupling of particles to $\boldsymbol{\sigma}$-field

$$\Omega = \int d^3\boldsymbol{x} \left[\frac{(\boldsymbol{\nabla}\sigma)^2}{2} + \frac{m_\sigma^2}{2}\sigma^2 + \frac{\lambda_3}{3}\sigma^3 + \frac{\lambda_4}{4}\sigma^4 + \ldots \right]$$

$$\delta f_{\boldsymbol{p}} = \delta f_{\boldsymbol{p}}^0 + \frac{\partial n_{\boldsymbol{p}}}{\partial m} g\,\delta\sigma$$

$$\delta m = g\delta\sigma$$

- Correlation length diverges as the $\boldsymbol{\sigma}$ mass vanishes: $\xi = m_\sigma^{-1}$
  - Higher order fluctuations depend on larger powers of ξ, introduce higher point couplings

$$\kappa_2 = \langle \sigma_V^2 \rangle = VT\,\xi^2\,; \qquad \kappa_3 = \langle \sigma_V^3 \rangle = 2\lambda_3 VT^2\,\xi^6$$

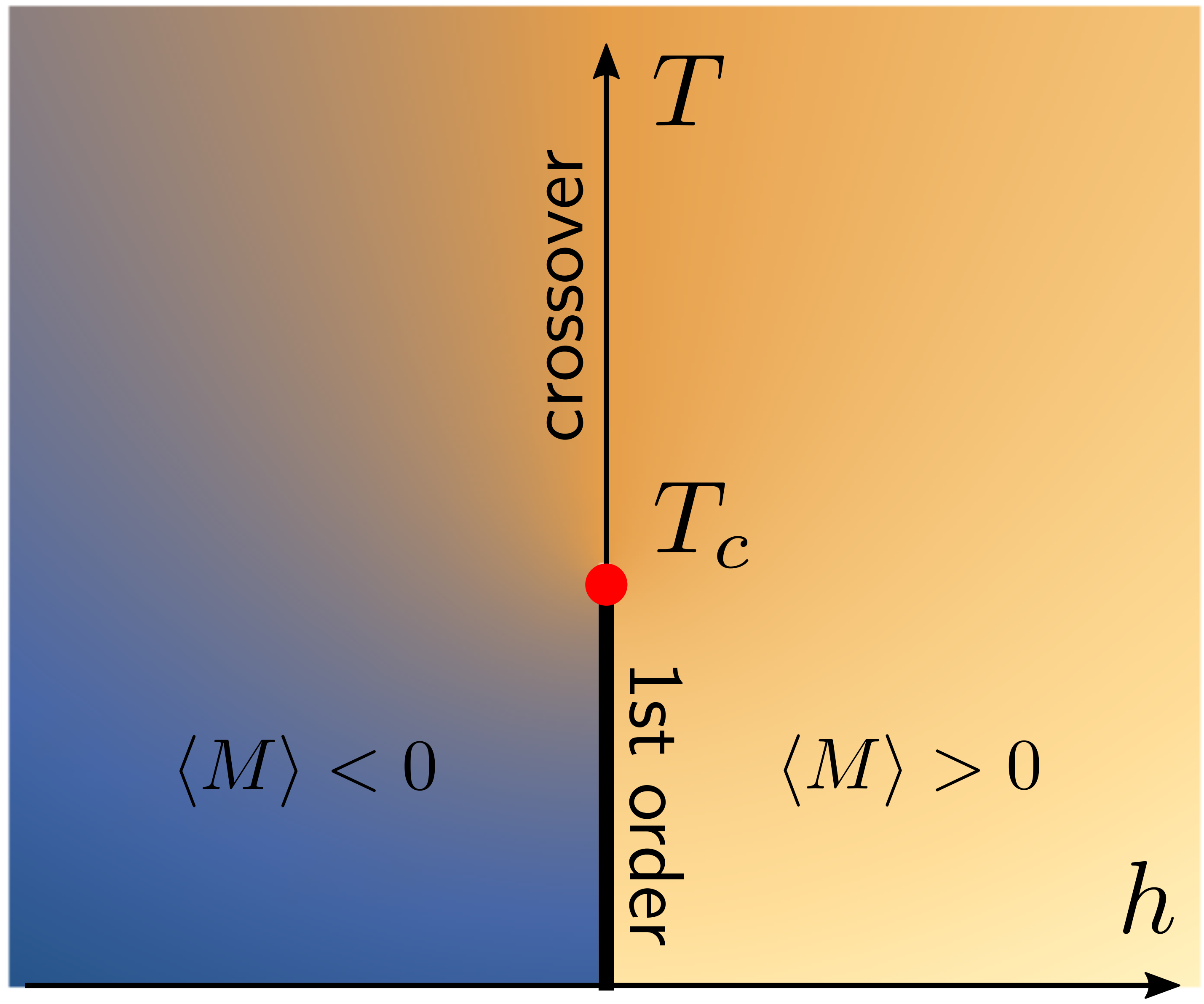
$$\kappa_4 = \langle \sigma_V^4 \rangle_c \equiv \langle \sigma_V^4 \rangle - 3\langle \sigma_V^2 \rangle^2 = 6VT^3\,[\,2(\lambda_3\xi)^2 - \lambda_4\,]\,\xi^8$$

M. Stephanov, K. Rajagopal and E. Shuryak, PRD (1999)
C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)
M. Stephanov, PRL (2011)

Universal Scaling EOS

- Average critical fluctuations of σ give rise to “magnetization”: $M = \langle \sigma \rangle$
- Universal critical scaling behavior given by the 3D Ising model equation of state:
  - Magnetic field: $h = h_0 R^{\beta\delta} H(\theta)$, $H(\theta) = \theta(3 - 2\theta^2)$
  - Reduced temperature: $t = R(1 - \theta^2)$
  - Magnetization: $M = M_0 R^{\beta} \theta$
- Critical fluctuations calculated in 3D Ising EOS

$$\kappa_{n+1}^{\mathrm{eq}} \propto \left(\frac{\partial^n M^{\mathrm{eq}}(t,h)}{\partial h^n} \right)_t$$



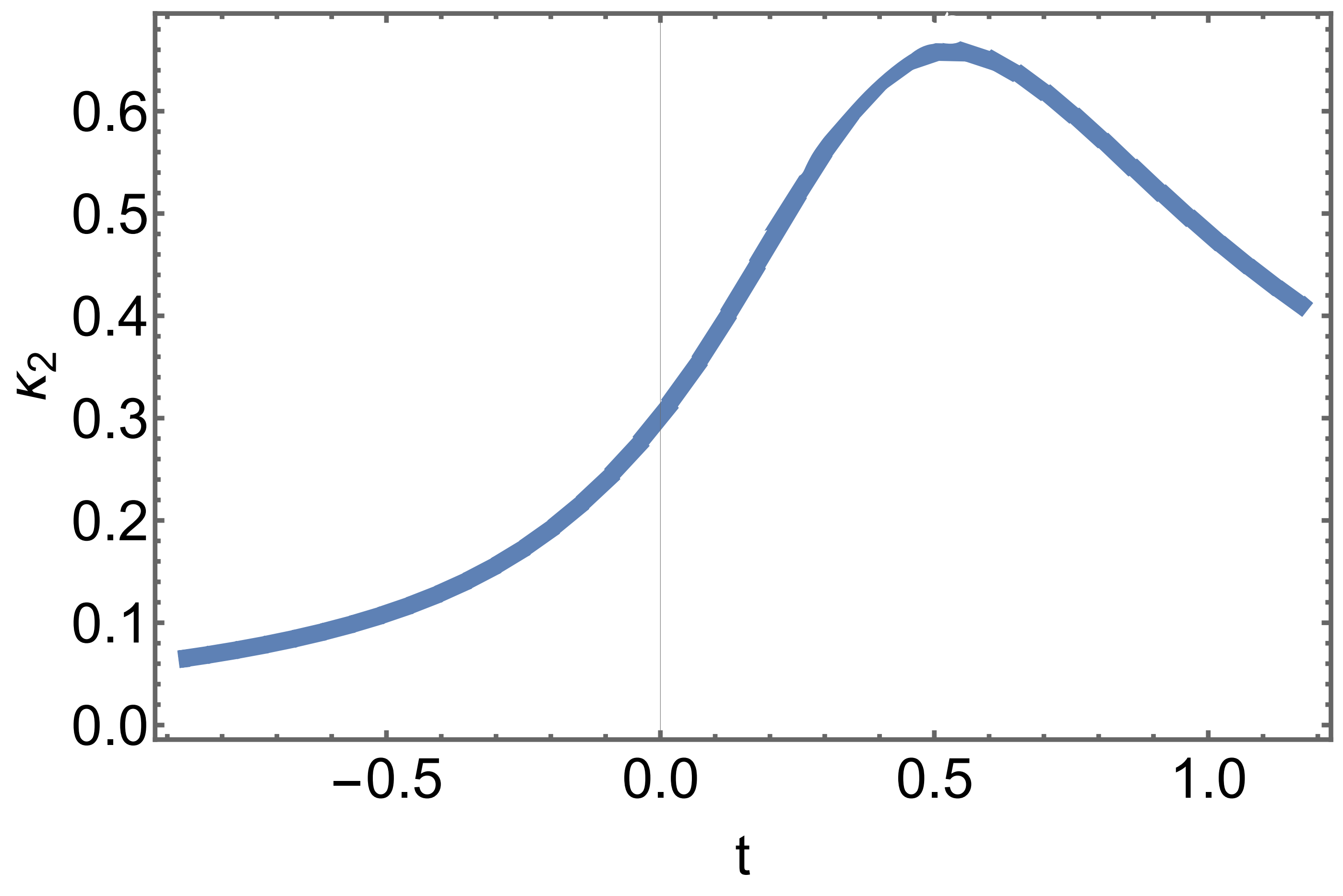
K. Rajagopal and F. Wilczek, Nucl. Phys. B (1993)
J. Zinn-Justin, Quantum Field Theory and Critical Phenomena
S. Mukherjee, R. Venugopalan, Y. Yin, PRC (2015)
A. Bzdak et al, Phys. Rep. (2020)

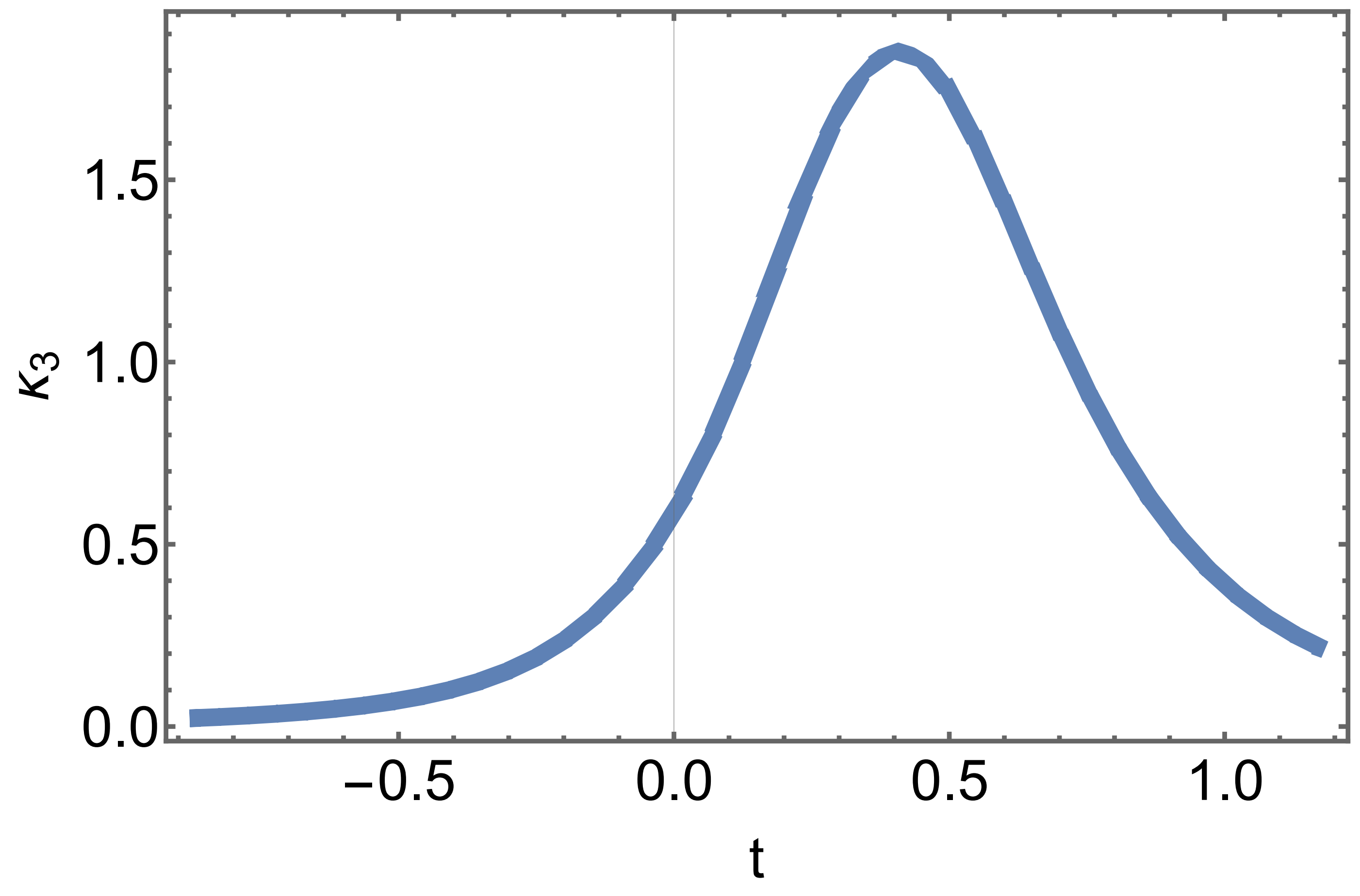
Equilibrium Fluctuations in 3D Ising Model

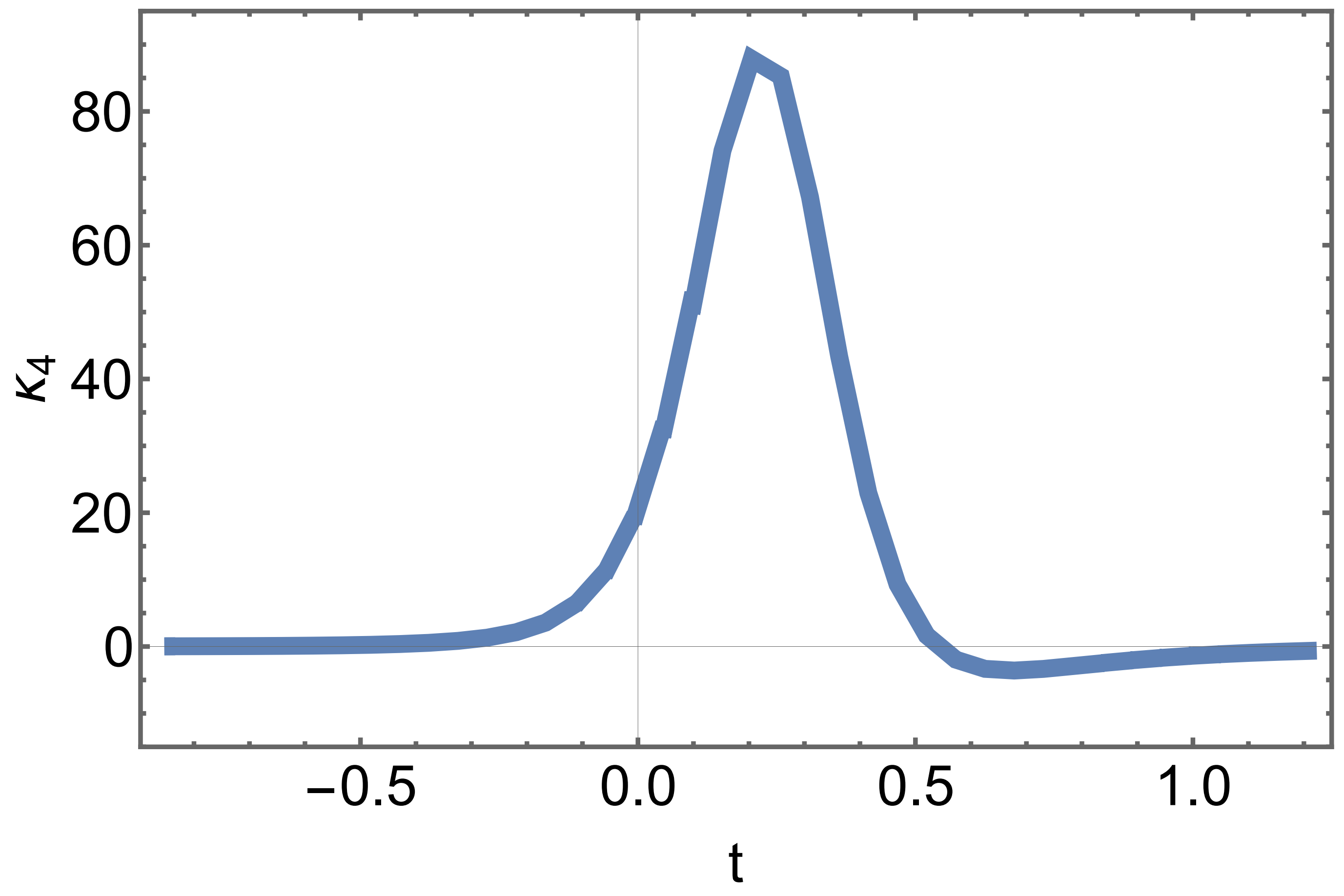
- Calculate critical fluctuations as parametric derivatives of universal EOS utilizing approximate critical exponents

$$\kappa_{n+1}^{\mathrm{eq}} \propto \left(\frac{\partial^n M^{\mathrm{eq}}(t,h)}{\partial h^n} \right)_t$$

$$\kappa_2 = \frac{M_0}{h_0} \frac{1}{R^{4/3}\left(3+2\theta^2\right)} \qquad \kappa_3 = \frac{-M_0}{h_0^2} \frac{4\theta\left(9+\theta^2\right)}{R^3\left(3-\theta^2\right)(3+2\theta)^3} \qquad \kappa_4 = \frac{-M_0}{h_0^3} \frac{12\left(2\theta^8-5\theta^6+105\theta^4-783\theta^2+81\right)}{R^{14/3}\left(3-\theta^2\right)^3\left(3+2\theta^2\right)^5}$$







M. Stephanov, PRL (2011)
S. Mukherjee, R. Venugopalan, Y. Yin, PRC (2015)

Equilibrium Correlation Length in 3D Ising Model

- 3D Ising EOS also provides a parametrization of the correlation length in the ϵ-expansion

$$\xi^2(M,t) = R^{-2\nu} g_\xi(\theta)$$

  - New equilibrium calculation to $\mathcal{O}(\epsilon^2)$

$$g_\xi(\theta) = g_\xi(0)\left(1 - \frac{5}{18}\epsilon\theta^2 + \left[\frac{1}{972}(24I - 25)\theta^2 + \frac{1}{324}(4I + 41)\theta^4\right]\epsilon^2)\right)$$

$$\text{where: } I \equiv \int_0^1 \frac{\ln[x(1-x)]}{1 - x(1-x)} dx \sim -2.3439$$

- Now with the true critical EOS determine the higher order couplings

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J. Zinn-Justin, Quantum Field Theory and Critical Phenomena

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J. Zinn-Justin, Quantum Field Theory and Critical Phenomena

Mapping to QCD Phase Diagram

- Utilize the BEST EOS mapping between the Ising parametric variables and QCD

  - Linear map:

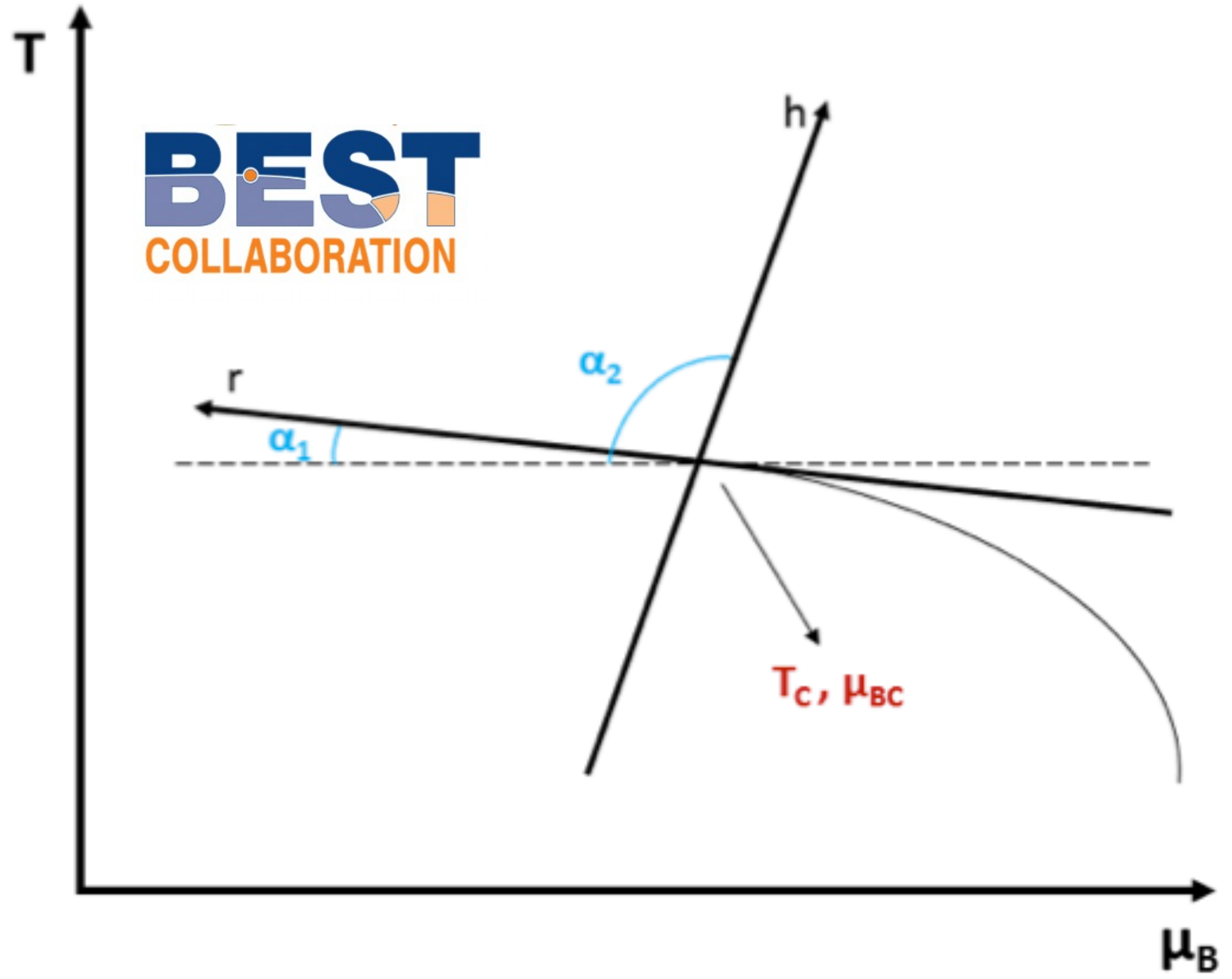
$$(\mathbf{r},\mathbf{h}) \longleftrightarrow (\mathbf{T},\mu_{\mathbf{B}}) : \quad \frac{T-\mathbf{T_C}}{\mathbf{T_C}} = \mathbf{w}\,(r\rho\,\sin\alpha_1 + h\,\sin\alpha_2)$$

$$\frac{\mu_B-\mu_{\mathbf{BC}}}{\mathbf{T_C}} = \mathbf{w}\,(-r\rho\,\cos\alpha_1 - h\,\cos\alpha_2)$$

  - Reduce free parameters by imposing constraints from Lattice QCD

$$T = T_0 + \kappa T_0 \left(\frac{\mu_B}{T_0}\right)^2 + O(\mu_B^4), \qquad \alpha_1 = \tan^{-1}\left(2\frac{\kappa}{T_0}\mu_{BC}\right)$$

  - Parameter choice consistent with BEST EOS: $\mu_{B,c} = 350\,\text{MeV}, w = 1,\ \rho = 2,\ \alpha_2 - \alpha_1 = 90^o$



P. Parotto et al, PRC (2020),
J. M. Karthein et al, EPJ+ (2021)

Estimating Equilibrium Fluctuations with BEST EOS

- Re-evaluate equilibrium estimates for normalized cumulants $\omega_{ip} \equiv \frac{\kappa_{ip}}{\langle N_p \rangle}$ with realistic critical EOS
  - Updates: $\xi, \lambda_3, \lambda_4$ (dimensionless, ξ-independent: $\tilde{\lambda}_3 = \lambda_3 T^{1/2} \xi^{3/2}$, $\tilde{\lambda}_4 = \lambda_4 T \xi$)
  - Remaining dependence on coupling: g_p

$$\omega_{4p,\sigma} = \frac{6(2\tilde{\lambda}_3^2 - \tilde{\lambda}_4)}{T^2 n_p} \xi^7 \left(d_p \, g_p \int_k \frac{v_k^2}{\gamma_k} \right)^4 \quad \xrightarrow{\textit{generalize}} \quad \omega_{ip} = 1 + \omega_{ip}^{\text{prefactor}} \left(\frac{n_p}{n_0} \right)^{i-1} \left(\frac{\xi}{\xi_{\max}} \right)^{\frac{5}{2}i-3}$$

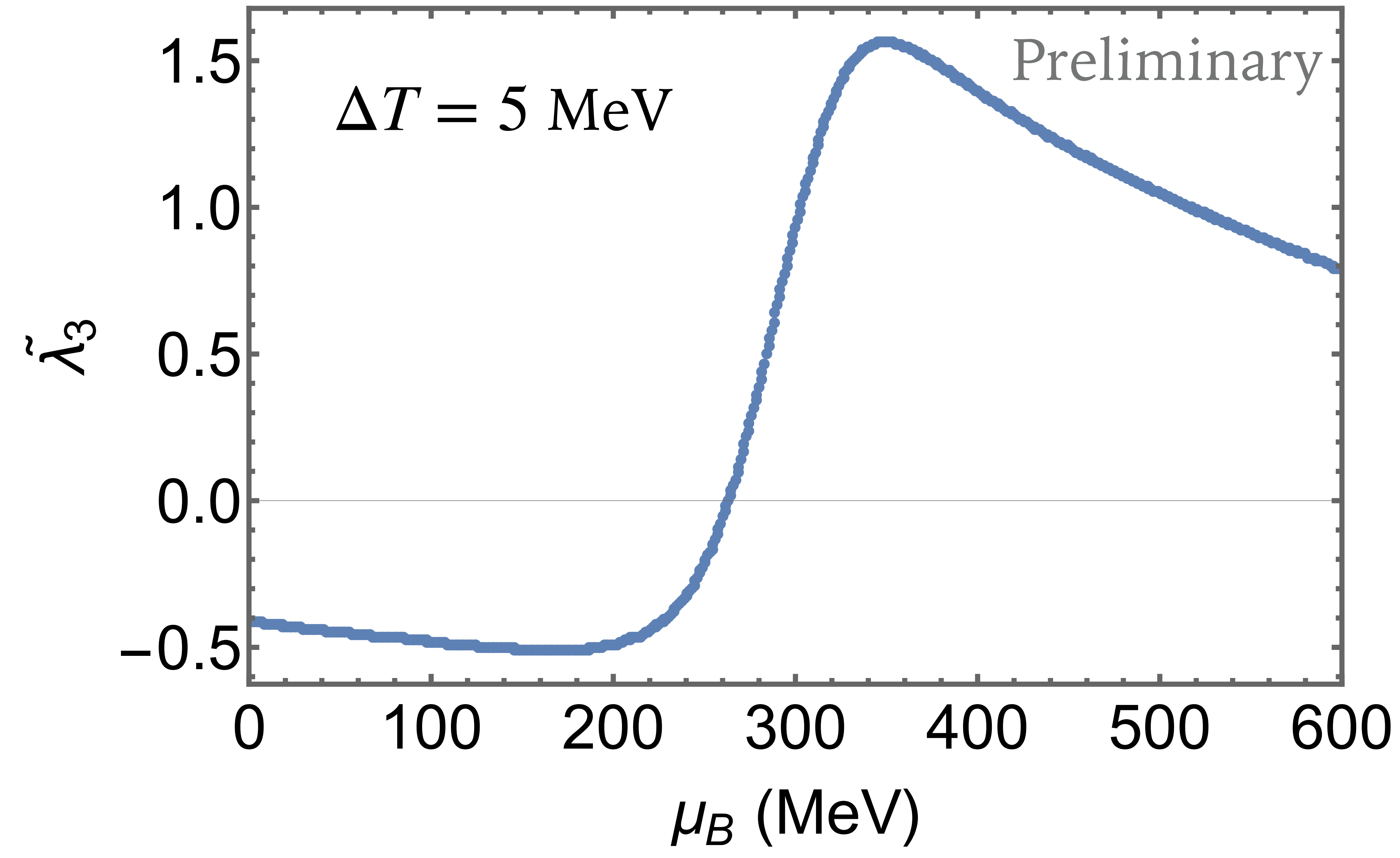
$$\omega_{ip}^{\text{prefactor}} = \frac{\tilde{\lambda}_i'(i-1)!\,\xi_{\max}^{\frac{5}{2}i-3}}{T^{i/2} n_p} \left(\int_k d_p \, g_p \frac{v_k^2}{\gamma_k} \right)^i \left(\frac{n_0}{n_p} \right)^{i-1}$$

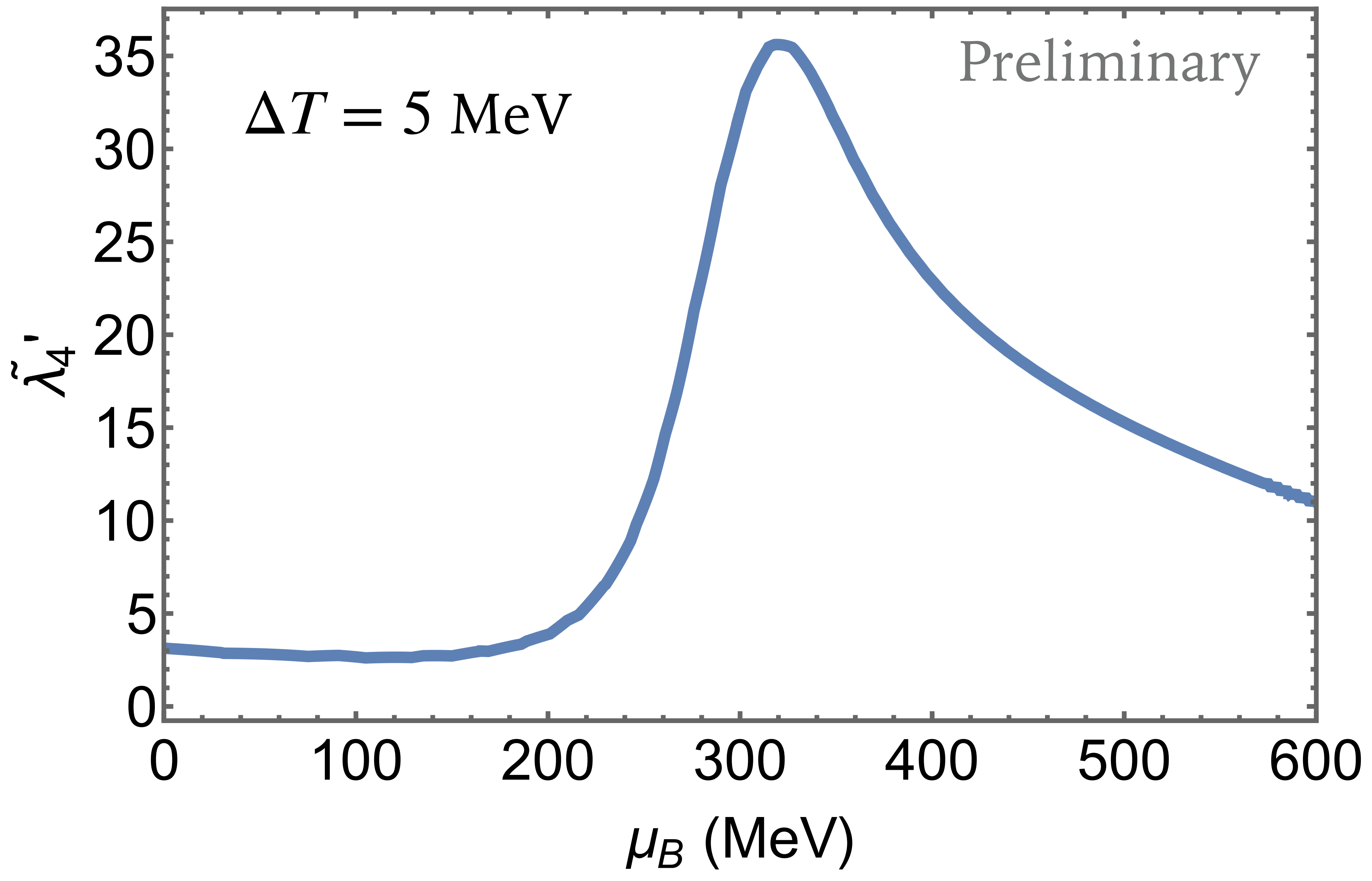
$$\tilde{\lambda}_3' \equiv \tilde{\lambda}_3 \quad \text{and} \quad \tilde{\lambda}_4' \equiv 2\tilde{\lambda}_3^2 - \tilde{\lambda}_4$$

C. Athanasiou, K. Rajagopal, M. Stephanov, PRD (2010)

Extracting Higher-point Couplings

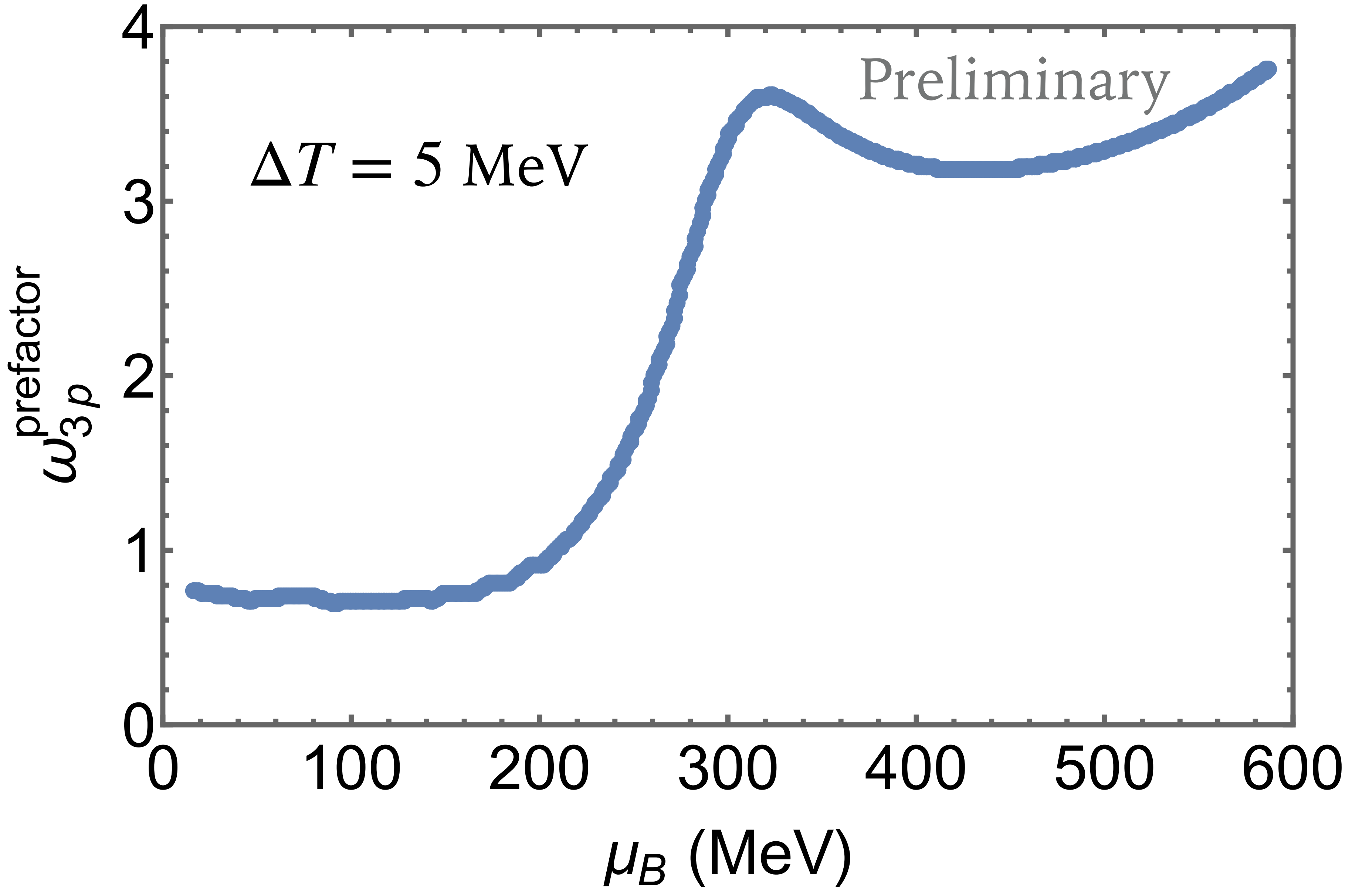
➤ Determine dimensionless couplings and their μ_B-dependence along chemical freeze-out lines parallel to the transition line from Lattice QCD $\Delta T = 5$ MeV below critical point

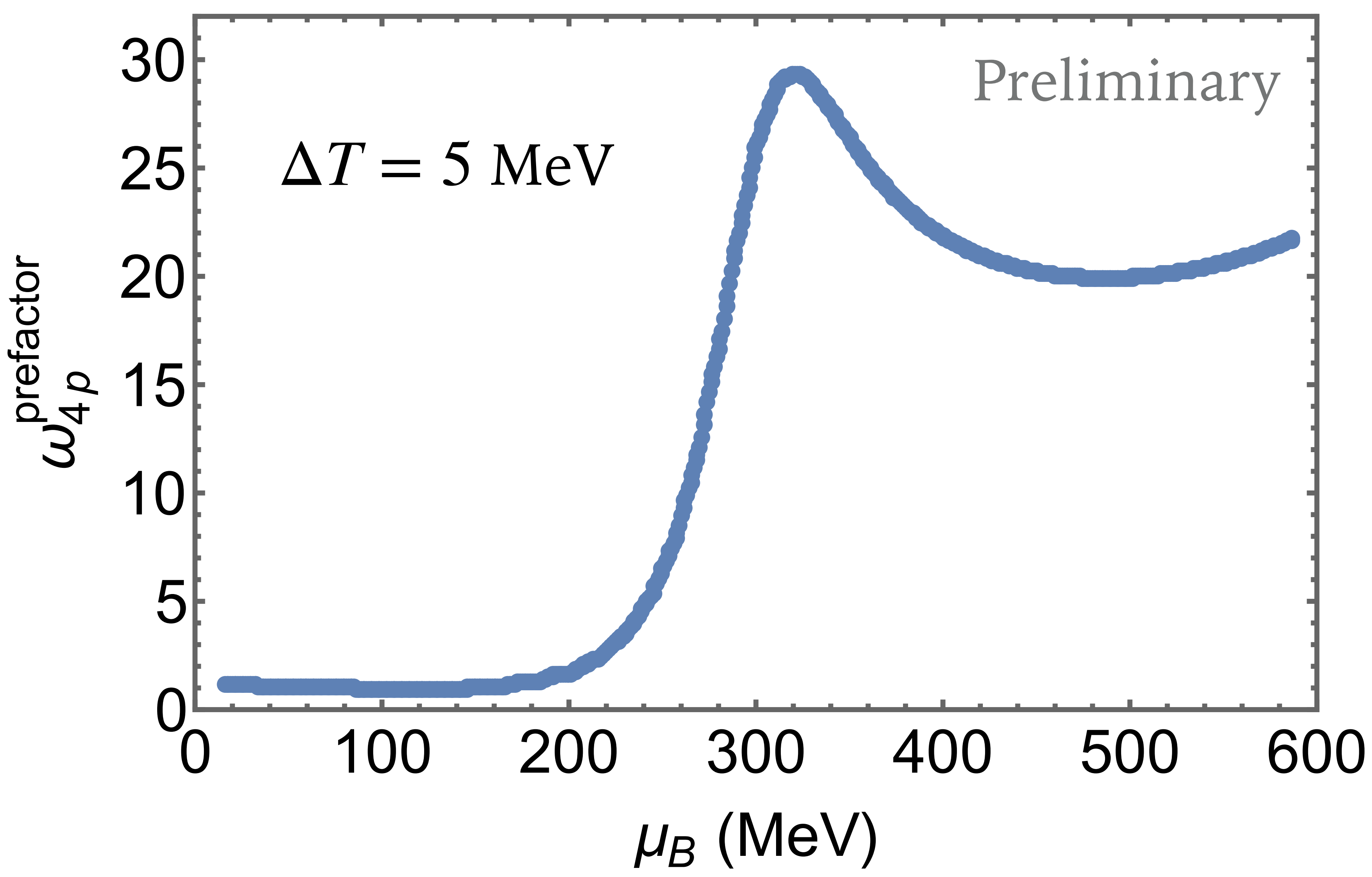




Pre-factors for Equilibrium Normalized Cumulants

- Update non-critical pre-factors along the same freeze-out line
  - Carry stronger μ_B-dependence than early estimates due to λ's

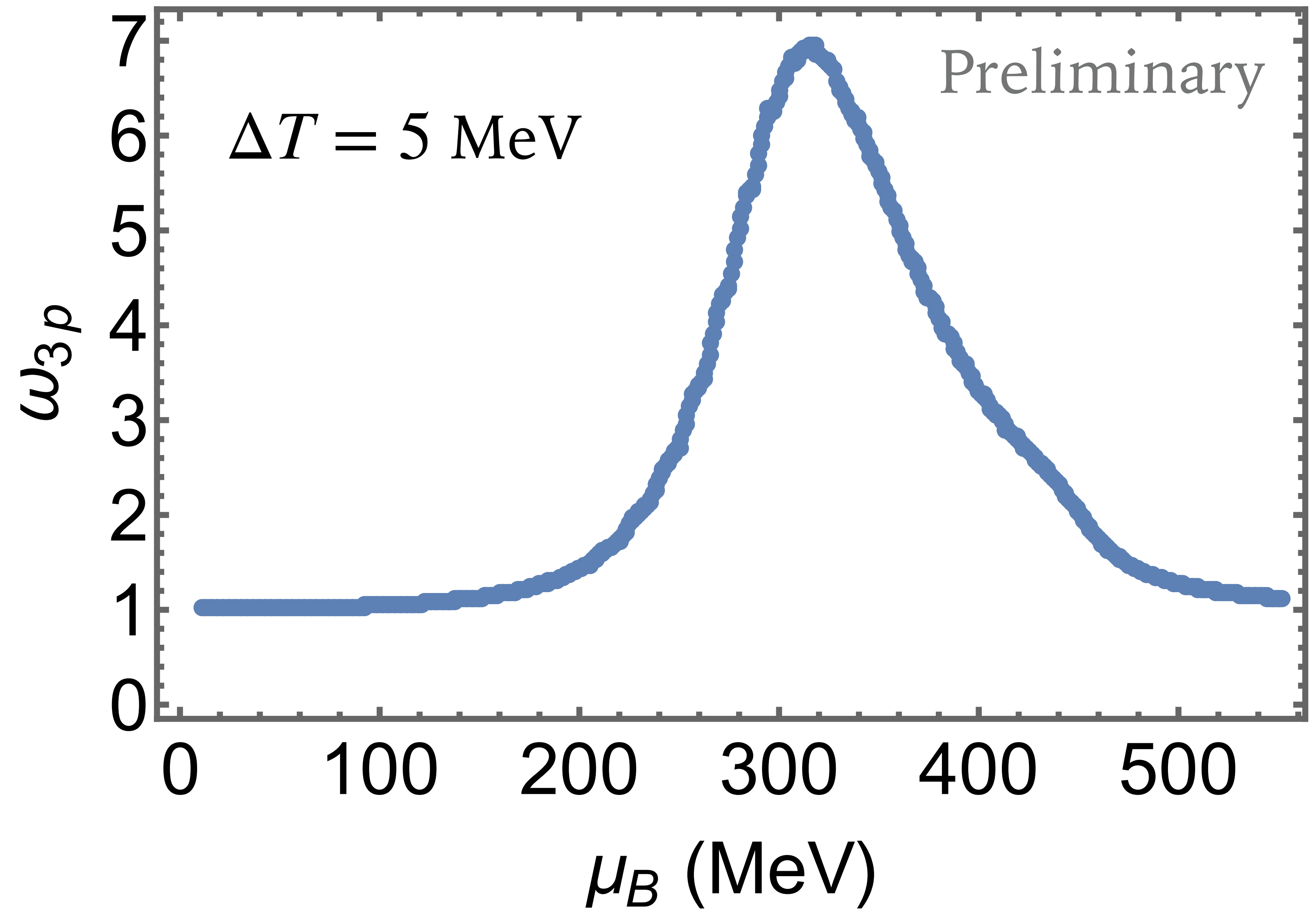


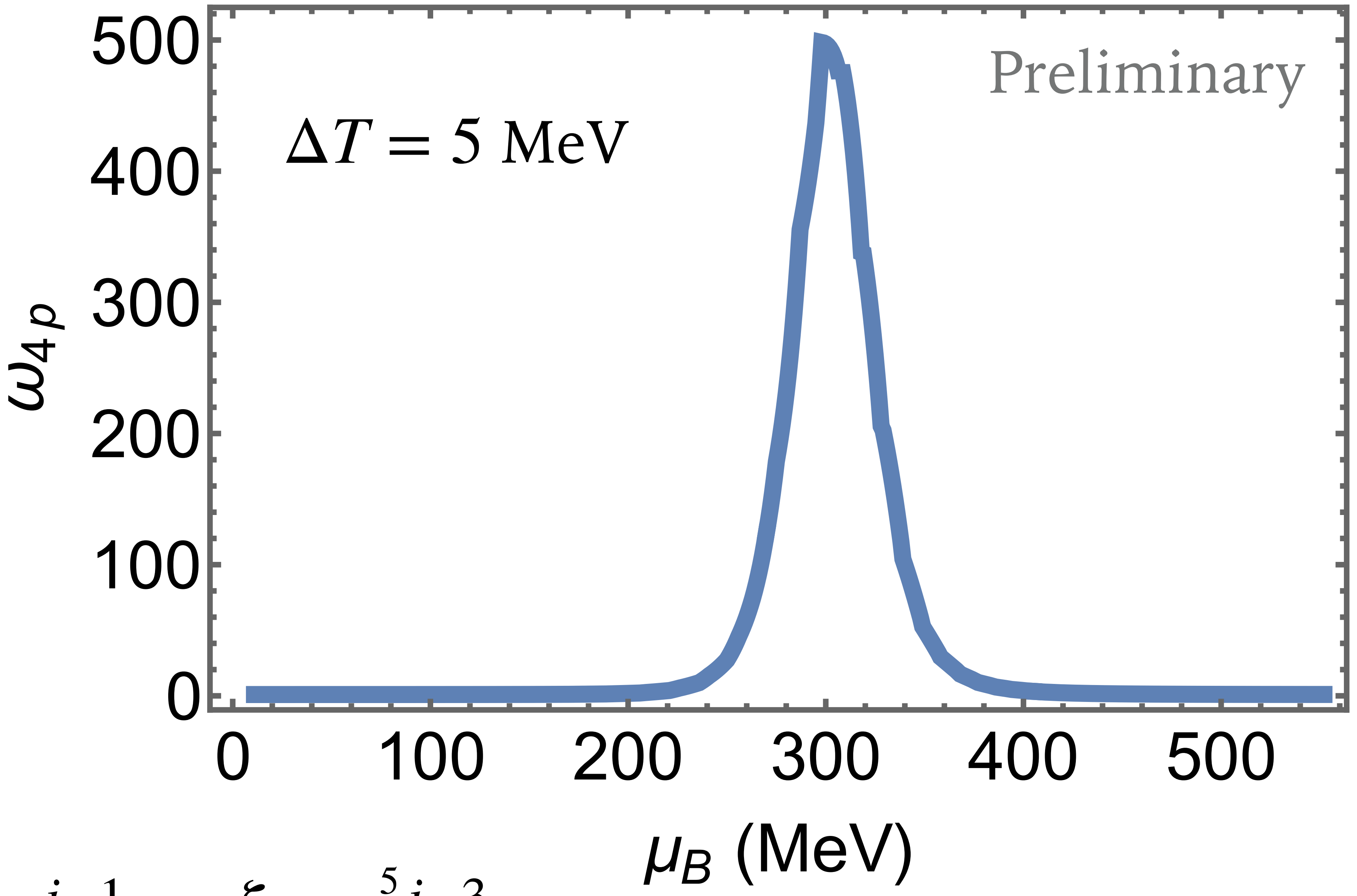


$$\omega_{ip}^{\text{prefactor}} = \frac{\tilde{\lambda}_i'(i-1)!\,\xi_{\max}^{\frac{5}{2}i-3}}{T^{i/2}\,n_p}\left(\int_k d_p\, g_p \frac{v_k^2}{\gamma_k}\right)^i \left(\frac{n_0}{n_p}\right)^{i-1}$$

Equilibrium Normalized Cumulants with Realistic EOS

- Normalized non-Gaussian proton cumulants in equilibrium at freeze-out, $\Delta T = 5$ MeV below critical point
  - 2010 equilibrium estimates are rather robust: using BEST EOS plus new calculation of universal behavior of $\xi, \lambda_3, \lambda_4$ yields predictions of comparable magnitude





$$\omega_{ip} = 1 + \omega_{ip}^{\text{prefactor}} \left(\frac{n_p}{n_0}\right)^{i-1} \left(\frac{\xi}{\xi_{\max}}\right)^{\frac{5}{2}i-3}$$

Conclusions

- Improvements of equilibrium results on fluctuations made possible by groundwork laid with BEST EOS
  - sensitivity to ΔT, g_p remains
- The updated equilibrium calculations reported here use the BEST EOS rather than the arbitrary ansätze used in 2010; will form the basis of better out-of-equilibrium estimates
- Next talk: what factors of suppression to expect in going from equilibrium to out-of-equilibrium
- Still to come: implement these new updates into Hydro+/freeze-out

Back-up

Further Lambda Plots