Evolving Non-Gaussian Fluctuations in Dynamical Relativistic Fluid

Xin An

with G. Başar, M. Stephanov and H.-U. Yee, to appear

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Motivation

- We are searching for signs of critical point from fluid.

Static fluid & static fluctuations
Stephanov, 2011
Mroczek, Acuna, Noronha-Hostler, Parotto, Ratti & Stephanov, 2020
see also talk by Karthein (Tue)

Static or uniformly varying fluid & dynamic fluctuations
Berdnikov & Rajagopal 1999
Mukherjee, Venugopalan & Yin, 2015
Nahrgang, Bluhm, Schafer & Bass, 2019
XA, Basar, Stephanov & Yee, 2020
see also talk by Pradeep (Tue), Sogabe (Wed)
Motivation

• The fluid is dynamic (neither static nor uniformly varying). Though it thermalizes rapidly, the fluctuations may not.

Dynamic fluid & dynamic fluctuations

Gaussian:
Stephanov & Yin, 2017
Akamatsu, Mazeliauskas & Teaney, 2017
XA, Basar, Stephanov & Yee, 2019
Rajagopal, Ridgway, Weller & Yin, 2019
Du, Heinz, Rajagopal & Yin, 2020
Pradeep, Rajagopal, Stephanov & Yin, 2022
...

Non-Gaussian:
XA, Basar, Stephanov & Yee, 2022

• We need to establish a covariant framework for dynamic fluid with non-Gaussian fluctuations, which may become slow near a critical point.
Stochastic equations

• Relativistically covariant Langevin equation for a set of variables $\ddot{\psi}_i$:

$$\ddot{u} \cdot \partial \ddot{\psi}_i = L_i[\ddot{\psi}] + \xi_i$$

where

$$\langle \xi_{i_1}(x_1)\xi_{i_2}(x_2) \rangle = 2Q_{i_1i_2} \delta^4(x_1 - x_2). \quad \text{(FDT)}$$

• All variables are measured in local rest frame – but, the frame is fluctuating!
Deterministic equations

• We want an equation for connected correlation function

\[ G_c^c \equiv G_{i_1 \ldots i_n}^c \equiv \langle \phi_{i_1}(x_1) \ldots \phi_{i_n}(x_n) \rangle_c \], where \( \phi = \bar{\psi} - \psi \):

\[ u(x) \cdot \partial(x) G_n^c = \text{polynomials of } G_2^c, G_3^c, \ldots, G_\infty^c. \]

• The evolution of \( G_n^c \) takes into account the evolution of fluctuations at each point, and is considered as their midpoint \( x \) moves.

\[ x = \frac{x_1 + x_2 + \ldots + x_n}{n} \]
Small parameters

- Small parameters in hydrodynamics: Akamatsu et al., 2017; An et al., 2019

<table>
<thead>
<tr>
<th>parameter</th>
<th>expression</th>
<th>meaning</th>
<th>role</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta$</td>
<td>$\ell_{\text{mic}}/\ell$</td>
<td>Knudsen number</td>
<td>controls gradient expansion</td>
</tr>
<tr>
<td>$\varepsilon$</td>
<td>$(\ell_{\text{mic}}/\ell)^3$</td>
<td>inverse of uncorrelated DOF</td>
<td>controls loop expansion</td>
</tr>
</tbody>
</table>

- $\ell_{\text{mic}} \ll b < \ell \ll L \gg \Lambda > q \gg k$
Truncated equations

• Power counting:

\[ G_n^c \sim \varepsilon^{n-1}, \quad L_i \sim \delta + \delta^2, \quad Q_{i_1 i_2} \sim \varepsilon \delta^2. \]

• Truncated equations for \( G_n^c \):

\[ u \cdot \partial^{(x)} G_n^c = \text{polynomials of tree like } G_2^c, G_3^c, \ldots, G_n^c. \]
Truncated equations

- Equations for $\psi$ and $G_n^c$ when $n = 2, 3, 4$:

\[
\begin{align*}
(\bigcirc) & = D \\
(\bigstar) & = D \bigstar + \triangle \\
(\bigtriangleup) & = D \bigtriangleup + D \bigstar + D \bigcirc \\
(\blacklozenge) & = D \blacklozenge + D \bigtriangleup + D \bigstar + D \bigcirc \\
\end{align*}
\]

\[
G_{i_1i_2\ldots}^c \equiv \ \\
L_{i_1,j_1j_2\ldots} \equiv \ \\
Q_{i_1i_2,j_1j_2\ldots} \equiv \\
\]
Confluent formalism

- **Confluent formalism**: adjusting relativistic equations along flow to preserve properties such as equal-time constraint. \( XA \textit{et al.}, 2019, 2022 \)

I. Confluent \( n \)-pt correlator:

\[
\bar{G}_{i_1...i_n}(x_1, \ldots, x_n) \equiv \Lambda_{i_1}^{j_1}(x_1 - x) \cdots \Lambda_{i_n}^{j_n}(x_n - x) G_{j_1...j_n}(x_1, \ldots, x_n).
\]
Confluent formalism

- **Confluent formalism**: adjusting relativistic equations along flow to preserve properties such as equal-time constraint. \( \text{XA et al., 2019, 2022} \)

II. Confluent derivative for \( \vec{G}_n \):

\[
\Delta x \cdot \vec{\nabla} \vec{G}_{i_1 \ldots i_n} \equiv \Lambda(\Delta x)^{j_1}_{i_1} \ldots \vec{G}_{j_1 \ldots j_n}(x + \Delta x + \Lambda(\Delta x)^{-1} y_1(x), \ldots) - \vec{G}_{i_1 \ldots i_n}(x + y_1(x), \ldots),
\]

where \( x^\mu_i - x^\mu \equiv y_i^\mu(x) = e^\mu_a(x) y^a_i \).
Confluent formalism

- **Confluent formalism**: adjusting relativistic equations along flow to preserve properties such as equal-time constraint. *XA et al., 2019, 2022*

### III. Confluent $n$-pt Wigner function:

\[
W_n(q_1, \ldots, q_n) = \int \left[ \prod_{i=1}^{n} d^3 y_i^a e^{-i q^a y_i^a} \right] \delta^{(3)} \left( \frac{1}{n} \sum_{i=1}^{n} y_i^a \right) \tilde{G}_n^c(y_1^a, \ldots, y_n^a).
\]

\[u(x) \cdot y_i = 0 \quad \& \quad y_1 + y_2 + \cdots + y_n = 0\]  
(a)  

\[u(x) \cdot q_i = 0 \quad \& \quad q_1 + q_2 + \cdots + q_n = 0\]  
(b)
First step towards fluctuating hydrodynamics

- What is the slowest variable near critical point?
  
  **Specific entropy (SE)** \( m \equiv s/n \): purely diffusive, slowest relaxation rate.
  
  \[
  \psi = m, \quad L_m = m_n \partial_\mu \lambda \Delta^{\mu\nu} \partial_\nu \alpha, \quad Q_{mm} = m_n \partial_\mu \lambda \Delta^{\mu\nu} \partial_\nu m_n,
  \]
  
  where \( m_n \equiv (\partial m/\partial n)_\epsilon \), \( \lambda \equiv \) conductivity, \( \alpha \equiv \) chemical potential.

- In critical regime, the most significant non-equilibrium effects are due to SE fluctuations.  
  
  \[ \xi_1^{3 \sim \omega} \xi_2^{2 \sim \omega} \xi_3^{1 \sim \omega} \]

\[ T \]
\[ \mu \]

**XA et al., 2019**
Equation for Gaussian SE correlators

- Evolution equation for 2-pt SE correlator:

\[
(\bullet \bullet) = \mathcal{D}\bullet + \Delta
\]

\[
\mathcal{L}[W_{mm}(q_1, q_2)] = (\partial \cdot u)W_{mm}(q_1, q_2) + 2[L_{m,m}(q_1, -q_1)W_{mm}(q_1, q_2) + Q_{mm}(q_1, q_2)]_{\text{perm}},
\]

where

\[
\mathcal{L}[W] \equiv \left( u \cdot \bar{\nabla} + f_i \cdot \frac{\partial}{\partial q_i} \right) W \quad \text{(Liouville operator)}
\]

and

\[
L_{m,m}(q_1, q_2) = \gamma_{mm} q_1 \cdot q_2, \quad Q_{mm}(q_1, q_2) = -m^2 \lambda q_1 \cdot q_2.
\]

- The way the dynamic background affects the Gaussian fluctuation dynamics is expected (i.e., agrees with Hydro+ in dynamic fluid): Stephanov & Yin, 2017
  - Background evolution competes with fluctuation relaxation. Akamatsu et al., 2017
  - Hubble force describes red shift of fluctuation wavelength. XA et al., 2019
Equation for non-Gaussian SE correlators

- Evolution equation for 3-pt SE correlator:

\[
\begin{align*}
\left( \begin{array}{c}
\text{•} \\
\text{•} \\
\text{•}
\end{array} \right) \cdot &= \quad \begin{array}{c}
\text{•} \\
\text{•} \\
\text{•}
\end{array} + \quad \begin{array}{c}
\text{•} \\
\text{•} \\
\text{•}
\end{array} + \quad \begin{array}{c}
\text{•} \\
\text{•} \\
\text{•}
\end{array} + \quad \begin{array}{c}
\text{•} \\
\text{•} \\
\text{•}
\end{array}
\end{align*}
\]

\[
\mathcal{L}[W_{mmm}(q_1, q_2, q_3)] = 2(\partial \cdot u)W_{mmm}(q_1, q_2, q_3)
+ 3[L_{m,m}(q_1, -q_1)W_{mmm}(q_1, q_2, q_3) + L_{m,p}(q_1, -q_1)W_{ppm}(q_1, q_2, q_3)]
+ L_{m,mm}(q_1, q_2, q_3)W_{mm}(-q_2, q_2)W_{mm}(-q_3, q_3)
+ 2Q_{mm,m}(q_1, q_2, q_3)W_{mm}(-q_3, q_3)\]_{perm},
\]

where

\[
\begin{align*}
L_{m,p}(q_1, q_2) &= \gamma_{mp}q_1 \cdot q_2, \\
L_{m,mm}(q_1, q_2, q_3) &= -(\gamma_{mm})_m q_1^2 + 2(\log m_n)_m \gamma_{mm} q_2 \cdot q_3, \\
Q_{mm,m}(q_1, q_2, q_3) &= -(m_n^2 \lambda)_m q_1 \cdot q_2 + (\log m_n)_m m_n^2 \lambda (q_1 + q_2)^2.
\end{align*}
\]

- We find that for non-Gaussian SE fluctuations: \cite{XA et al., 2022}
  
  - \(n\)-pt functions expand with rate \((n - 1)\partial \cdot u\).
  - \(\log m_n\) terms result from \(m\) not being a conserved quantity like \(n\).
Equation for non-Gaussian SE correlators

- Slow variable $m$ provides instantaneously frozen background for fast variable $p$; fast variable $p$ affects the evolution of slow variable $m$ like noise. \textit{XA et al., 2022}

\[
\begin{align*}
\left( \begin{array}{c}
\delta m \\
\delta p
\end{array} \right)^* &= \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array} \\
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array} \\
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array} \\
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array} \\
\end{array}
\end{align*}
\]

\[
W_{mmp} = \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array}
\]

\[
W_{pp}^{eq} = \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array} \quad W_{mm} = \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array} \quad S_{mmp} = \begin{array}{c}
\begin{array}{c}
\text{D} \\
\text{D}
\end{array}
\end{array}
\]

In Hydro+ regime, $\delta p$ correlators oscillate and relax parametrically fast towards their \textit{partially} equilibrated state, which is determined by the slowly varying $\delta m$. 

\begin{tikzpicture}
\end{tikzpicture}
Equilibrium solution

- The equilibrium solutions nontrivially match thermodynamics.

\[
\begin{align*}
W_{mm}^{\text{eq}} &= \quad \quad \quad \quad \quad \quad \quad \quad \\
W_{mp}^{\text{eq}} &= \quad \quad \quad \quad \quad \quad \quad \\
W_{pp}^{\text{eq}} &= \quad \quad \quad \quad \quad \quad \quad \\
W_{mmm}^{\text{eq}} &= \quad \quad \quad \quad \quad \quad \quad \\
W_{mmp}^{\text{eq}} &= \\
W_{mmmm}^{\text{eq}} &= \quad + \quad \quad + \quad \quad \\
W_{mmmp}^{\text{eq}} &= \quad + \quad \quad + \quad \quad 
\end{align*}
\]
Recap and outlook

Recap

• A relativistically covariant framework for non-Gaussian fluctuation dynamics is established, with:
  • dynamic fluid background;
  • confluent formalism.
• Hydro+ is now for non-Gaussian fluctuations in arbitrary flow:
  • fluctuations of pressure, albeit fast, play an important role;
  • additional terms due to $m$ being a nonlinear function of $n$.

Outlook

• Numerical implementation.
• Adding more fluctuation DOFs (extension to Hydro++).
• Freeze-out procedure.
• First-order phase transitions.
• …

Thank You!
Backup
Equation for SE correlators

\[
\begin{align*}
(\cdots)^* &= \quad + \\
(\cdots)^* &= \quad + \\
(\cdots)^* &= \quad + \\
&\quad + \\
&\quad + \\
&\quad + \\
\end{align*}
\]

\[\begin{array}{c}
\text{peq} = \\
\text{peq} = \end{array}\]
Critical behavior

- In critical regime \((T^{-1} \ll \xi \ll q^{-1})\): Hohenberg et al., 1977

\[
\lambda_{\substack{m \ldots m \ k}} \sim \xi^{\frac{1}{2}}(k+2), \quad \alpha_{\substack{m \ldots m \ k}} \sim \xi^{\frac{1}{2}}(k-5),
\]

\[
(\gamma_{\substack{m \cdot m \ k}})_{\substack{m \ldots m \ k}} \sim \xi^{\frac{1}{2}}(k-2), \quad (\log m_n)_{\substack{m \ldots m \ k \leq 5}} \sim 1,
\]

- Leading singular behavior:

\[
\mathcal{L}[W_k] \sim q^2 \xi^{\frac{1}{2}}(5k-8).
\]

\(\log m_n\) terms are subleading near critical point, but are important for comparison to the non-critical part.