Evolving Non-Gaussian Fluctuations in Dynamical Relativistic Fluid

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with G. Başar, M. Stephanov and H.-U. Yee, to appear

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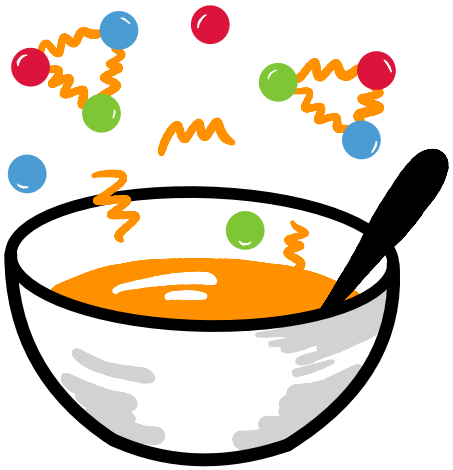
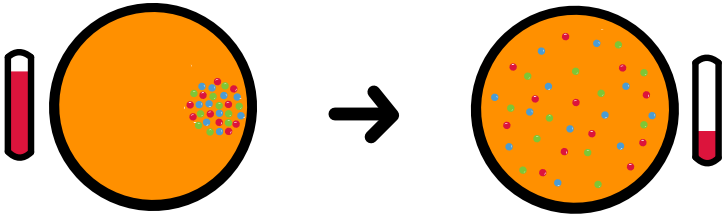
NCBJ UNC

Motivation

- We are searching for signs of critical point from fluid .

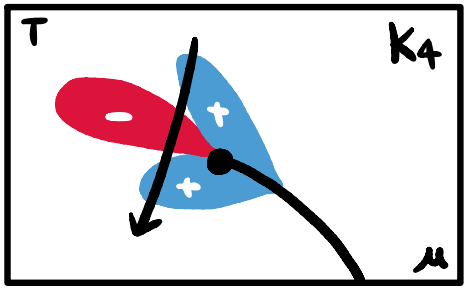


static fluid



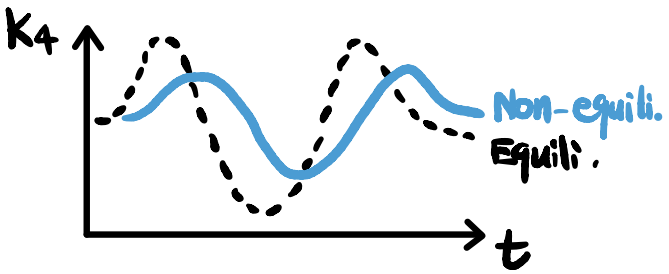
Static fluid & static fluctuations

Stephanov, 2011
Mroczek, Acuna, Noronha-Hostler, Parotto, Ratti & Stephanov, 2020
see also talk by Karthein (Tue)
...



Static or uniformly varying fluid & dynamic fluctuations

Berdnikov & Rajagopal 1999
Mukherjee, Venugopalan & Yin, 2015
Nahrgang, Bluhm, Schafer & Bass, 2019
XA, Basar, Stephanov & Yee, 2020
see also talk by Pradeep (Tue), Sogabe (Wed)
...



Motivation

- The fluid is dynamic (neither static nor uniformly varying). Though it thermalizes rapidly, the fluctuations may *not*.

dynamic fluid

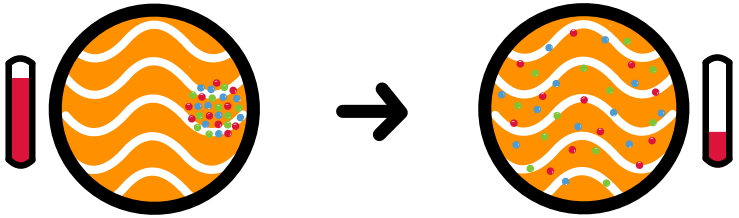
Dynamic fluid & dynamic fluctuations

Gaussian:

Stephanov & Yin, 2017
Akamatsu, Mazeliauskas & Teaney, 2017
XA, Basar, Stephanov & Yee, 2019
Rajagopal, Ridgway, Weller & Yin, 2019
Du, Heinz, Rajagopal & Yin, 2020
Pradeep, Rajagopal, Stephanov & Yin, 2022
...

Non-Gaussian:

XA, Basar, Stephanov & Yee, 2022



This work

- We need to establish a *covariant* framework for *dynamic* fluid with *non-Gaussian* fluctuations, which may become *slow* near a critical point.

Stochastic equations

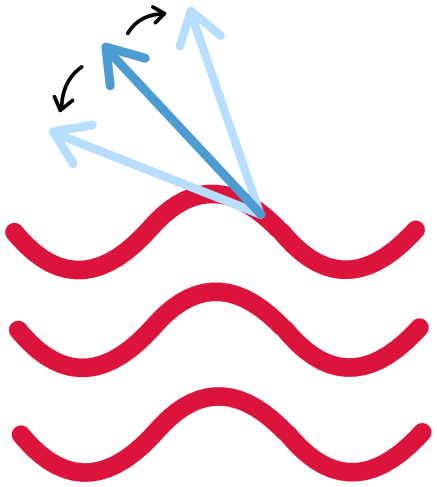
- Relativistically *covariant* Langevin equation for a set of variables $\breve{\psi}_i$:

$$\breve{u} \cdot \partial \breve{\psi}_i = L_i[\breve{\psi}] + \xi_i$$

where

$$\langle \xi_{i_1}(x_1) \xi_{i_2}(x_2) \rangle = 2 Q_{i_1 i_2} \delta^4(x_1 - x_2) \,. \qquad \text{(FDT)}$$

- All variables are measured in local rest frame – but , the frame is fluctuating !



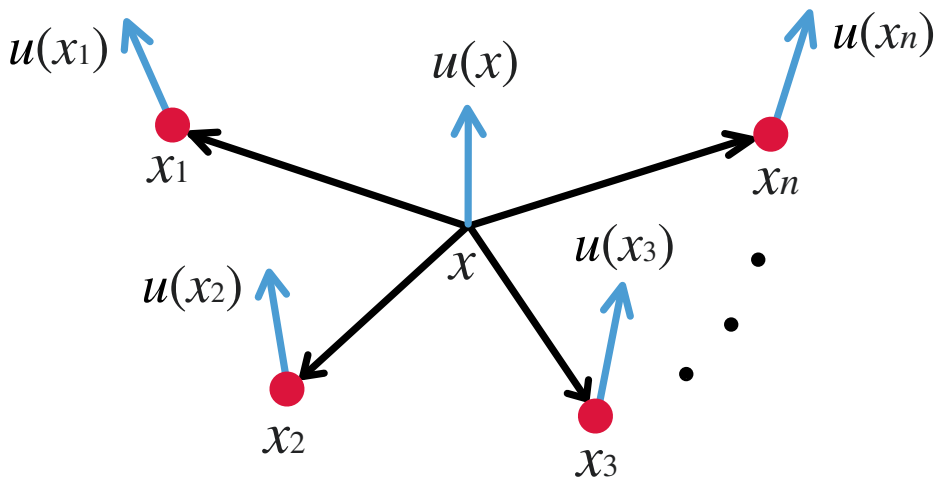
Deterministic equations

- We want an equation for connected correlation function

  $G^{\rm c}_n \equiv G^{\rm c}_{i_1\ldots i_n} \equiv \langle \phi_{i_1}(x_1)\ldots\phi_{i_n}(x_n)\rangle_{\rm c}$, where $\phi = \breve{\psi} - \psi$:

$$u(x)\cdot\partial^{(x)}G^{\rm c}_n = \text{polynomials of } G^{\rm c}_2,\ G^{\rm c}_3,\ \ldots,\ G^{\rm c}_\infty\,.$$

- The evolution of $G^{\rm c}_n$ takes into account the evolution of fluctuations at each point, and is considered as their midpoint x moves .

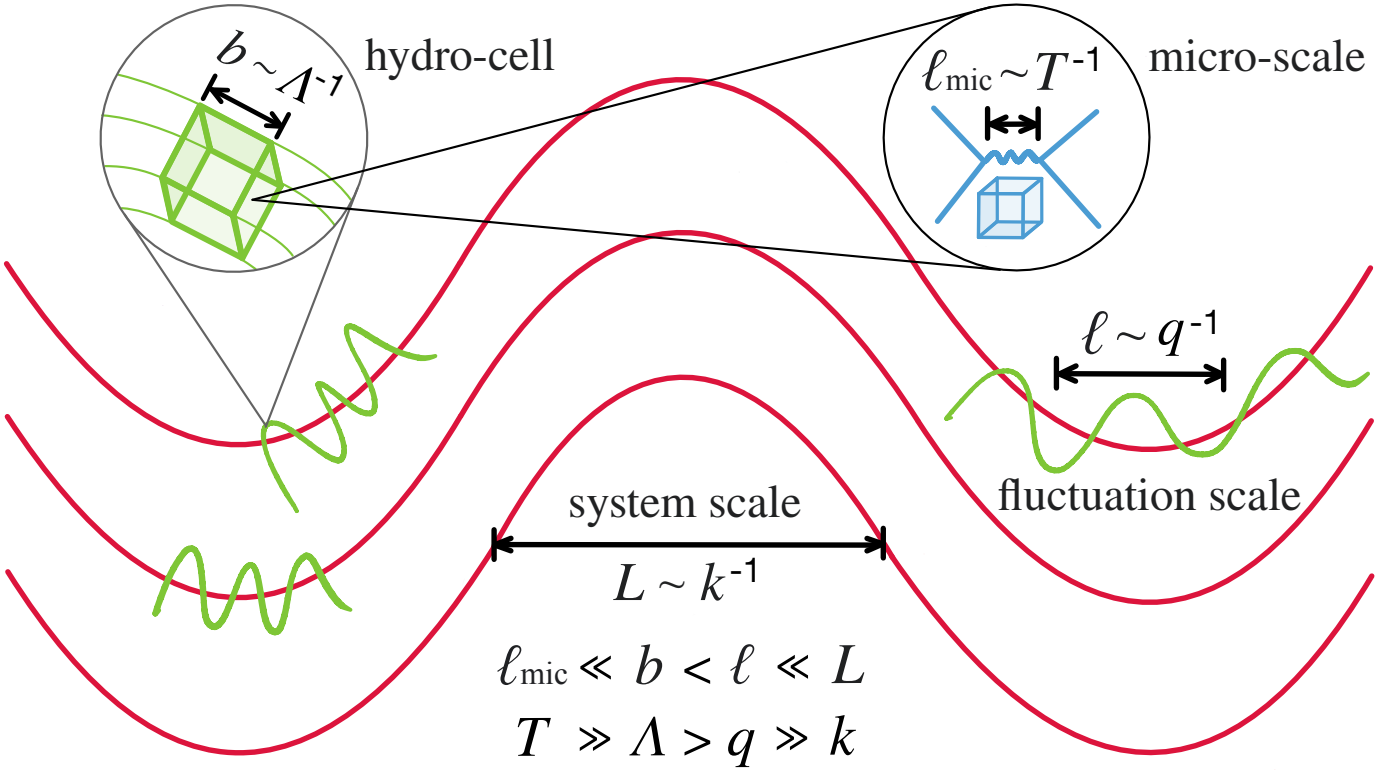


$$x = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

Small parameters

- Small parameters in hydrodynamics : Akamatsu *et al.*, 2017; An *et al.*, 2019

parameter	expression	meaning	role
δ	$\ell_{\rm mic}/\ell$	Knudsen number	controls gradient expansion
ε	$(\ell_{\rm mic}/\ell)^3$	inverse of uncorrelated DOF	controls loop expansion



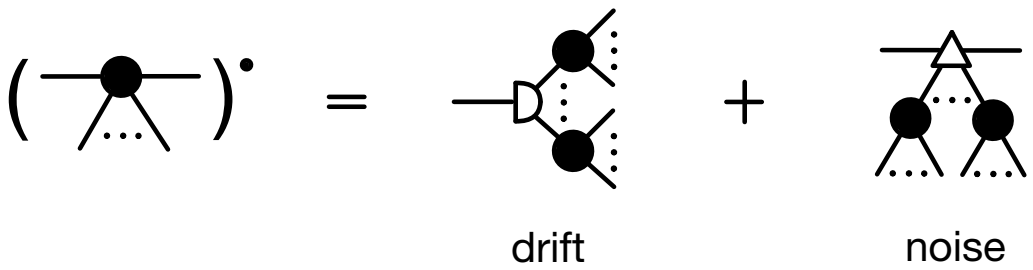
Truncated equations

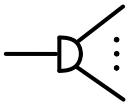
- Power counting :

$$G_n^{\mathrm{c}} \sim \varepsilon^{n-1}\,, \qquad L_i \sim \delta + \delta^2\,, \qquad Q_{i_1 i_2} \sim \varepsilon\delta^2\,.$$

- Truncated equations for G_n^{c} :

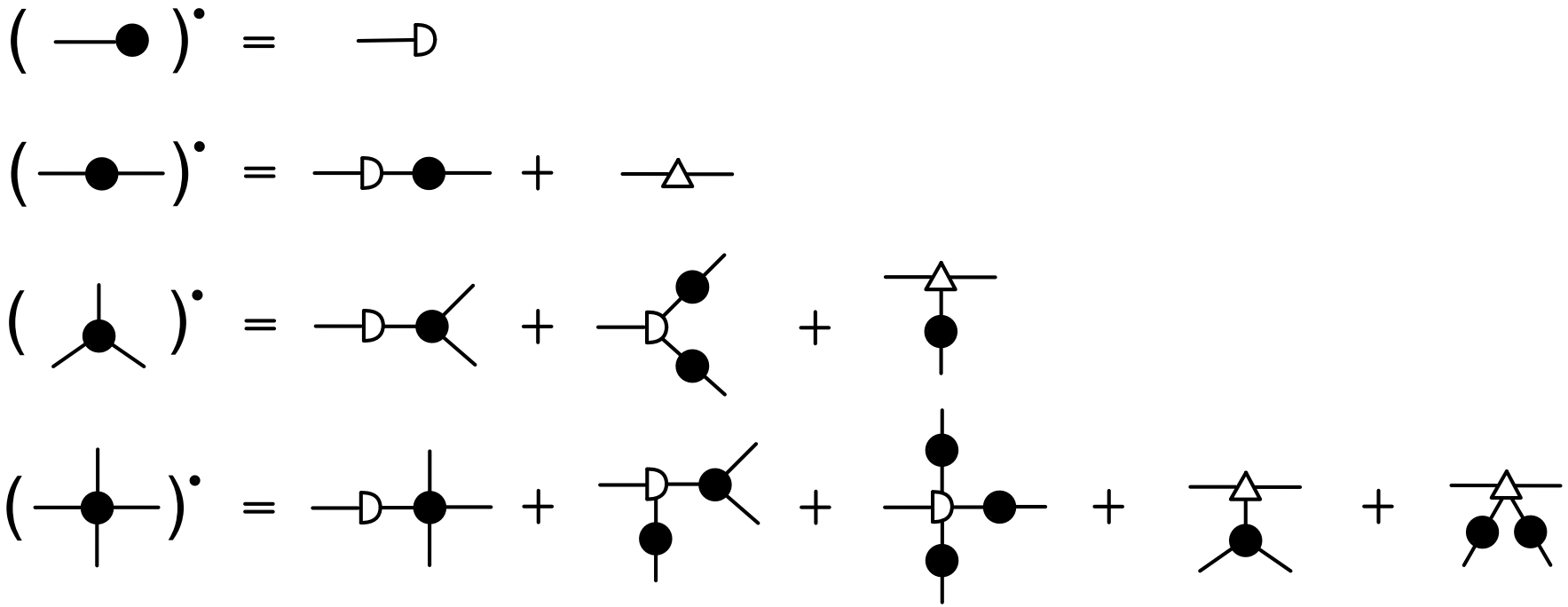
$$u \cdot \partial^{(x)} G_n^{\mathrm{c}} = \text{polynomials of tree like } G_2^{\mathrm{c}},\, G_3^{\mathrm{c}},\, \ldots,\, G_n^{\mathrm{c}}\,.$$

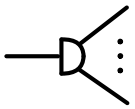


$G^{\mathrm{c}}_{i_1 i_2 \ldots} \equiv$  $L_{i_1, j_1 j_2 \ldots} \equiv$  $Q_{i_1 i_2, j_1 j_2 \ldots} \equiv$ 

Truncated equations

- Equations for ψ and G_n^{c} when $n = 2, 3, 4$:



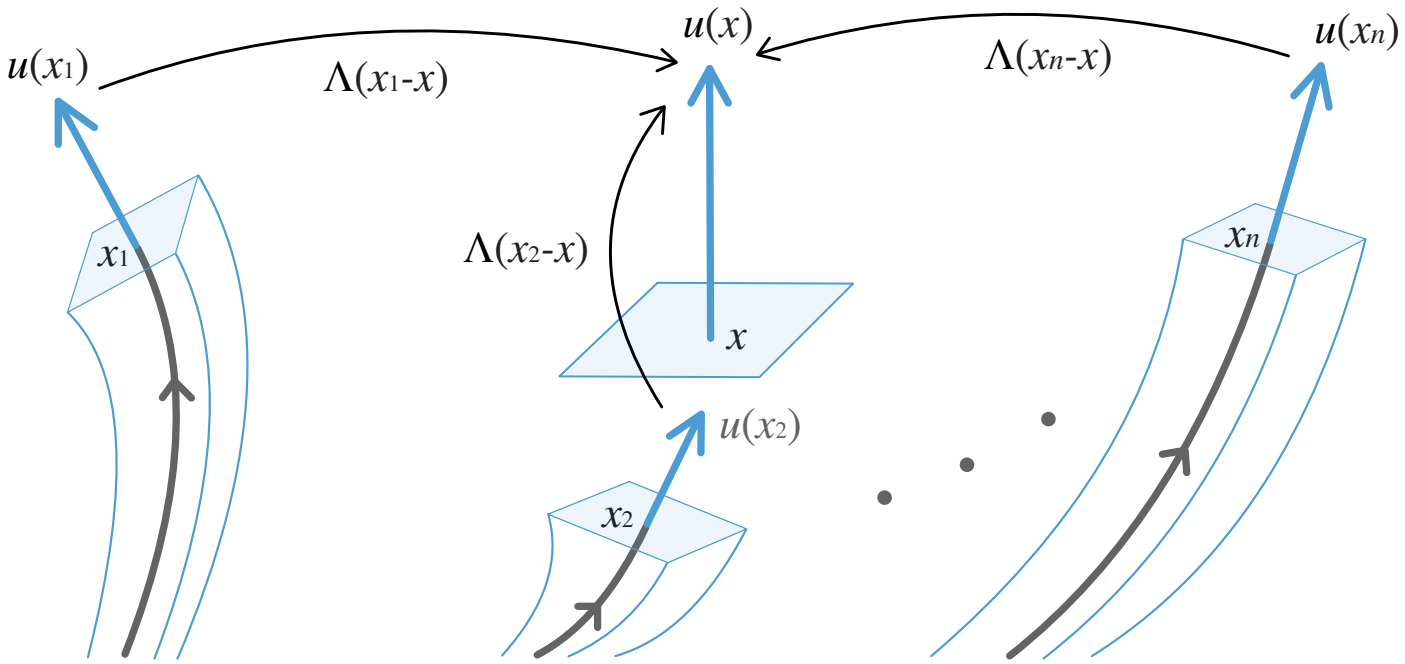
$G^{\mathrm{c}}_{i_1 i_2 \ldots} \equiv$  $L_{i_1, j_1 j_2 \ldots} \equiv$  $Q_{i_1 i_2, j_1 j_2 \ldots} \equiv$ 

Confluent formalism

- **Confluent formalism** : adjusting relativistic equations along flow to preserve properties such as equal-time constraint . XA *et al.*, 2019, 2022

I. Confluent n-pt correlator :

$$\bar{G}_{i_1\ldots i_n}(x_1,\ldots,x_n) \equiv \Lambda_{i_1}^{\ j_1}(x_1-x)\ldots\Lambda_{i_n}^{\ j_n}(x_n-x)G_{j_1\ldots j_n}(x_1,\ldots,x_n)\,.$$



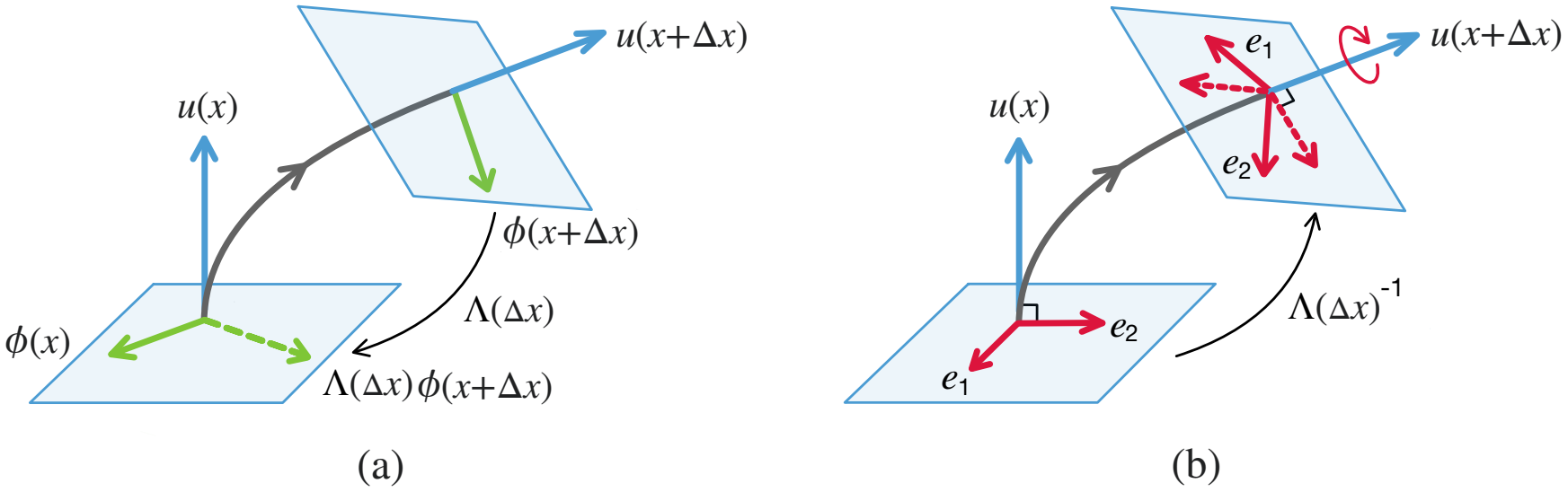
Confluent formalism

- **Confluent formalism** : adjusting relativistic equations along flow to preserve properties such as equal-time constraint . XA *et al.*, 2019, 2022

II. Confluent derivative for $\bar{G}_n$:

$$\Delta x \cdot \bar{\nabla} \bar{G}_{i_1 \dots i_n} \equiv \Lambda(\Delta x)_{i_1}^{\ j_1} \dots \bar{G}_{j_1 \dots j_n}(x + \Delta x + \Lambda(\Delta x)^{-1} y_1(x), \dots) - \bar{G}_{i_1 \dots i_n}(x + y_1(x), \dots) ,$$

where $x_i^\mu - x^\mu \equiv y_i^\mu(x) = e_a^\mu(x) y_i^a$.

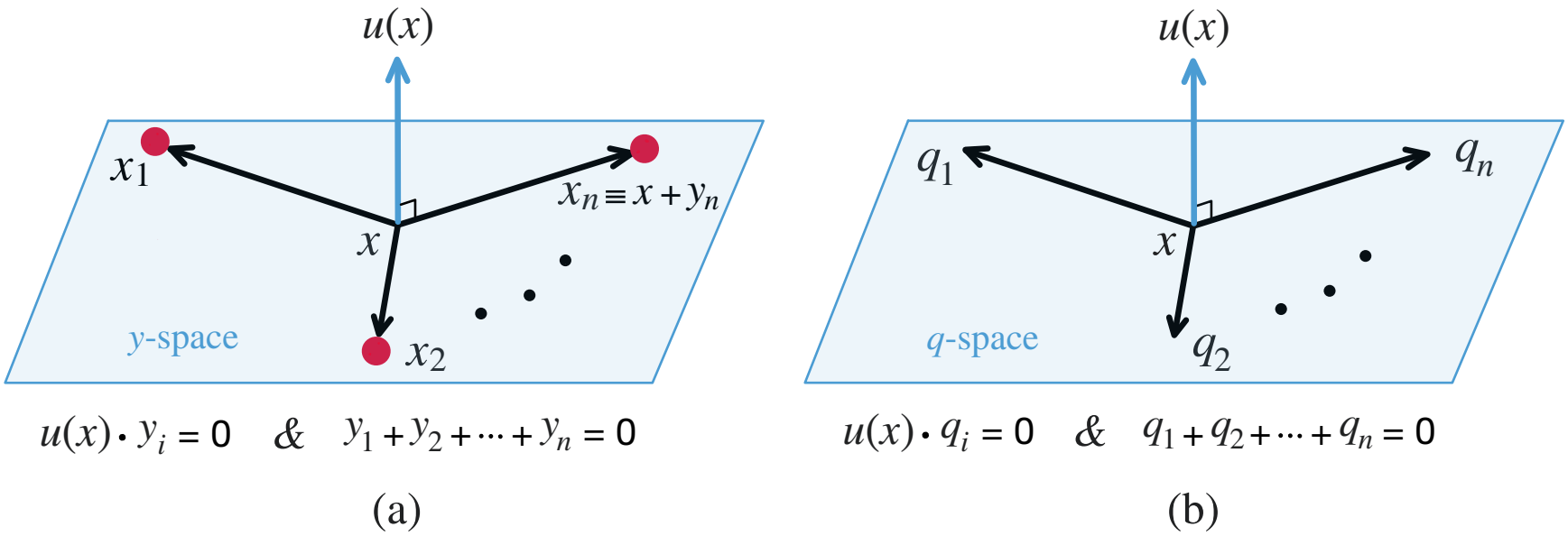


Confluent formalism

- **Confluent formalism** : adjusting relativistic equations along flow to preserve properties such as equal-time constraint . XA *et al.*, 2019, 2022

III. Confluent n-pt Wigner function :

$$W_n(\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n) = \int \left[\prod_{i=1}^{n} d^3 y_i^a \, e^{-i q_{ia} y_i^a} \right] \delta^{(3)} \left(\frac{1}{n} \sum_{i=1}^{n} y_i^a \right) \bar{G}_n^{\mathsf{c}}(y_1^a, \ldots, y_n^a) \, .$$



First step towards fluctuating hydrodynamics

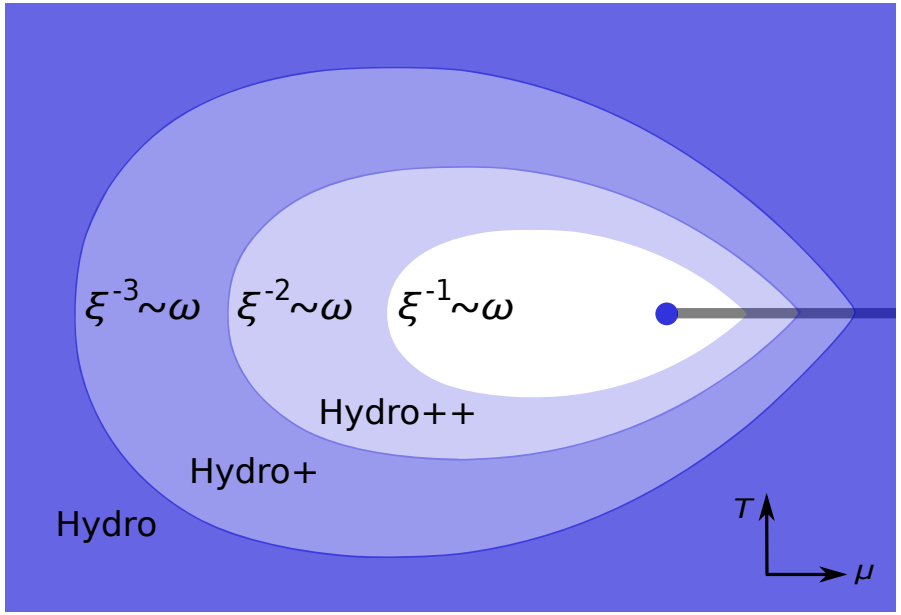
- What is the *slowest* variable near critical point ?

  Specific entropy (SE) $m \equiv s/n$: purely diffusive, slowest relaxation rate .

$$\psi = m\,, \qquad L_m = m_n \partial_\mu \lambda \Delta^{\mu\nu} \partial_\nu \alpha\,, \qquad Q_{mm} = m_n \partial_\mu \lambda \Delta^{\mu\nu} \partial_\nu m_n\,,$$

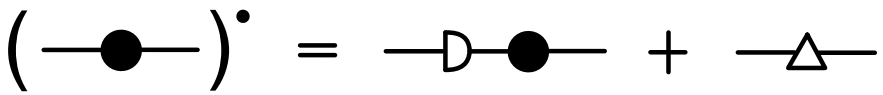
  where $m_n \equiv \left(\frac{\partial m}{\partial n}\right)_\epsilon$, $\lambda \equiv$ conductivity , $\alpha \equiv$ chemical potential .

- In critical regime, the most significant non-equilibrium effects are due to SE fluctuations . XA *et al.*, 2019



Equation for Gaussian SE correlators

- Evolution equation for 2-pt SE correlator :



$$\mathcal{L}[W_{mm}(\boldsymbol{q}_1,\boldsymbol{q}_2)] = (\partial\cdot u)W_{mm}(\boldsymbol{q}_1,\boldsymbol{q}_2) + 2\big[L_{m,m}(\boldsymbol{q}_1,-\boldsymbol{q}_1)W_{mm}(\boldsymbol{q}_1,\boldsymbol{q}_2) + Q_{mm}(\boldsymbol{q}_1,\boldsymbol{q}_2)\big]_{\overline{\text{perm}}}\,,$$

where

$$\mathcal{L}[W] \equiv \left(u\cdot\bar{\nabla} + f_i\cdot\frac{\partial}{\partial q_i}\right)W \qquad \text{(Liouville operator)}$$

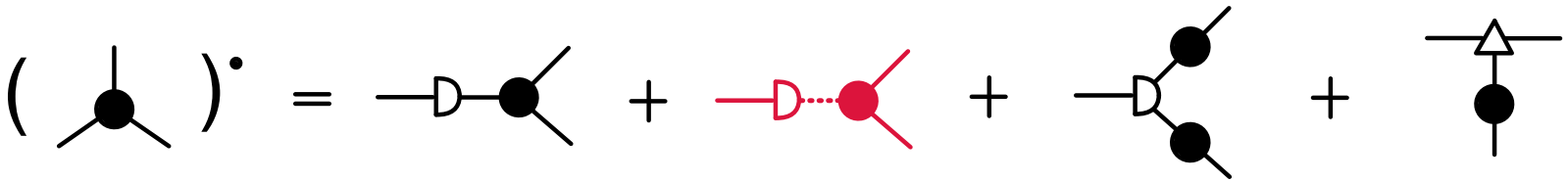
and

$$L_{m,m}(\boldsymbol{q}_1,\boldsymbol{q}_2) = \gamma_{mm}\boldsymbol{q}_1\cdot\boldsymbol{q}_2\,, \qquad Q_{mm}(\boldsymbol{q}_1,\boldsymbol{q}_2) = -m_n^2\lambda\boldsymbol{q}_1\cdot\boldsymbol{q}_2\,.$$

- The way the dynamic background affects the *Gaussian* fluctuation dynamics is expected (i.e., agrees with *Hydro+* in dynamic fluid) : Stephanov & Yin, 2017
  - Background evolution competes with fluctuation relaxation . Akamatsu *et al.*, 2017
  - Hubble force describes red shift of fluctuation wavelength . XA *et al.*, 2019

Equation for non-Gaussian SE correlators

- Evolution equation for 3-pt SE correlator :



$$\mathcal{L}[W_{mmm}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)] = 2(\partial\cdot u)W_{mmm}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)$$
$$+\,3\big[L_{m,m}(\boldsymbol{q}_1,-\boldsymbol{q}_1)W_{mmm}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3) + L_{m,p}(\boldsymbol{q}_1,-\boldsymbol{q}_1)W^{\rm peq}_{pmm}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)$$
$$+\,L_{m,mm}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_2,\boldsymbol{q}_2)W_{mm}(-\boldsymbol{q}_3,\boldsymbol{q}_3)$$
$$+\,2Q_{mm,m}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3)W_{mm}(-\boldsymbol{q}_3,\boldsymbol{q}_3)\big]_{\overline{\rm perm}}\,,$$

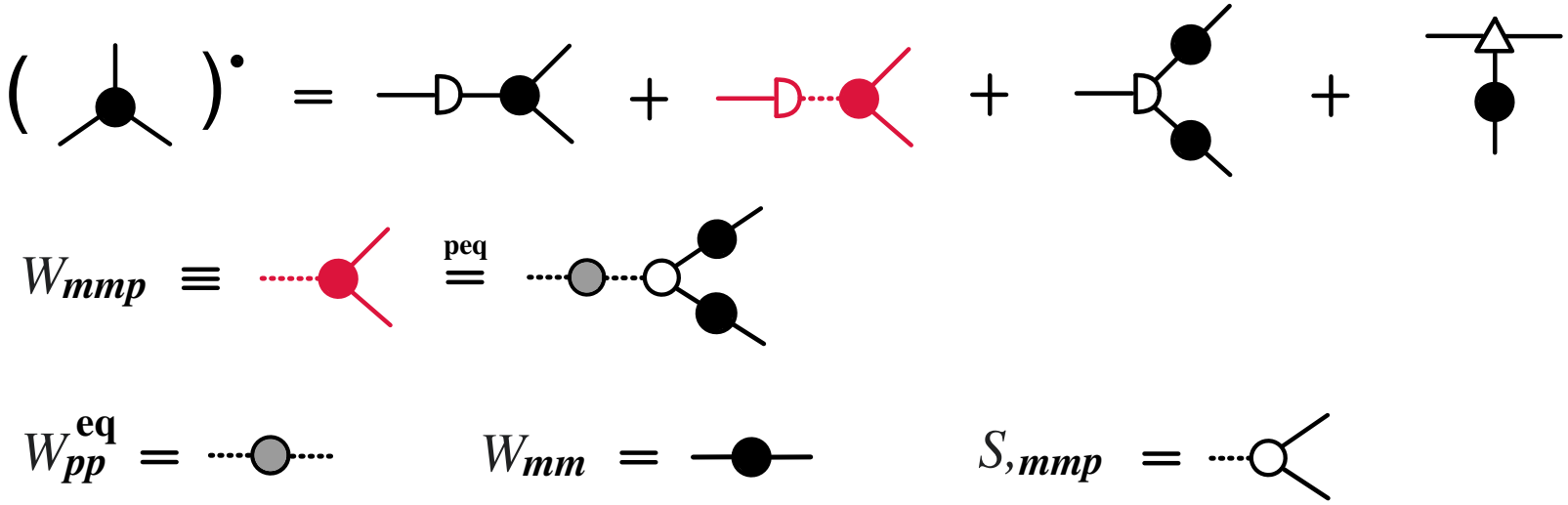
where

$$L_{m,p}(\boldsymbol{q}_1,\boldsymbol{q}_2) = \gamma_{mp}\boldsymbol{q}_1\cdot\boldsymbol{q}_2\,,$$
$$L_{m,mm}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3) = -(\gamma_{mm})_{,m}\boldsymbol{q}_1^2 + 2(\log m_n)_{,m}\gamma_{mm}\boldsymbol{q}_2\cdot\boldsymbol{q}_3\,,$$
$$Q_{mm,m}(\boldsymbol{q}_1,\boldsymbol{q}_2,\boldsymbol{q}_3) = -(m_n^2\lambda)_{,m}\boldsymbol{q}_1\cdot\boldsymbol{q}_2 + (\log m_n)_{,m}m_n^2\lambda(\boldsymbol{q}_1+\boldsymbol{q}_2)^2\,.$$

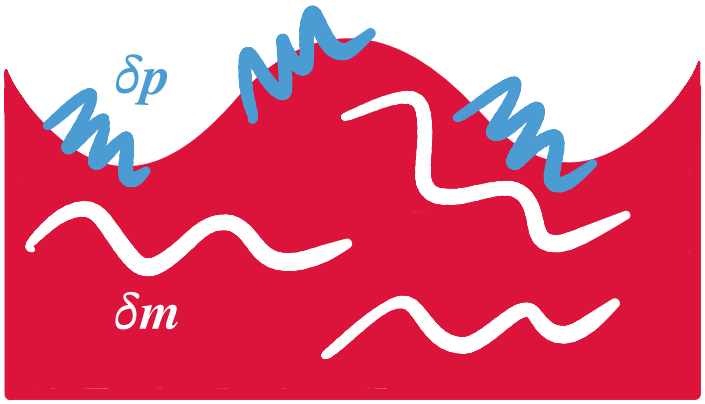
- We find that for *non-Gaussian* SE fluctuations : XA *et al.*, 2022
  - n-pt functions expand with rate $(n-1)\partial\cdot u$.
  - $\log m_n$ terms result from m *not* being a conserved quantity like n .

Equation for non-Gaussian SE correlators

- Slow variable m provides instantaneously frozen background for fast variable p; fast variable p affects the evolution of slow variable m like noise. XA *et al.*, 2022

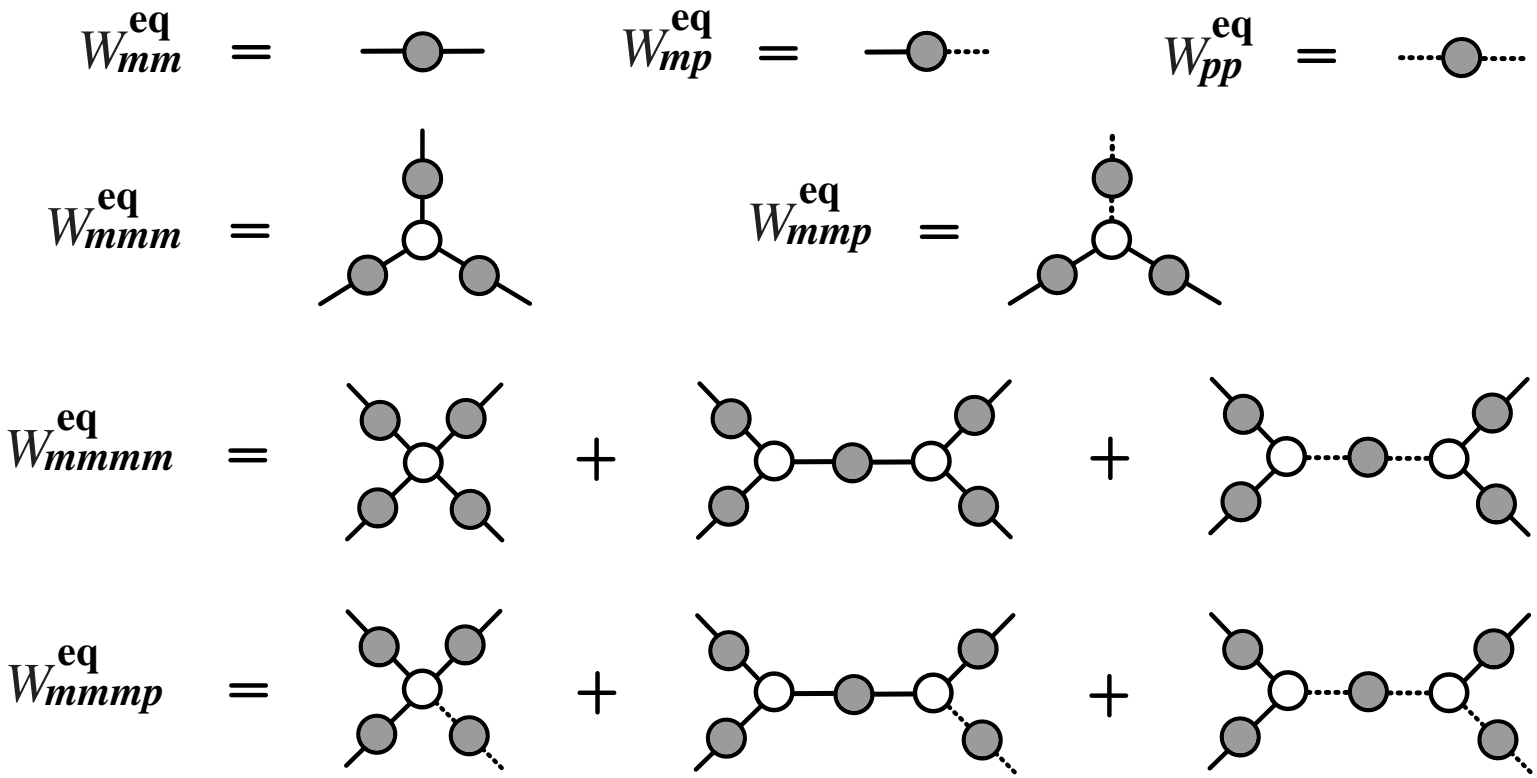


In Hydro+ regime, δp correlators oscillate and relax parametrically fast towards their *partially* equilibrated state, which is determined by the slowly varying δm.



Equilibrium solution

- The equilibrium solutions nontrivially match thermodynamics .



Recap and outlook

Recap

- A relativistically covariant framework for non-Gaussian fluctuation dynamics is established , with :
  - dynamic fluid background ;
  - confluent formalism .
- Hydro+ is now for non-Gaussian fluctuations in arbitrary flow :
  - fluctuations of pressure , albeit fast , play an important role ;
  - additional terms due to m being a nonlinear function of n .

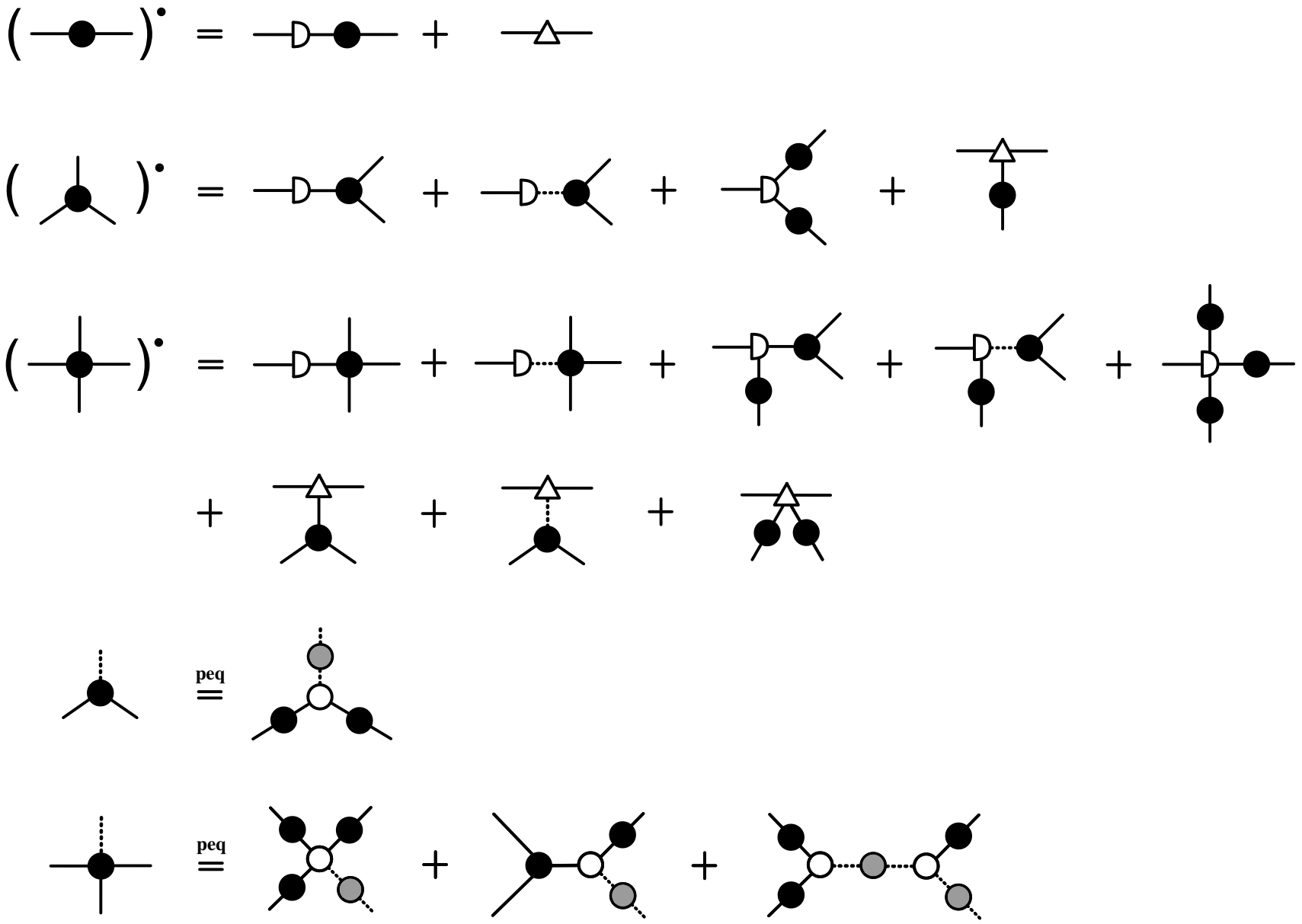
Outlook

- Numerical implementation .
- Adding more fluctuation DOFs (extension to Hydro++) .
- Freeze-out procedure .
- First-order phase transitions .
- . . .

Thank You !

Backup

Equation for SE correlators



Critical behavior

- In critical regime $(T^{-1} \ll \xi \ll q^{-1})$: Hohenberg *et al.*, 1977

$$\lambda_{,\underbrace{m\ldots m}_{k}} \sim \xi^{\frac{1}{2}(k+2)}\,, \quad \alpha_{\underbrace{m\ldots m}_{k}} \sim \xi^{\frac{1}{2}(k-5)}\,,$$

$$(\gamma_{mm})_{,\underbrace{m\ldots m}_{k}} \sim \xi^{\frac{1}{2}(k-2)}\,, \qquad (\log m_n)_{,\underbrace{m\ldots m}_{k\leqslant 5}} \sim 1\,,$$

- Leading singular behavior :

$$\mathcal{L}[W_k] \sim q^2 \xi^{\frac{1}{2}(5k-8)}\,.$$

$\log m_n$ terms are subleading near critical point , but are important for comparison to the non-critical part .