# Longer-term outlook on Lattice Flavour Physics

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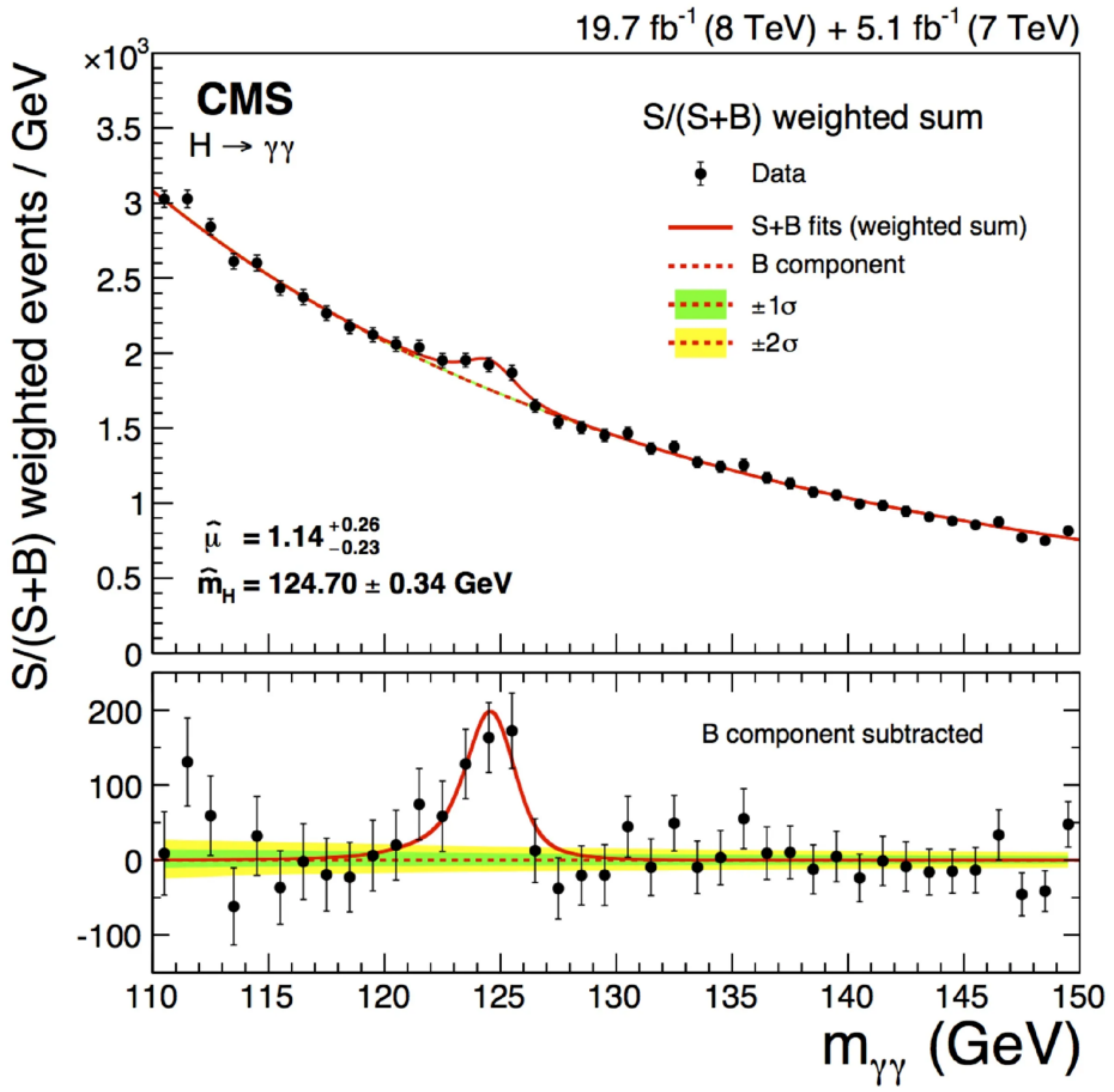
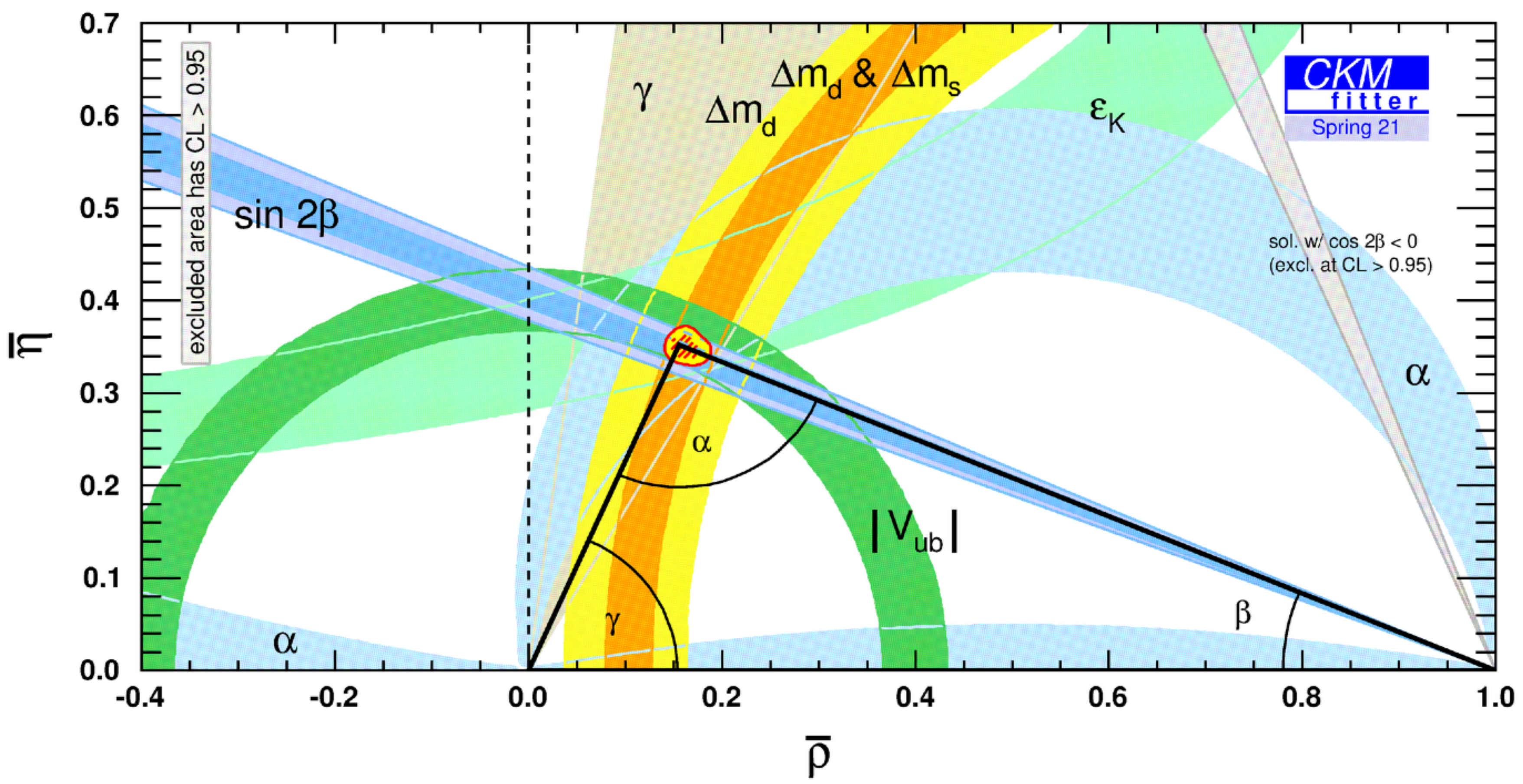
# Outline

- Lattice flavour physics and its relevance for testing the Standard Model
- State-of-the-art
- Outlook on a few selected topics that will allow us to go beyond the status quo

# Testing the Standard Model

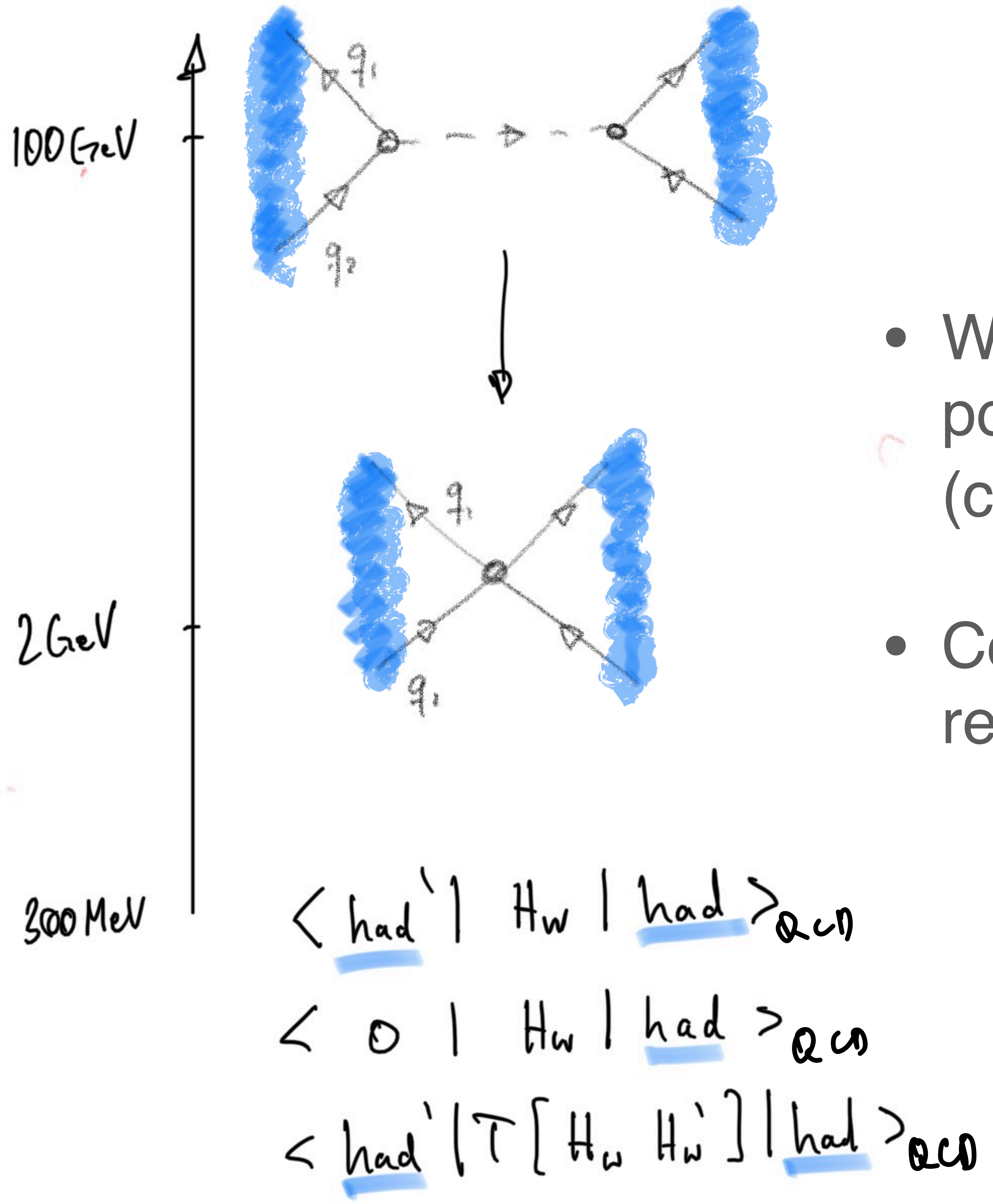
- searches for new physics
  - *direct searches* — 'bump in the spectrum'
  - *indirect searches* — SM provides relations between processes; we can therefore use experiment + theory to over-constrain SM



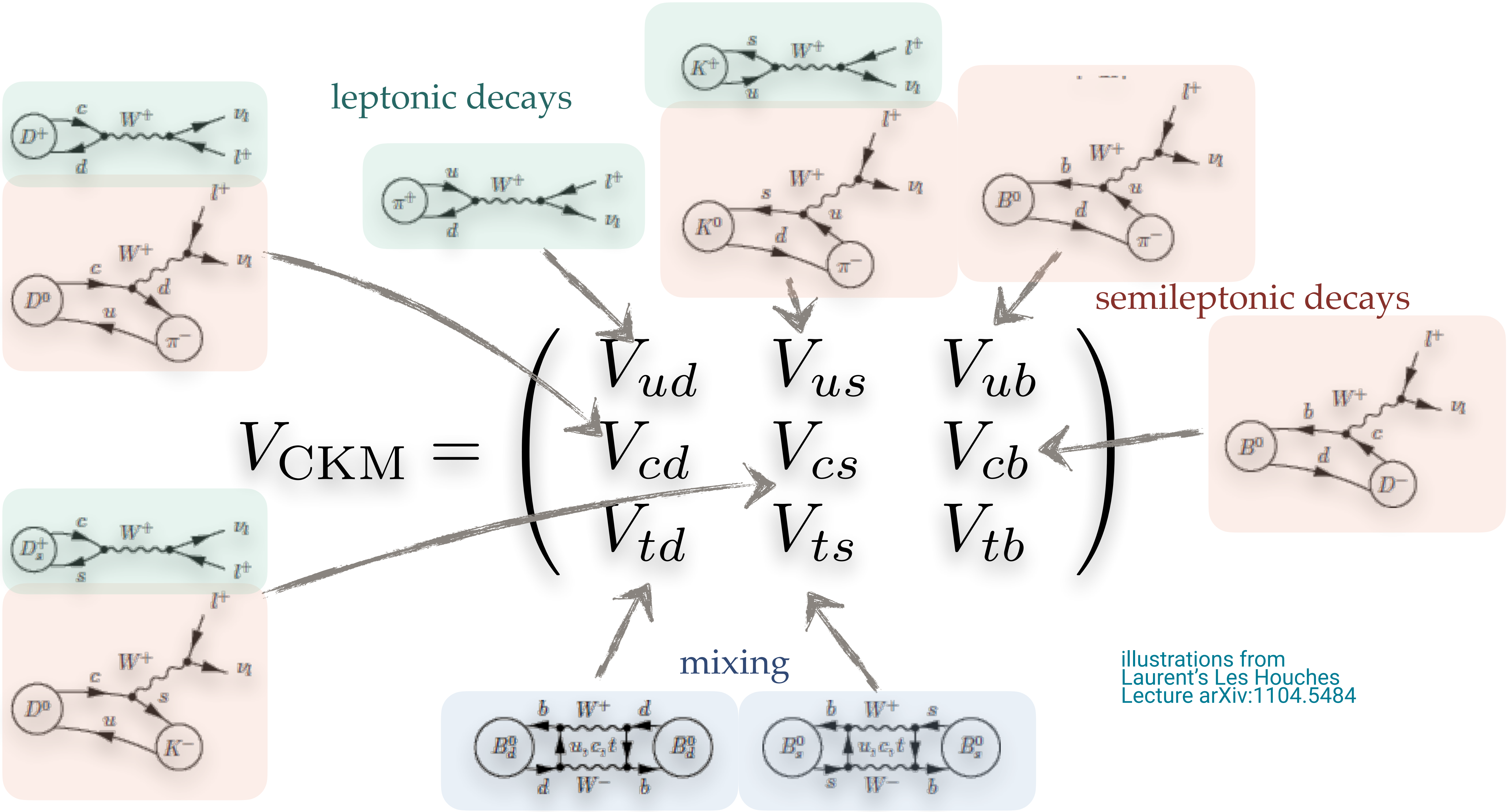

We will now look at Lattice QCD's role in *indirect searches*

# Indirect SM tests




- Weak gauge bosons so heavy that we can replace them by point-interaction described by an effective Hamiltonian $H_W$ (conveniently we thereby *get rid* of a very high energy scale)
- Computing weak processes between hadrons therefore requires computing matrix elements of $H_W$ between hadronic states

# Indirect SM tests




illustrations from
Laurent's Les Houches
Lecture arXiv:1104.5484

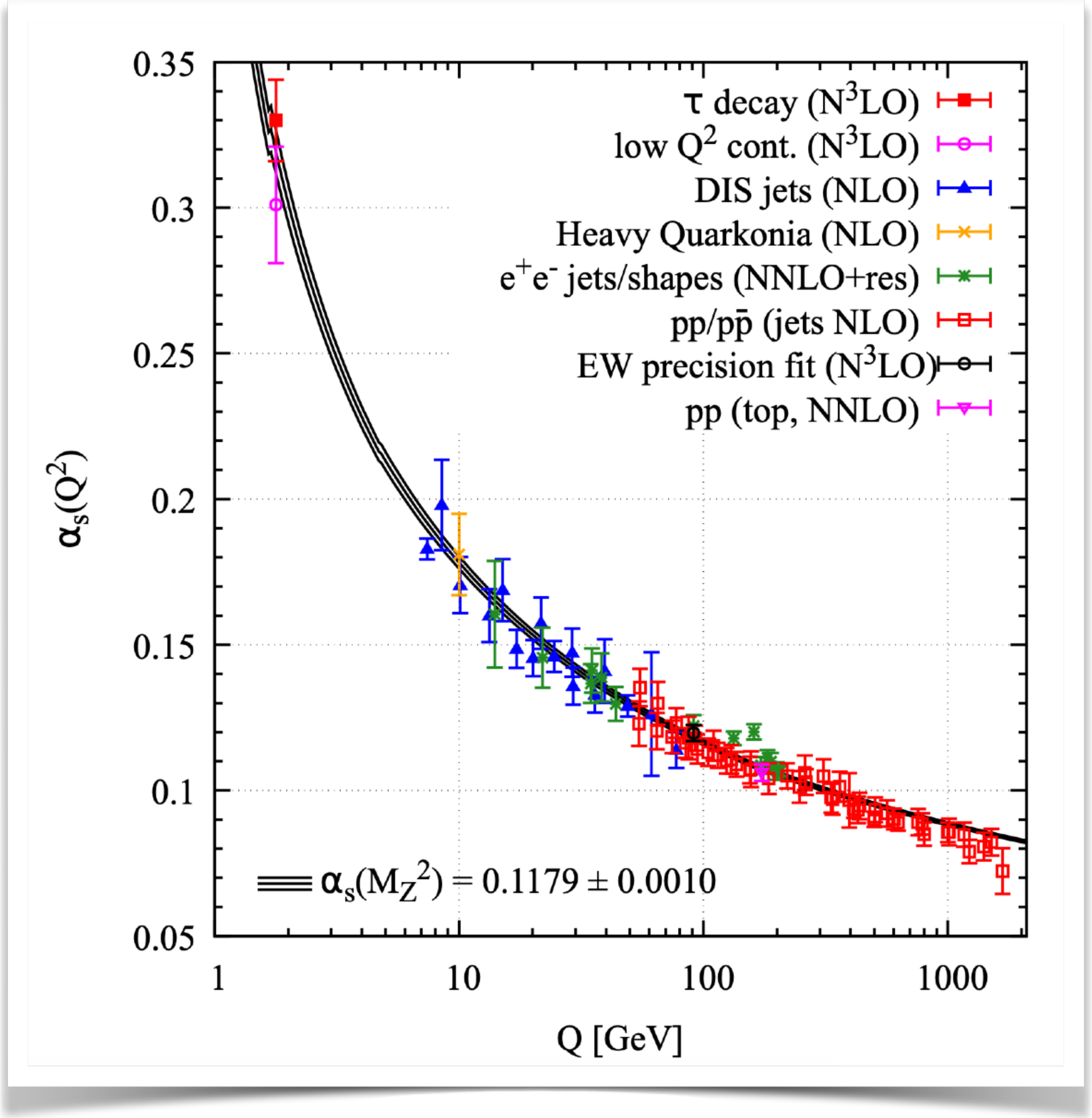
over-constraining the CKM-matrix elements is hoped to eventually reveal cracks in the SM, e.g.:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1\,?$$

It is a high-precision game…

# Indirect SM tests

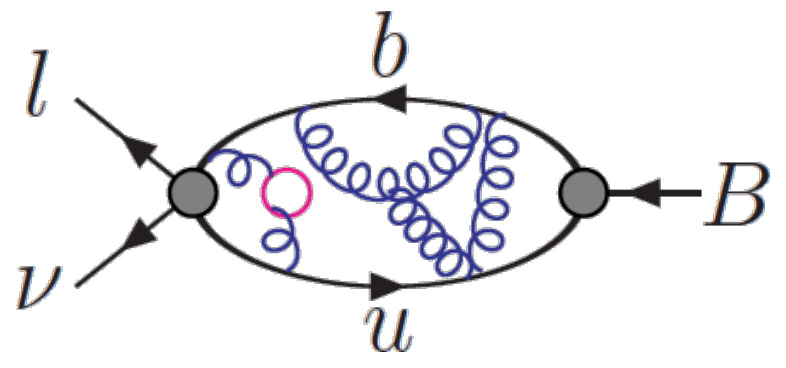
| SM-sector | typical coupling | mediator |
|---|---|---|
| WEAK | $10^{-5}\,\mathrm{GeV}^{-2}$ | $Z, W^{\pm}$ |
| EM | 1/137 | $\gamma$ |
| QCD | **0-*O(1)*** | gluons |




hadronic (QCD) uncertainties often still dominating error budget

# Indirect SM tests (1): EW tree level

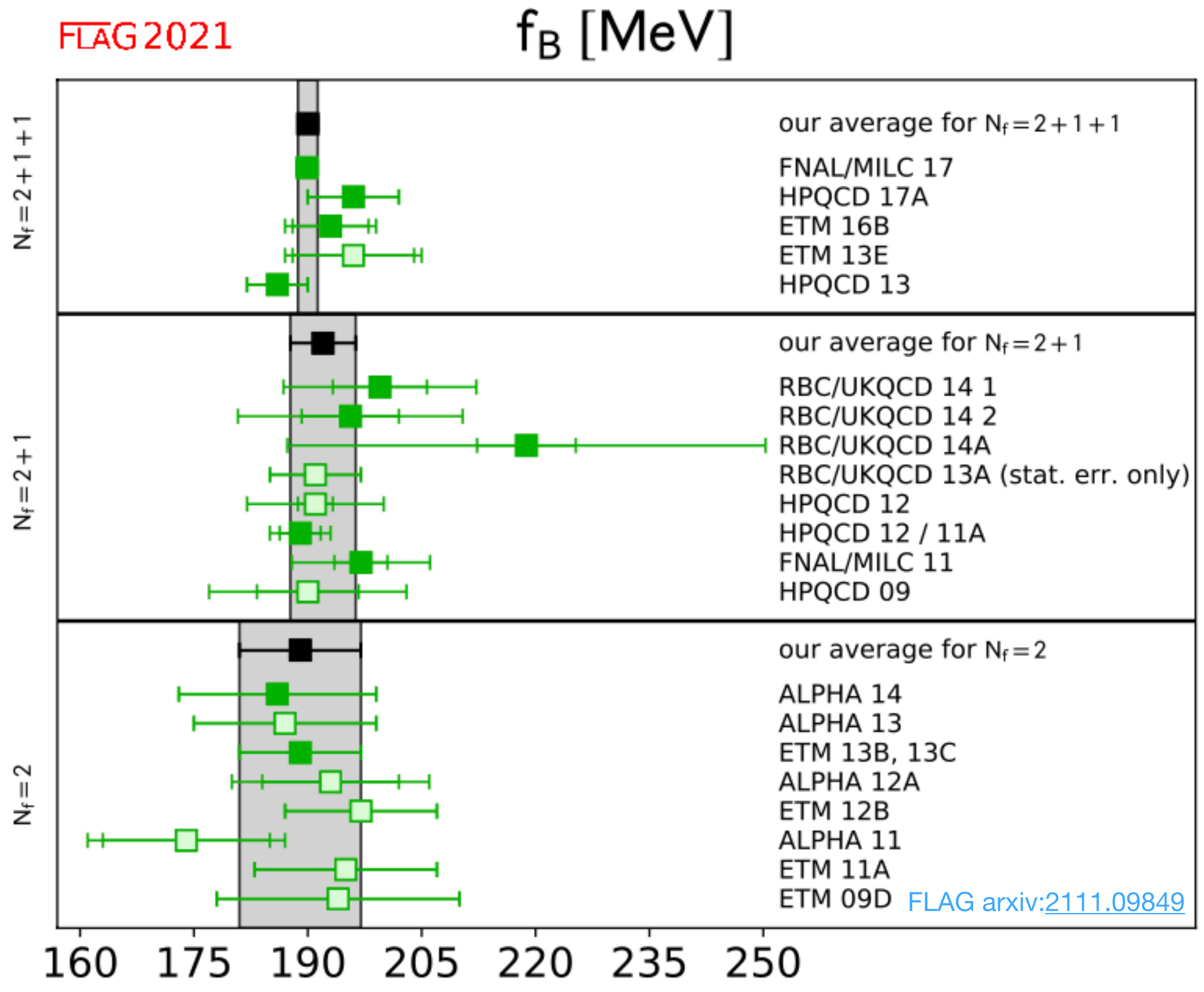
e.g tree level leptonic $B$ decay:




$(\langle 0 | H_w | \text{had} \rangle)$

$$\underbrace{\Gamma_{\text{exp.}}(B \to l\nu_l)}_{\text{experiment}} \overset{???}{=} V_{\text{CKM}}(\text{WEAK})(\text{EM})\boxed{(\text{STRONG})}$$

$$\approx \underbrace{|V_{\text{ub}}|^2}_{\text{output}} \underbrace{\frac{m_B}{8\pi} G_F^2 m_l^2 \left(1 - \frac{m_l^2}{m_B^2}\right)^2 \boxed{f_B^2}}_{\text{theory prediction/kinematics}}$$

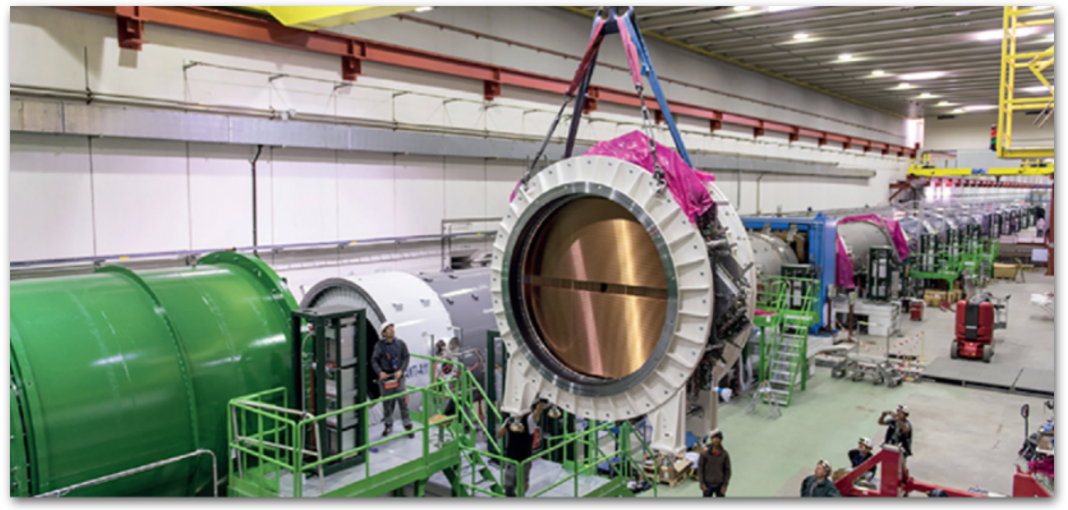


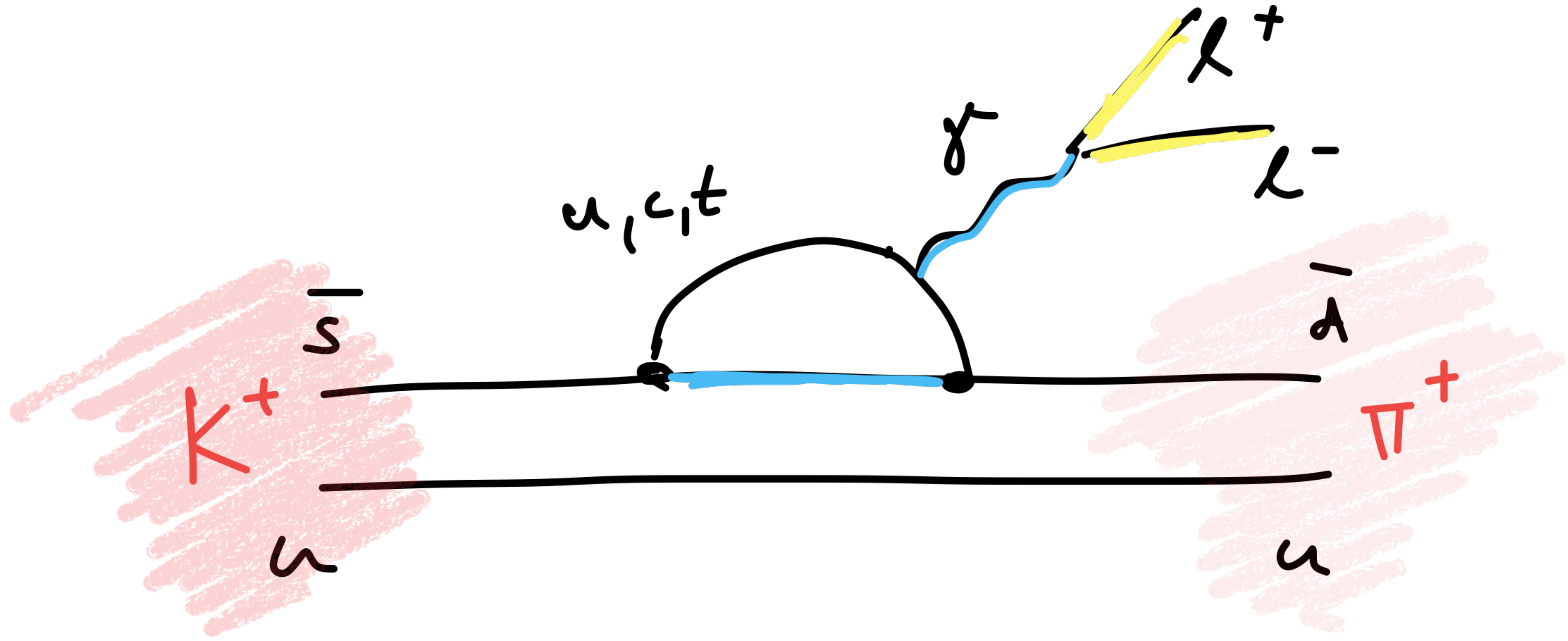

- Experimental measurement + theory prediction allows for extraction of CKM MEs and for strong tests of the SM
- similar for $\pi, K, D_{(s)}, B_{(s)}, \ \dots$
- similar for semiletonic

# Indirect SM tests (2): rare decay, e.g. kaon

$K^+ \to \pi^+ \nu \bar{\nu}$

NA62 (CERN)
small **LD contribution**, candidate for lattice






$K^+ \to \pi^+ l^+ l^-$

1-photon exchange **LD dom.**

$$\mathcal{A}_\mu = (q^2) \int d^4x \langle \pi(p) | T \left[ J_\mu(0) H_W(x) \right] | K(k) \rangle$$

- process is highly suppressed in the SM ( $\sim 10^{-10}$) and therefore sensitive to potential New-Physics contributions
- a first lattice computation came out recently


Isidori, Martinelli, Turchetti: *PLB* 633 (2006) 75-83
RBC/UQKCD:
PRD 92 (2015) 9, 094512
PRD 93 (2016) 11, 114517
Phys.Rev.D 94 (2016) 11, 114516
Phys.Rev.Lett. 118 (2017) 25, 252001
Phys.Rev.D 98 (2018) 7, 074509
Phys.Rev.D 100 (2019) 11, 114506
arXiv:2202.08795

# Lattice QCD

$$\langle 0 | O | 0 \rangle = \frac{1}{\mathcal{Z}} \int D[A, \bar{\psi}, \psi]\, O\, e^{iS_{\mathrm{QCD}}[A, \bar{\psi}, \psi]}$$

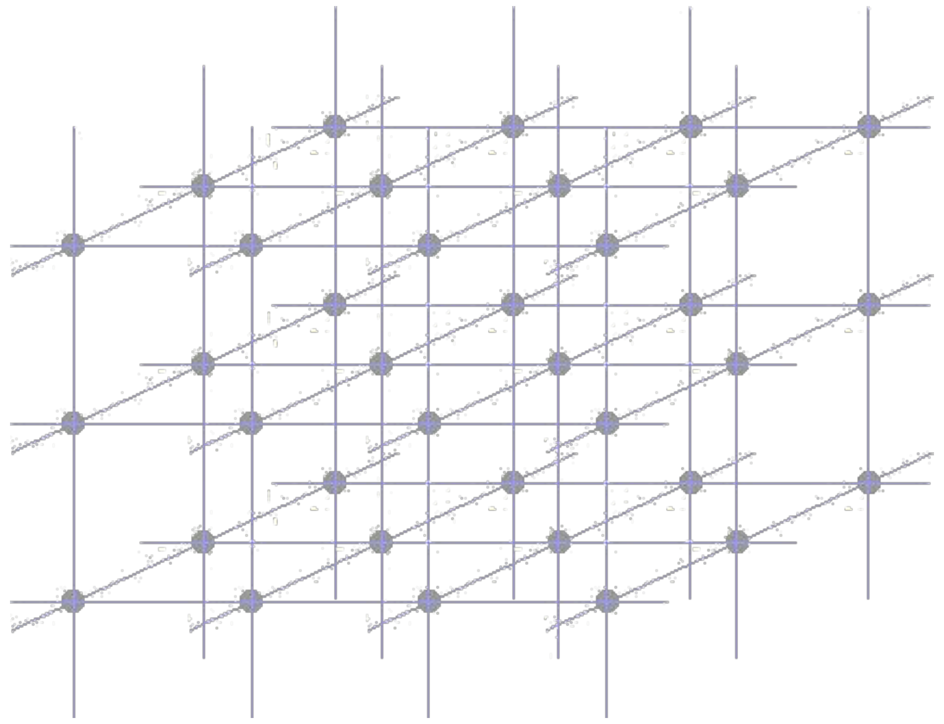
$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + \sum_f \bar{\psi}_f \left( i\gamma^\mu D_\mu - m_f \right) \psi_f$$

Free parameters:

- gauge coupling $g \rightarrow \alpha_s = g^2/4\pi$
- quark masses $m_f = u,d,s,c,b,t$

- Lagrangian of massless gluons and *almost massless quarks*
- What experiment sees are bound states, e.g. $m_\pi, m_P \gg m_{u,d}$
- Underlying physics non-perturbative

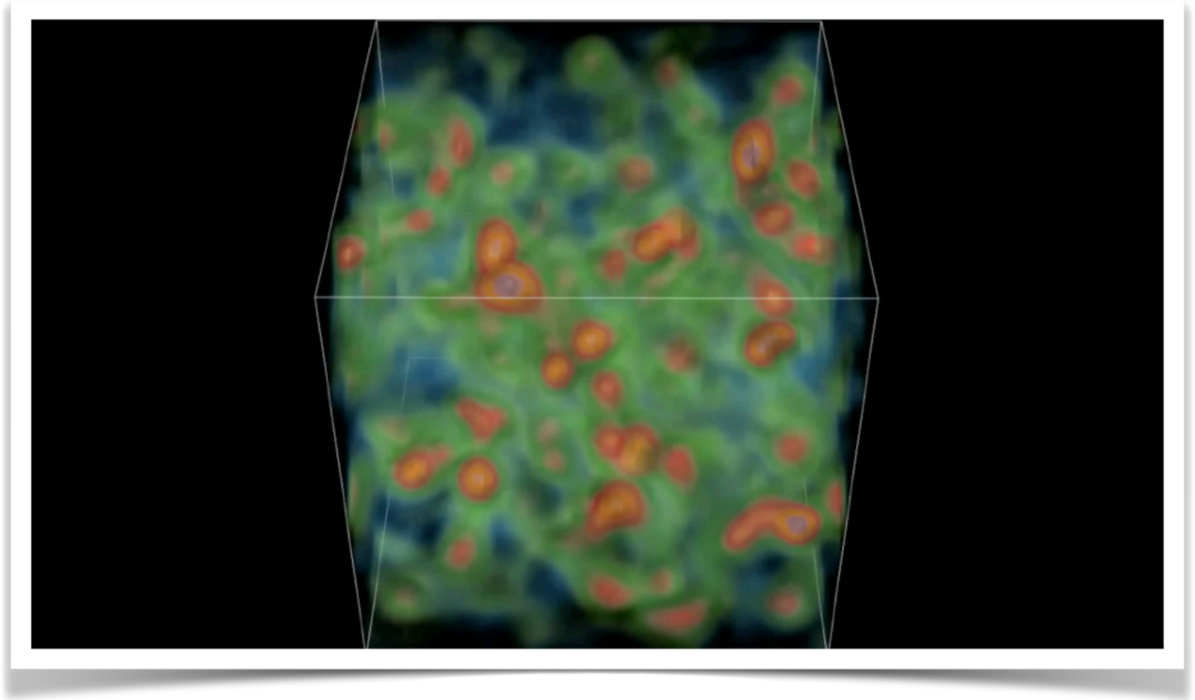
# Lattice QCD



finite volume, space-time grid (IR and UV regulators)

$\propto L^{-1}$ $\propto a^{-1}$

→ Well defined, **finite dimensional** path integral



Path integral still very high-dimensional — can't be integrated analytically

$$\langle 0|O|0\rangle = \frac{1}{Z}\int \mathcal{D}[U,\psi,\bar{\psi}]Oe^{iS_{\text{lat}}[U,\psi,\bar{\psi}]}$$
$$\langle 0|O|0\rangle = \frac{1}{Z}\int \mathcal{D}[U,\psi,\bar{\psi}]O\boxed{e^{-S_{\text{lat}}[U,\psi,\bar{\psi}]}}$$

Euclidean space-time
Boltzmann factor

Very precise computations possible, rely on availability of state-of-the-art HPC resource

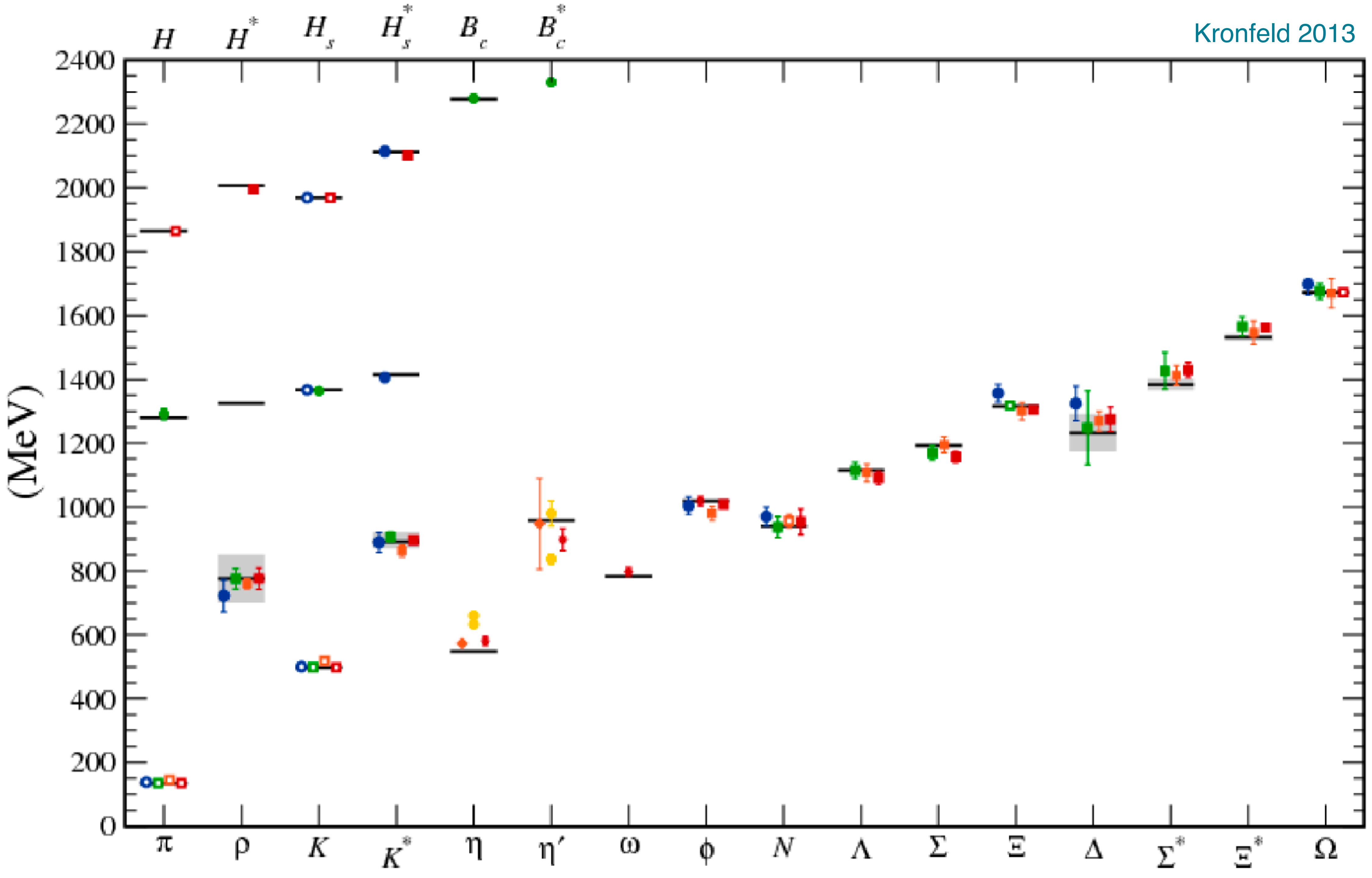
# Lattice QCD

**QCD parameter tuning**

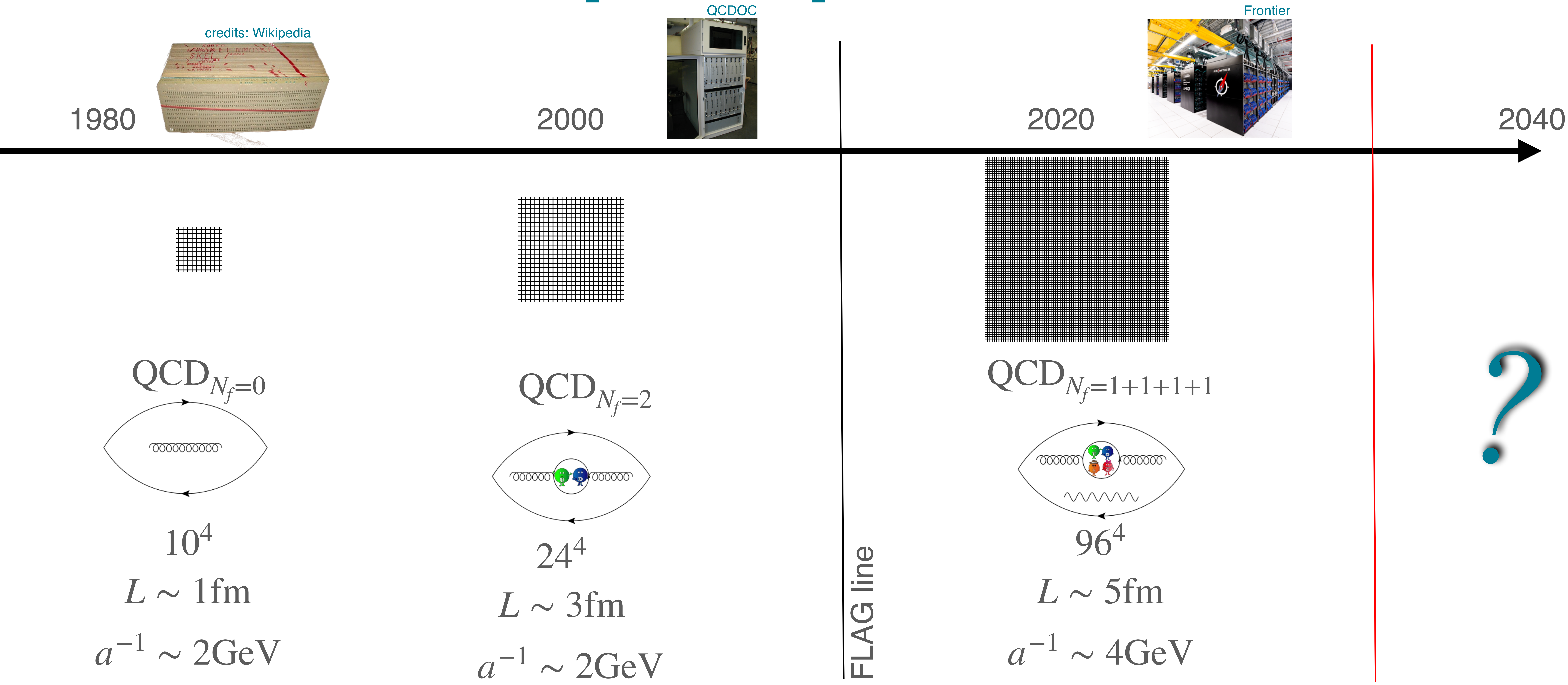
$g, m_u = m_d, m_s, m_c, m_b$
start from *educated guesses* and, e.g.
for $N_f = 2 + 1$:

- tune bare $am_{ud}$ such that $\frac{am_\pi}{am_P} = \frac{m_\pi^{PDG}}{m_P^{PDG}}$
- tune bare $am_s$ such that $\frac{am_\pi}{am_K} = \frac{m_\pi^{PDG}}{m_K^{PDG}}$
- determine physical lattice spacing $a = \frac{af_\pi}{f_\pi^{PDG}}$

Once the QCD-parameters are tuned no further parameters need to be fixed and we can make fully predictive simulations of QCD:



# Lattice's past-present-future



# State-of-the-art

- **Standard:**
  - meson ME with single incoming and/or outgoing pseudo-scalar states
    $\pi, K, D_{(s)}, B_{(s)} \to \mathrm{QCD-vacuum} \;\; \pi \to \pi, K \to \pi, D \to K, B \to \pi, ... \;\; B_K, (B_D), B_B$
  - QCD parameters: quark masses, strong coupling constant
  - meson/baryon spectroscopy of stable (in QCD) states

- **Challenging:**
  - two initial/final hadronic states, one channel $\pi\pi \to \pi\pi, K\pi \to K\pi, K \to \pi\pi$
  - elm. effects in spectra
  - long-distance contributions in e.g. rare Kaon decays, *K*-mixing

- **Very challenging — new ideas needed:**
  - multi-channel final states (hadronic $D, B$)
  - transition MEs with unstable in/out states
  - electromagnetic effects in hadronic MEs
  - inclusive hadron decays
  - ...

# What's currently the best lattice value for a particular quantity?

FLAG

Flavour Lattice Averaging Group

http://flag.unibe.ch

- summary of lattice results
- evaluation according to FLAG quality criteria (colour coding)
- averages or best values where possible (if sensible)
- detailed summary of properties of individual simulations
- target audience: wider phenomenology community

FLAG 11 Eur.Phys.J.C 71 (2011) 1695
FLAG 13 Eur.Phys.J.C 74 (2014) 2890
FLAG 16 Eur.Phys.J.C 77 (2017) 2, 112
FLAG 19 Eur.Phys.J.C 80 (2020) 2, 113
FLAG 21 2111.09849, accepted by EPJC

# Quantities discussed in FLAG21

[FLAG 21 2111.09849](https://arxiv.org/abs/2111.09849)

quark masses

kaon decay

low-energy constants

kaon mixing

| Quantity | Sec. | $N_f=2+1+1$ | Refs. | $N_f=2+1$ | Refs. | $N_f=2$ | Refs. |
|---|---|---|---|---|---|---|---|
| $m_{ud}$[MeV] | 3.1.4 | 3.410(43) | [6, 7] | 3.381(40) | [8–12] | | |
| $m_s$[MeV] | 3.1.4 | 93.43(70) | [6, 7, 13, 14] | 92.2(1.1) | [8–11, 15] | | |
| $m_s/m_{ud}$ | 3.1.5 | 27.250(64) | | | ] | | |
| $m_u$[MeV] | 3.1.6 | 2.14(8) | | | | | |
| $m_d$[MeV] | 3.1.6 | 4.70(5) | | | | | |
| $m_u/m_d$ | 3.1.6 | 0.465(24) | [19, 21] | 0.485(19) | [20] | | |
| $\overline{m}_c$(3 GeV)[GeV] | 3.2.2 | 0.988(11) | [6, 7, 14, 22, 23] | 0.992(6) | [11, 24, 25] | | |
| $m_c/m_s$ | 3.2.3 | 11.768(34) | [6, 7, 14] | 11.82(16) | [24, 26] | | |
| $\overline{m}_b(\overline{m}_b)$[GeV] | 3.3 | 4.203(11) | [6, 27–30] | 4.164(23) | [11] | | |
| $f_+(0)$ | 4.3 | 0.9698(17) | | | | 0.9560(57)(62) | [35] |
| $f_{K^\pm}/f_{\pi^\pm}$ | 4.3 | 1.1932(21) | | | | 1.205(18) | [44] |
| $f_{\pi^\pm}$[MeV] | 4.6 | | | | | | |
| $f_{K^\pm}$[MeV] | 4.6 | 155.7(3) | | | | 157.5(2.4) | [44] |
| $\Sigma^{1/3}$[MeV] | 5.2.4 | 286(23) | [45, 46] | 272(5) | [12, 47–51] | 266(10) | [45, 52–54] |
| $F_\pi/F$ | 5.2.4 | 1 | | | | 3(15) | [52–54, 57] |
| $\bar{\ell}_3$ | 5.2.4 | 3 | | | | (82) | [52, 53, 57] |
| $\bar{\ell}_4$ | 5.2.4 | 4 | | | | (28) | [52, 53, 57, 58] |
| $\bar{\ell}_6$ | 5.2.4 | | | | | 15.1(1.2) | [53, 57] |
| $\hat{B}_K$ | 6.3 | 0.717(18)(16) | [59] | | [8, 60–62] | 0.727(22)(12) | [63] |
| $B_2$ | 6.4 | 0.46(1)(3) | | | 4] | 0.47(2)(1) | [63] |
| $B_3$ | 6.4 | 0.79(2)(5) | | | 4] | 0.78(4)(2) | [63] |
| $B_4$ | 6.4 | 0.78(2)(4) | | | 4] | 0.76(2)(2) | [63] |
| $B_5$ | 6.4 | 0.49(3)(3) | [59] | 0.720(38) | [62, 64] | 0.58(2)(2) | [63] |

$D_{(s)}$-meson decay

$B_{(s)}$-meson decay

$B_{(s)}$-mixing

strong-coupling constant

| Quantity | Sec. | $N_f=2+1+1$ | Refs. | $N_f=2+1$ | Refs. | $N_f=2$ | Refs. |
|---|---|---|---|---|---|---|---|
| $f_D$[MeV] | 7.1 | 212.0(7) | [17, 37] | 209.0(2.4) | [65–67] | 208(7) | [68] |
| $f_{D_s}$[MeV] | 7.1 | 249.9(5) | | | 69] | 246(4) | [68, 70] |
| $\frac{f_{D_s}}{f_D}$ | 7.1 | 1.1783(16 | | | | 1.20(2) | [68] |
| $f_+^{D\pi}(0)$ | 7.2 | 0.612(35) | | | | | |
| $f_+^{DK}(0)$ | 7.2 | 0.7385(44) | [71] | 0.747(19) | [73] | | |
| $f_B$[MeV] | 8.1 | 190.0(1.3 | | | | 188(7) | [68, 80] |
| $f_{B_s}$[MeV] | 8.1 | 230.3(1.3 | | | | 225.3(6.6) | [68, 70, 80] |
| $\frac{f_{B_s}}{f_B}$ | 8.1 | 1.209(5) | | | 81] | 1.206(23) | [68, 80] |
| $f_{B_d}\sqrt{\hat{B}_{B_d}}$[MeV] | 8.2 | | | 225(9) | [78, 82, 83] | 216(10) | [68] |
| $f_{B_s}\sqrt{\hat{B}_{B_s}}$[MeV] | 8.2 | | | | 8, 82, 83] | 262(10) | [68] |
| $\hat{B}_{B_d}$ | 8.2 | | | | 8, 82, 83] | 1.30(6) | [68] |
| $\hat{B}_{B_s}$ | 8.2 | | | | 8, 82, 83] | 1.32(5) | [68] |
| $\xi$ | 8.2 | | | 1.206(17) | [78, 83] | 1.225(31) | [68] |
| $B_{B_s}/B_{B_d}$ | 8.2 | | | 1.032(38) | [78, 83] | 1.007(21) | [68] |
| Quantity | Sec. | $N_f=2+1$ and $N_f=2+1+1$ | | | Refs. | | |
| $\alpha_{\overline{\rm MS}}^{(5)}(M_Z)$ | 9.11 | | | | | | |
| $\Lambda_{\overline{\rm MS}}^{(5)}$[MeV] | 9.11 | | | | | | |
| $\Lambda_{\overline{\rm MS}}^{(4)}$[MeV] | 9.11 | | | | | | |
| $\Lambda_{\overline{\rm MS}}^{(3)}$[MeV] | 9.11 | 339(12) | | | [11, 14, 25, 84–87] | | |

nucleon matrix elements

| Quantity | Sec. | $N_f=2+1+1$ | Refs. | $N_f=2+1$ | Refs. | $N_f=2$ | Refs. |
|---|---|---|---|---|---|---|---|
| $g_A^{u-d}$ | 10.3.1 | 1.246(28) | [88–90] | 1.248(23) | [91, 92] | | |
| $g_S^{u-d}$ | 10.3.2 | 1.02(10) | [88] | | | | |
| $g_T^{u-d}$ | 10.3.3 | 0.989(34) | [88] | | | | |
| $g_A^u$ | 10.4.1 | | | | | | |
| $g_A^d$ | 10.4.1 | | | | | | |
| $g_A^s$ | 10.4.1 | | | | | | |
| $\sigma_{\pi N}$[MeV] | 10.4.4 | | | | | 37(8)(6) | [97] |
| $\sigma_s$[MeV] | 10.4.4 | 41.0(8.8) | [98] | 52.9(7.0) | [94–96, 98, 99] | | |
| $g_T^u$ | 10.4.5 | 0.784(28)(10) | [100] | | | | |
| $g_T^d$ | 10.4.5 | −0.204(11)(10) | [100] | | | | |
| $g_T^s$ | 10.4.5 | −0.0027(16) | [100] | | | | |

$D_{(s)}$ SL decays

$D \to \pi\ell\nu \qquad D \to K\ell\nu$

$B_{(s,c)}$ SL decays

$B \to \pi\ell\nu \qquad B_s \to K\ell\nu \qquad R(D_s)$

$B_{(s)} \to D_{(s)}\ell\nu \qquad B \to D^*\ell\nu \qquad R(D)$

$B_c \to (\eta_c, J/\psi)\ell\nu$

$\Lambda_b$ SL decays

$\Lambda_b \to (p, \Lambda_c^{(*)})\ell\bar{\nu} \qquad \Lambda_b \to \Lambda^{(*)}\ell\ell$

1st, 2nd row CKM ME

$|V_{ud}|, |V_{us}|, |V_{ub}|, |V_{cd}|, |V_{cs}|, |V_{cb}|$

# FLAG criteria

| Collaboration | Ref. | $N_f$ | publication status | continuum extrapolation | chiral extrapolation | finite volume | renormalization/matching | heavy-quark treatment | $f_{B_s}/f_{B^+}$ | $f_{B_s}/f_{B^0}$ | $f_{B_s}/f_B$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| FNAL/MILC 17 | [17] | 2+1+1 | A | ★ | ★ | ★ | ★ | ✓ | 1.2180(49) | 1.2109(41) | – |
| HPQCD 17A | [75] | 2+1+1 | A | ○ | ★ | ★ | ○ | ✓ | – | – | 1.207(7) |
| ETM 16B | [29] | 2+1+1 | A | ★ | ○ | ○ | ○ | ✓ | – | – | 1.184(25) |
| ETM 13E | [489] | 2+1+1 | C | ★ | ○ | ○ | ○ | ✓ | – | – | 1.201(25) |
| HPQCD 13 | [74] | 2+1+1 | A | ○ | ★ | ★ | ○ | ✓ | 1.217(8) | 1.194(7) | 1.205(7) |
| RBC/UKQCD 18A | [81] | 2+1 | P | ★ | ★ | ★ | ★ | ✓ | – | – | $1.1949(60)(^{+95}_{-175})$ |
| RBC/UKQCD 14 | [79] | 2+1 | A | ○ | ○ | ○ | ○ | ✓ | 1.223(71) | 1.197(50) | – |
| RBC/UKQCD 14A | [78] | 2+1 | A | ○ | ○ | ○ | ○ | ✓ | – | – | 1.193(48) |
| RBC/UKQCD 13A | [490] | 2+1 | C | ○ | ○ | ○ | ○ | ✓ | – | – | $1.20(2)^{\diamond}_{\rm stat}$ |
| HPQCD 12 | [77] | 2+1 | A | ○ | ○ | ○ | ○ | ✓ | – | – | 1.188(18) |
| FNAL/MILC 11 | [66] | 2+1 | A | ○ | ○ | ★ | ○ | ✓ | 1.229(26) | – | – |
| RBC/UKQCD 10C | [497] | 2+1 | A | ■ | ■ | ■ | ○ | ✓ | – | – | 1.15(12) |
| HPQCD 09 | [82] | 2+1 | A | ○ | ○ | ○ | ○ | ✓ | – | – | 1.226(26) |
| ALPHA 14 | [80] | 2 | A | ★ | ★ | ★ | ★ | ✓ | – | – | 1.203(65) |
| ALPHA 13 | [491] | 2 | C | ★ | ★ | ★ | ★ | ✓ | – | – | 1.195(61)(20) |
| ETM 13B, 13C† | [68, 492] | 2 | A | ★ | ○ | ★ | ○ | ✓ | – | – | 1.206(24) |
| ALPHA 12A | [493] | 2 | C | ★ | ★ | ★ | ★ | ✓ | – | – | 1.13(6) |
| ETM 12B | [494] | 2 | C | ★ | ○ | ★ | ○ | ✓ | – | – | 1.19(5) |
| ETM 11A | [221] | 2 | A | ○ | ○ | ★ | ○ | ✓ | – | – | 1.19(5) |

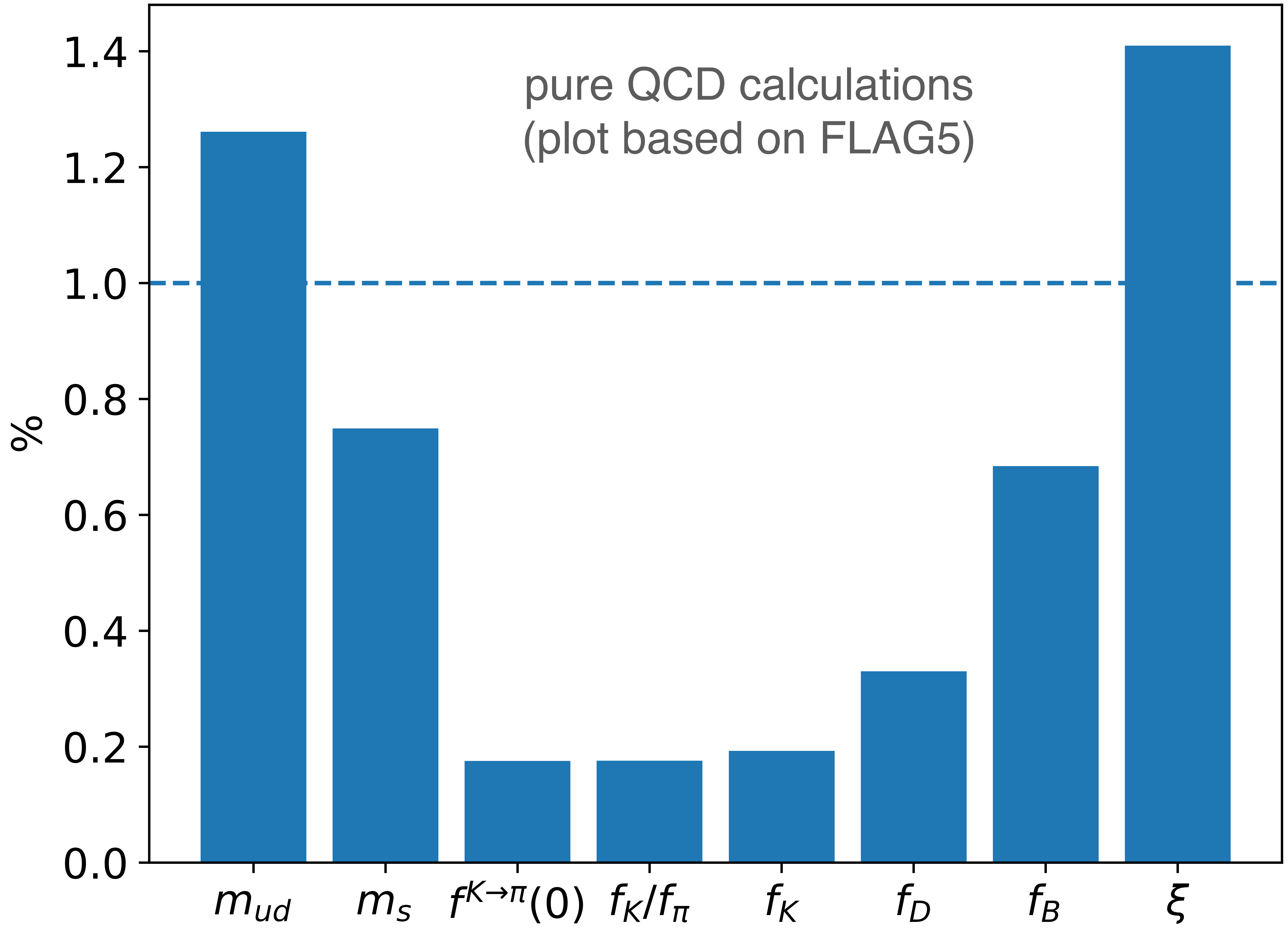
| Collaboration | Ref. | $N_f$ | publication status | chiral extrapolation | continuum extrapolation | finite volume | $a_0^2 M_\pi$ | $\ell^2_{\pi\pi}$ |
|---|---|---|---|---|---|---|---|---|
| ETM 15E | [325] | 2+1+1 | A | ○ | ★ | ★ | $-0.0442(2)(^{4}_{0})$ | $3.79(0.61)(^{+1.34}_{-0.11})$ |
| PACS-CS 13 | [331] | 2+1 | A | ★ | ■ | ■ | −0.04263(22)(41) | |
| Fu 13 | [324] | 2+1 | A | ■ | ■ | ★ | −0.04430(25)(40) | 3.27(0.77)(1.12) |
| Fu 11 | [350] | 2+1 | A | ■ | ■ | ★ | −0.0416(2) | 11.6(9) |
| NPLQCD 11A | [351] | 2+1 | A | ■ | ■ | ★ | −0.0417(07)(02)(16) | |
| NPLQCD 07 | [322] | 2+1 | A | ■ | ■ | ■ | $-0.04330(42)_{\rm tot}$ | |
| NPLQCD 05 | [352] | 2+1 | A | ■ | ■ | ■ | −0.0426(06)(03)(18) | |
| ETM 20B | [348] | 2 | A | ○ | ■ | ○ | −0.0481(86) | |
| Mai 19 | [347] | 2 | P | ■ | ■ | ○ | −0.0433(2) | |
| Culver 19 | [346] | 2 | P | ■ | ■ | ○ | −0.0445(14)(19) | |
| Yagi 11 | [353] | 2 | P | ○ | ■ | ■ | −0.04410(69)(18) | |
| ETM 09G | [323] | 2 | A | ○ | ○ | ○ | −0.04385(28)(38) | 4.65(0.85)(1.07) |
| CP-PACS 04 | [354] | 2 | A | ■ | ■ | ★ | −0.0413(29) | |
| Caprini 11 | [321] | | | | | | −0.0445(11)(4)(8) | |
| Colangelo 01 | [312] | | | | | | $-0.0444(10)_{\rm tot}$ | |

★ satisfactory control of systematic uncertainties possible

○ simulations allow for reasonable attempt at estimating systematic uncertainties, which however could be improved

■ simulations are unlikely to allow for a reasonable control of systematic uncertainties

# Current accuracy on some quantities



# Summary (1) and what next?

- Lattice Flavour Physics is a mature research field
- many independent groups competing
- (sub-)percent accuracy for *standard* quantities feasible in isospin-symmetric QCD
- FLAG summarises particularly mature quantities for use in SM and BSM phenomenology

Results listed in FLAG are from mature calculations where systematics are under control

- What stops us from increasing precision further?
- Can we extend the set of quantities to provide further, potentially more stringent SM tests?

# A look into the future

- Better, faster, stronger algorithms and computing
- Outlook 1 — QCD+QED
- Outlook 2 — Higher-order EW
- Outlook 3 — Flavour physics as an inverse problem

# Better, faster, stronger algorithms and computing

Frontier@OLCF

$a^{-1}$ << physics of interest << $L^{-1}$

finite lattice spacing — finite box size

UV and IR scales difficult to accommodate given limited computing resources

**finite lattice spacing (UV)**

- cutoff effects
- extrapolation potentially not that easy
  Husung, Markquardt, Sommer, 2022, Husung 2206.03536
- critical slowing down
- heavy quarks
- momentum reach (think e.g. $B \to \pi l \nu$)

**finite lattice volume (IR)**

- finite-size effects
- $e^{-M_\pi L}$ for many bread-and-butter quantities but also often $1/L$ (scattering, QED, …)
- momentum resolution

**Other systematics:** scale setting, renormalisation/matching,… (see **FLAG 21**)

# Better, faster, stronger algorithms and computing

Frontier@OLCF

- improved computing resource will allow to tackle larger volumes, smaller lattice spacings, higher statistics → smaller stat. errors, smaller systematic errors

- historically step-changes in quality due to improved computing algorithms **and** ideas in QFT

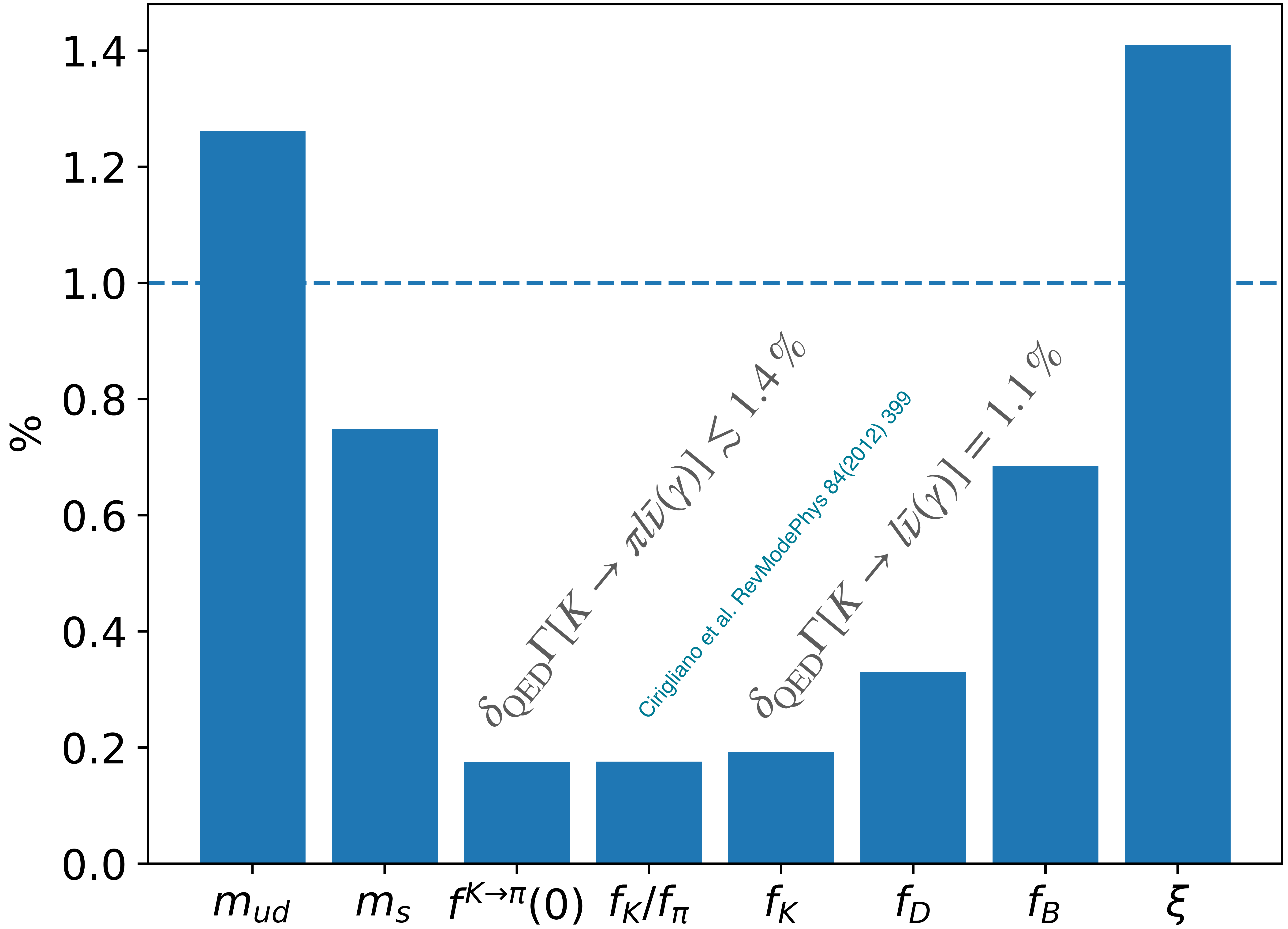
A. *bread-and-butter* quantities will profit most directly

B. historically difficult/impossible computations may become more feasible

# Outlook 1

## What stops us from further improving precision?

# A hint from EFT




for most quantities lattice still only able to achieve such high precision while **ignoring strong isospin breaking and QED effects!**!

# What stops us from further reducing precision?

- strong isospin breaking

Numbers from FLAG 21

$$m_u = 2.14(8)\mathrm{MeV} \qquad m_d = 4.70(5)\mathrm{MeV} \qquad \overline{\mathrm{MS}}(2\mathrm{GeV})$$

$$\frac{m_u - m_d}{\Lambda_{\mathrm{QCD}}} \sim O(1\%)$$

- QED

$$\alpha_{\mathrm{QED}} \approx \frac{1}{137} \sim O(1\%)$$

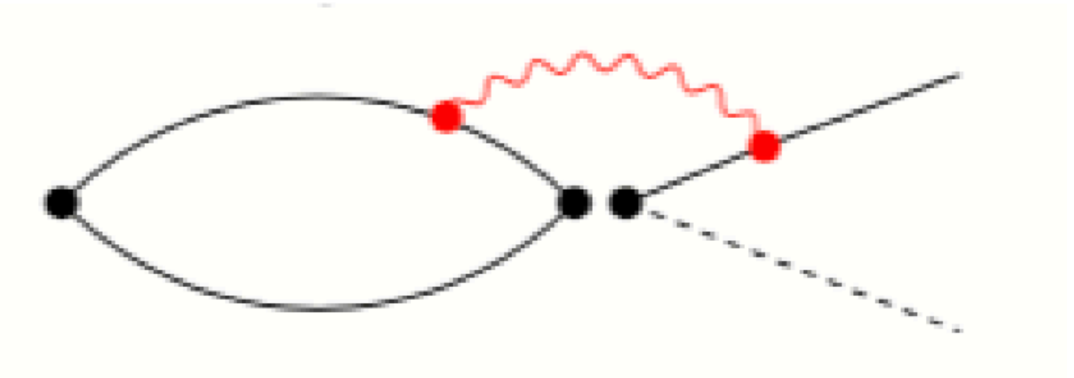
- Majority of results in FLAG 21 for isospin-symmetric QCD but at sub-percent precision we can no longer ignore QED and strong isospin
- higher order QED or weak not yet relevant for *tree*-level flavour changing decays (but may become): $G_F/\Lambda^2_{\mathrm{QCD}} \sim 10^{-6}$

Going forward we have to include QED/isospin breaking effects

# EM effects

$$\Gamma_{\text{exp.}} \stackrel{???}{=} V_{\text{CKM}}(\text{WEAK})(\text{EM})(\text{STRONG})$$

e.g. leptonic decay:



A number of (open) questions:

- How to formulate QED in finite volume
  Duncan et al. PRL 76 1996, Hayakawa, Uno Prog.Th.Ph 120 2008, Endres et al. PRL 117 2016, Lucini et al. JHEP 02 2016, …
- Photon is massless and induces power-suppressed finite-size effects
  BMWc Science 347 2015, Lubicz et al. PRD 95 2017, Davoudi, Savage, PRD 90 2014, Endres et al. PRL 117 2016, Lucini et al. JHEP 02 2016, Davoudi et al. PRD 99 2019, J. Bijnens et al. PRD 100, 014508 (2019), Hansen PRL 123 (2019) 172001
- Renormalising QCD+QED RM123+SOTON, PRD120 (2018), PRDD100 (2019)
- IR singularities need to be dealt with Carrasco et al. PRD 91 2016
- quark-disconnected diagrams
- …

Many challenging conceptual problems in adding QED attract a lot of attention in the community!

# *Outdated* observables?

$$\overbrace{\phantom{(\mathrm{STRONG})}}^{\langle 0 | A_\mu | \pi \rangle \sim f_\pi}$$

QCD $\quad \boxed{\Gamma_{\mathrm{exp.}} \overset{???}{=} V_{\mathrm{CKM}} (\mathrm{WEAK})(\mathrm{EM})(\mathrm{STRONG})}$

QCD+QED $\quad \boxed{\Gamma_{\mathrm{exp.}} \overset{???}{=} V_{\mathrm{CKM}} (\mathrm{WEAK})\underbrace{(\mathrm{EM})(\mathrm{STRONG})}}$

at NLO $\Gamma_{\mathrm{QED+QCD}} = \Gamma_{\mathrm{isoQCD}} + \Gamma_\gamma + \Gamma_{\mathrm{IB}}$

Very interesting discussion around how to define decomposition, see e.g. FLAG report or N. Tantalo's talk at Lattice 2022

# Hadronic matrix elements in QCD+QED

- leptonic decay at $O(\alpha^0)$:

$$\Gamma(\pi^+ \to l^+ \nu_l) = \frac{G_F^2 |V_{ud}|^2 f_\pi^2}{8\pi} m_\pi m_l^2 \left(1 - \frac{m_l^2}{m_\pi^2}\right)^2$$

- including elm. effects @ $O(\alpha)$:

$$\Gamma(\pi^+ \to l^+ \nu_l(\gamma)) = \Gamma(\pi^+ \to l^+ \nu_l) + \Gamma(\pi^+ \to l^+ \nu_l \gamma)$$

$$\equiv \qquad \Gamma_0 \qquad + \qquad \Gamma_1$$

IR div. cancel between terms on r.h.s.
between virtual and real photons
(Bloch Nordsieck)

# Hadronic matrix elements in QCD+QED

Carrasco et al. PRD 91 074506 (2015)
V. Lubicz et al, Phys. Rev. D95, 034504 (2017)

- cut on small photon momentum < *ΔE* → γ *sees* point-like π
  *ΔE*≈*20*MeV experimentally accessible and π point like

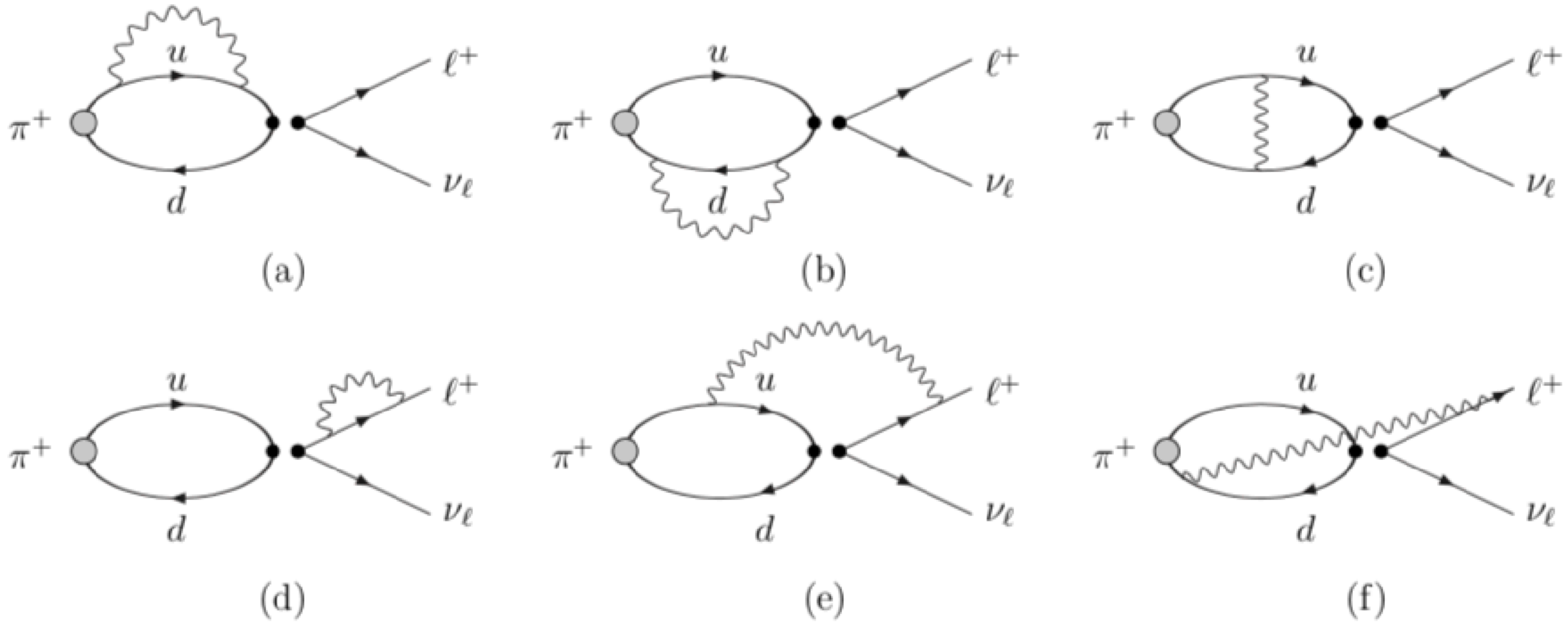
point approximation

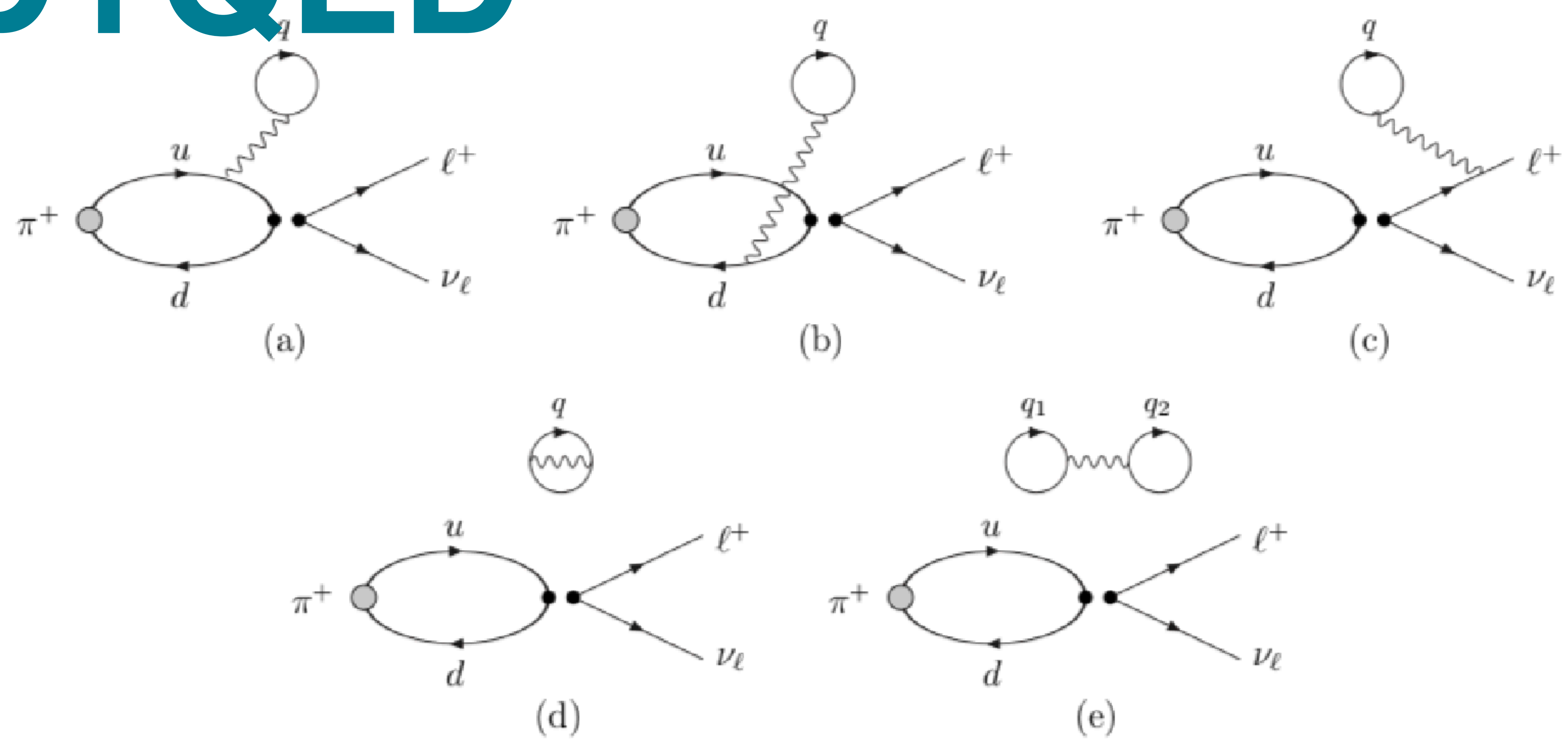
$$\Gamma(\Delta E) = \lim_{V\to\infty}\left(\Gamma_0 - \Gamma_0^{\mathrm{pt}}\right) + \lim_{V\to\infty}\left(\Gamma_0^{\mathrm{pt}} + \Gamma_1^{\mathrm{pt}}(\Delta E)\right)$$

$\Gamma(\pi^+ \to l^+ \nu_l)$
lattice and analytical finite *V*

$\Gamma(\pi^+ \to l^+ \nu_l \gamma(\Delta E))$
analytically in $V \to \infty$

# Hadronic matrix elements in QCD+QED




First complete calculation of connected pieces: (Rome-Southampton. PRD120 (2018) 7, 072001, PRD 100 (2019) 3, 034514)

$$\Gamma_P(E) = \Gamma_P^0\left(1 + \delta R_P(E)\right) \qquad \delta R_K(E_K^{\max}) = 0.0024(10), \qquad \delta R_\pi(E_K^{\max}) = 0.0153(19)$$

In progress: including IB effects in lattice QCD+QED yet to be understood conceptually for a wide range of quantities, e.g. semileptonic decays

QCD+QED is the way forward when going below %-level accuracy

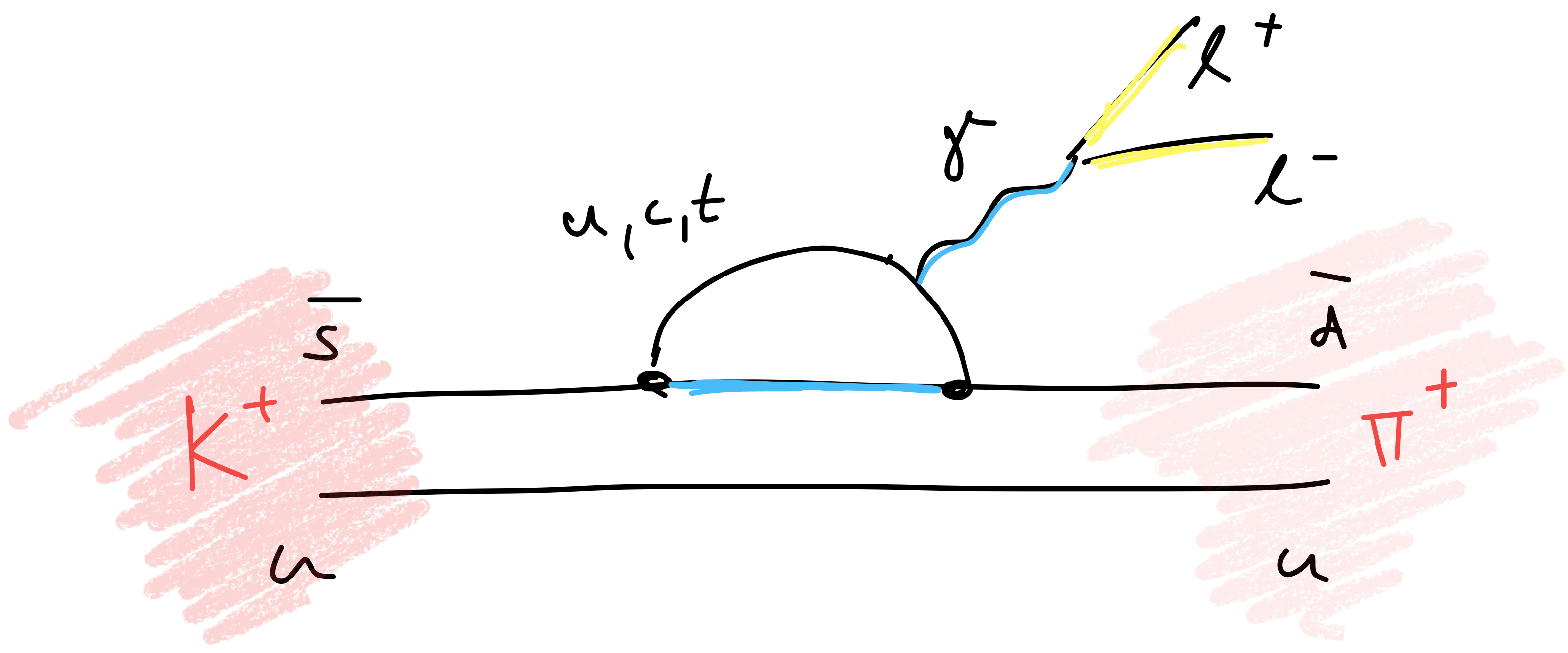
# Outlook 2

## 2nd order weak processes

$$\begin{matrix} \langle \mathrm{had}' | H_W | \mathrm{had} \rangle_{QCD} \\ \langle 0 | H_W | \mathrm{had} \rangle_{QCD} \end{matrix} \longrightarrow \langle \mathrm{had}' | T[H_W H_W'] | \mathrm{had} \rangle_{QCD}$$

# 2nd order weak processes

consider $K^+ \to \pi^+ l^+ l^-$ with dominant 1-photon contribution:




**2nd order weak decay**
**→ 2 insertions of *$H_W$/$J_\mu$***

$$\mathcal{A}_\mu = (q^2) \int d^4x \langle \pi(p) | T \left[ J_\mu(0) H_W(x) \right] | K(k) \rangle$$

# Difficulties

2nd order weak ME on the lattice is new development

**Complications**

1. **Spectral representation:** Euclidean space intermediate states lead to artefacts that need to be controlled

2. **Renormalisation:** EW operator contact terms lead to UV div.

3. **Finite volume effects:** The finite-volume corrections from intermediate on-shell states can be large

4. **Signal-to-noise:** require smart variance-reduction techniques

**Isidori et al.** PLBB 633 (2006) 75-83, **Christ et al.** PRD91 (2015), 114510, **RBC/UKQCD** PRD92 (2015) 094512, PRD94 (2016) 114516, PRD93 (2016) 114517, PRL118 (2017) 252001, PRD98 (2018) 7, 074509, arXiv:2202.08795

# Spectral representation - Minkowski

$$\mathcal{A}^c_\mu(q^2) = \int d^4x \langle \pi^c(p)| T\left[J_\mu(0) H_W(x)\right] |K^c(k)\rangle$$

non-strange intermediate states

$$\begin{aligned}\mathcal{A}^c_\mu(q^2) =& i\int_0^\infty dE\,\frac{\rho(E)}{2E}\,\frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E + i\epsilon}\\ &- i\int_0^\infty dE\,\frac{\rho_S(E)}{2E}\,\frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p}) + i\epsilon}\end{aligned}$$

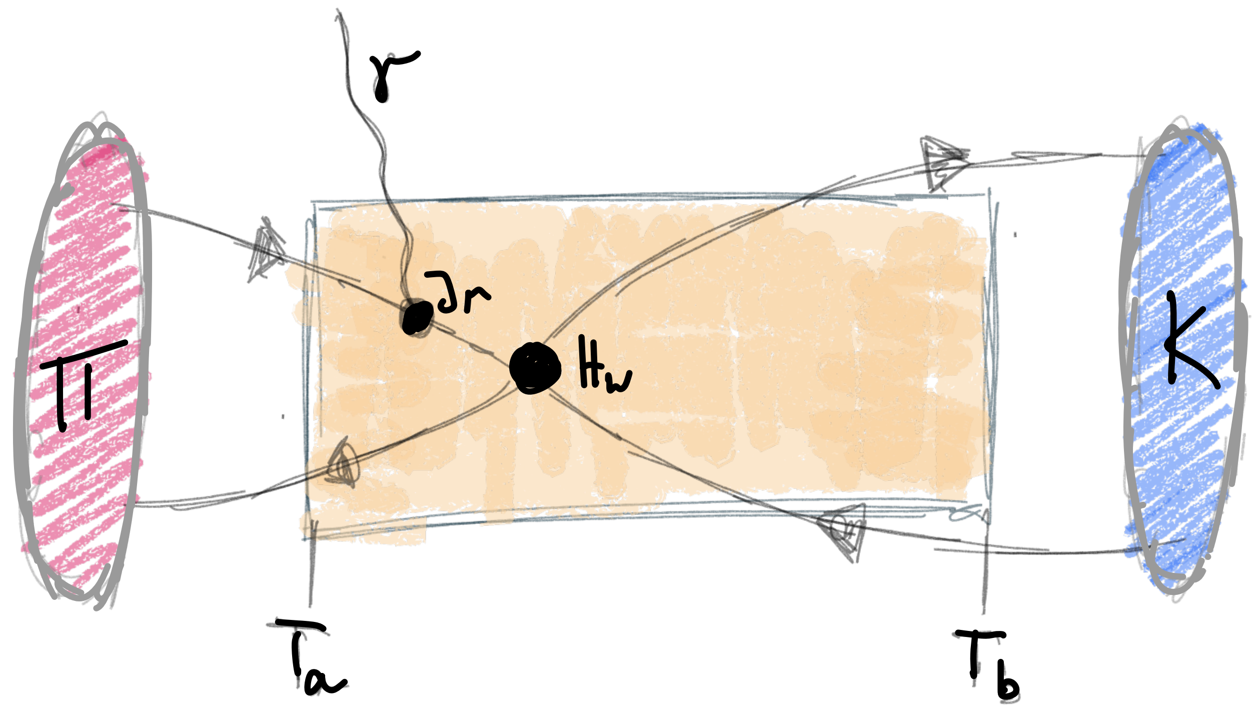
strange intermediate states

complications arise when considering the amplitude in **Euclidean space …**

# Spectral representation - Euclidean

$$A^c_\mu(T_a, T_b, q^2) = \int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k}) - E}\left(1 - e^{(E_K(\mathbf{k})-E)T_a}\right)$$

$$+ \int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E - E_\pi(\mathbf{p})}\left(1 - e^{-(E-E_\pi(\mathbf{k}))T_b}\right)$$




exponential in first terms on r.h.s.

- 1st line:
  - $E>E_K$: exponential term vanishes as $T_a \to \infty$
  - $E<E_K$: exponential term grows as $T_a \to \infty$, must be removed (possible intermediate states π, ππ, πππ)
- 2nd line: no problem, all intermediate states E larger $E_\pi$

# Spectral representation - Euclidean

$$A^c_\mu(T_a,T_b,q^2) = \int_0^\infty dE \frac{\rho(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k})-E}\left(1-e^{(E_K(\mathbf{k})-E)T_a}\right)$$

$$+\int_0^\infty dE \frac{\rho_S(E)}{2E} \frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E-E_\pi(\mathbf{p})}\left(1-e^{-(E-E_\pi(\mathbf{k}))T_b}\right)$$

subtraction of exponentially increasing states, e.g.

- $\pi$: either get amplitudes from 2pt and 3pt functions and subtract

  **or** replace

  $$H_W(x) \to H'_W(x) = H_W(x) + c_S(\mathbf{k})\bar{s}(x)d(x)$$

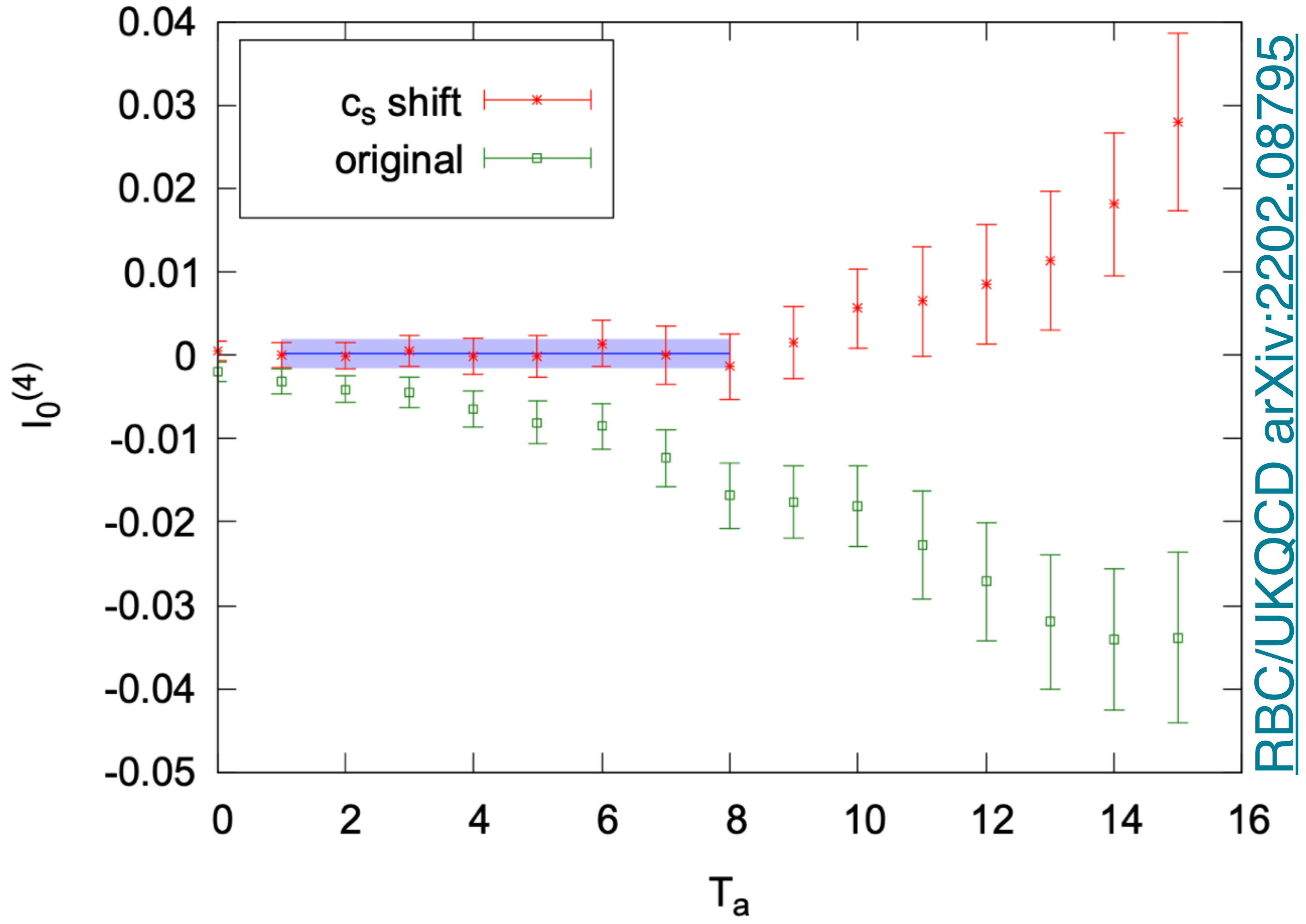
  where $c_S$ such that $\langle \pi^c(\mathbf{k})|H'_W(0,\mathbf{k})|K^c(\mathbf{k})\rangle = 0$ kills the unwanted divergent contribution and does not contribute to the amplitude itself

- $\pi\pi$, $\pi\pi\pi$ states either disallowed or small — need to be monitored

# Spectral representation - Euclidean

$$A_\mu^c(T_a,T_b,q^2) = \int_0^\infty dE\,\frac{\rho(E)}{2E}\,\frac{\langle \pi^c(\mathbf{p})|J_\mu(0)|E,\mathbf{k}\rangle\langle E,\mathbf{k}|H_W(0)|K^c(\mathbf{k})\rangle}{E_K(\mathbf{k})-E}\left(1-e^{(E_K(\mathbf{k})-E)T_a}\right)$$

$$+\int_0^\infty dE\,\frac{\rho_S(E)}{2E}\,\frac{\langle \pi^c(\mathbf{p})|H_W(0)|E,\mathbf{p}\rangle\langle E,\mathbf{p}|J_\mu(0)|K^c(k)\rangle}{E-E_\pi(\mathbf{p})}\left(1-e^{-(E-E_\pi(\mathbf{k}))T_b}\right)$$

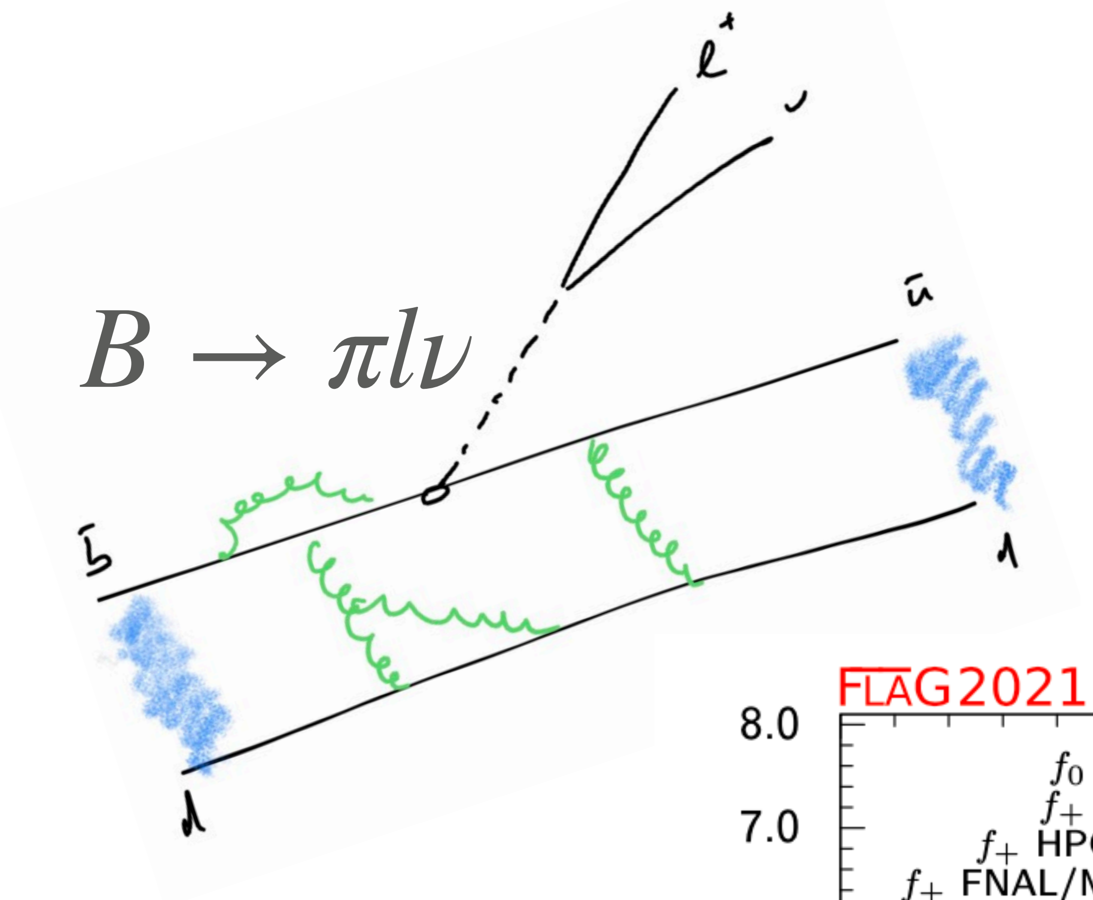



RBC/UKQCD arXiv:2202.08795

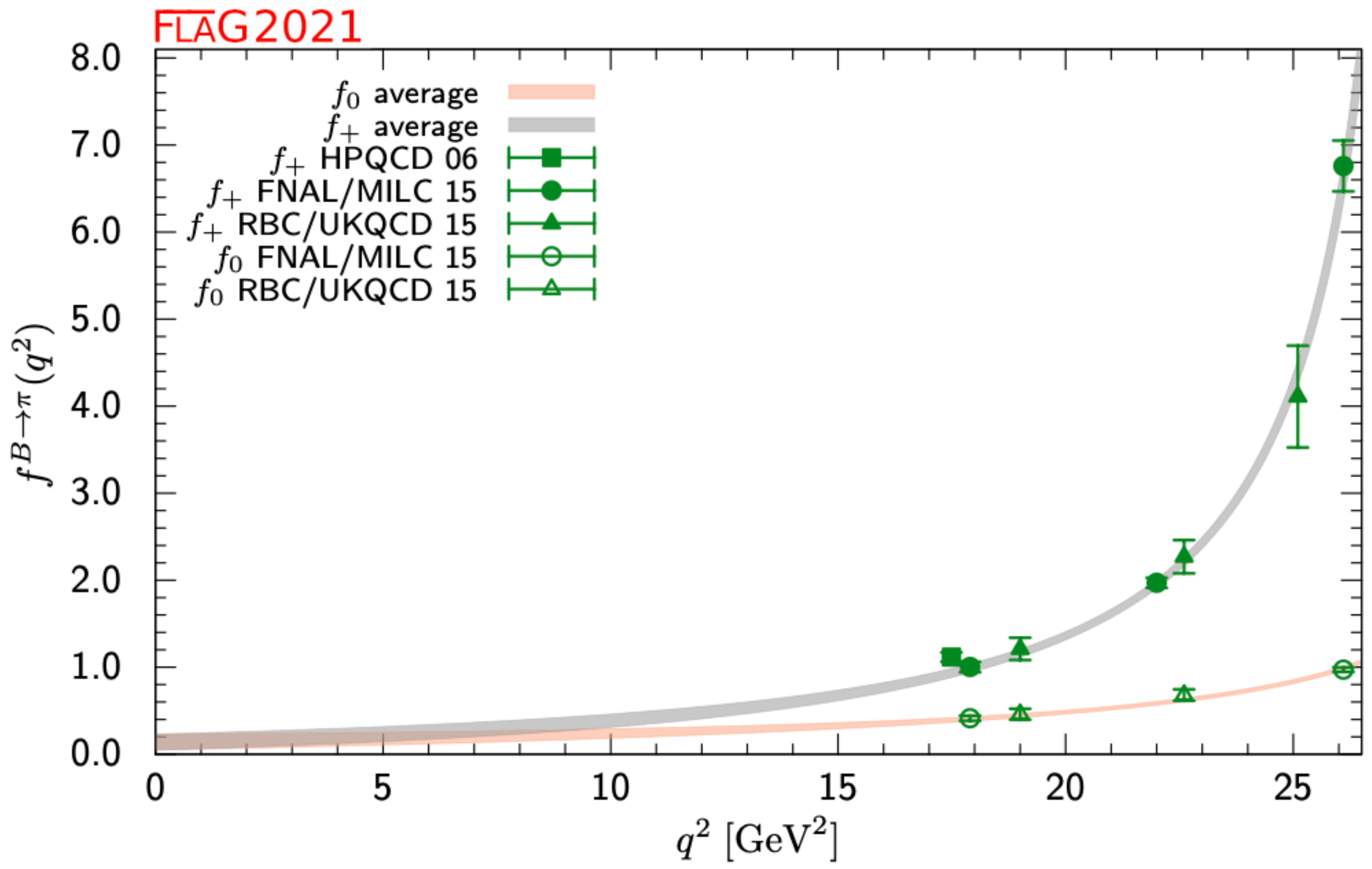
- technically very challenging computation
- first exploratory result were promising RBC/UKQCD PRDD 94 (2016) 11, 114516
- signal-to-noise plaguing simulation at physical point RBC/UKQCD arXiv:2202.08795
  Optimistic that significant error reduction possible in culprit-diagram (GIM subtraction), I expect substantial reduction in stat. error in coming years

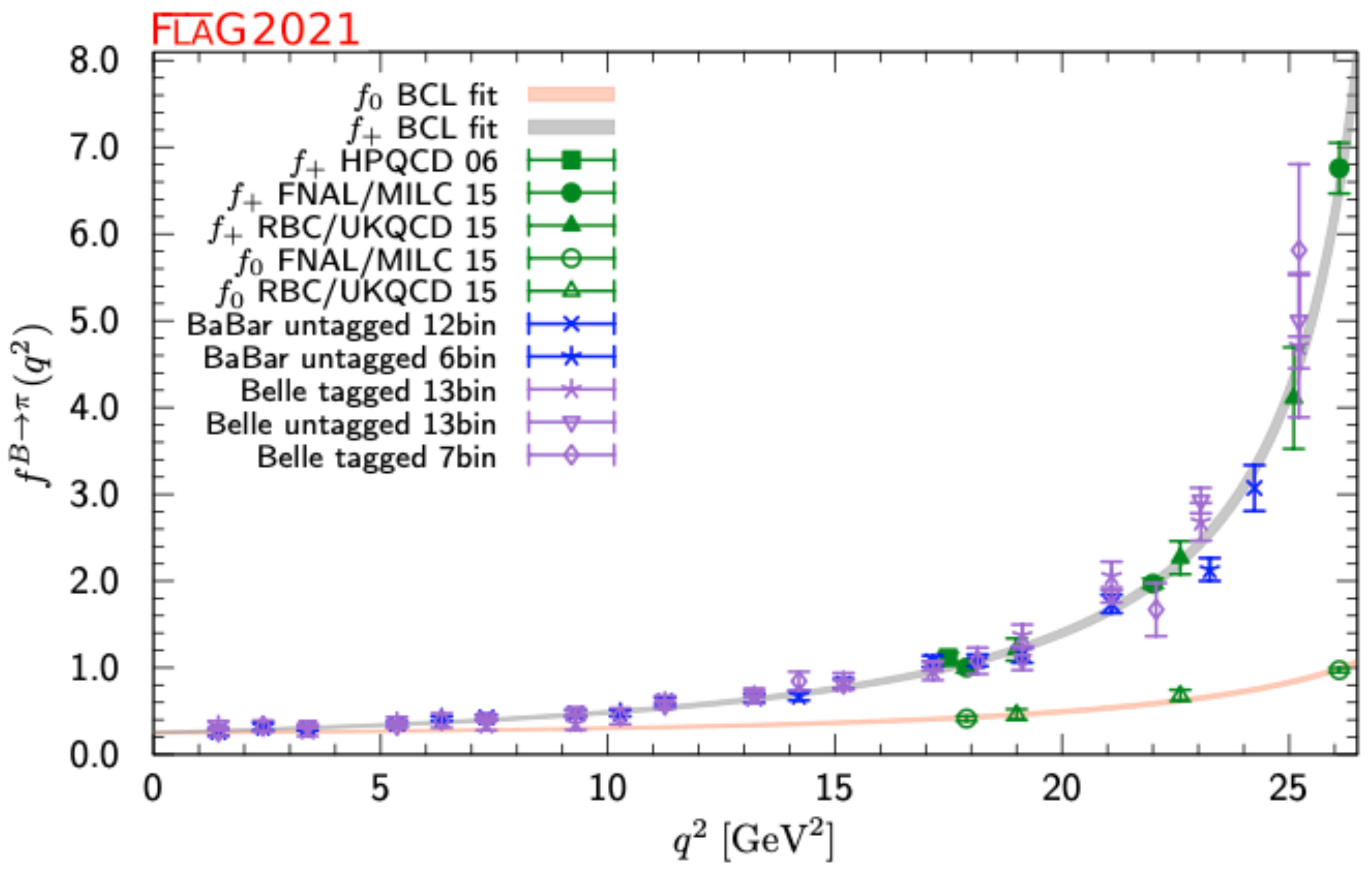
# Outlook 3

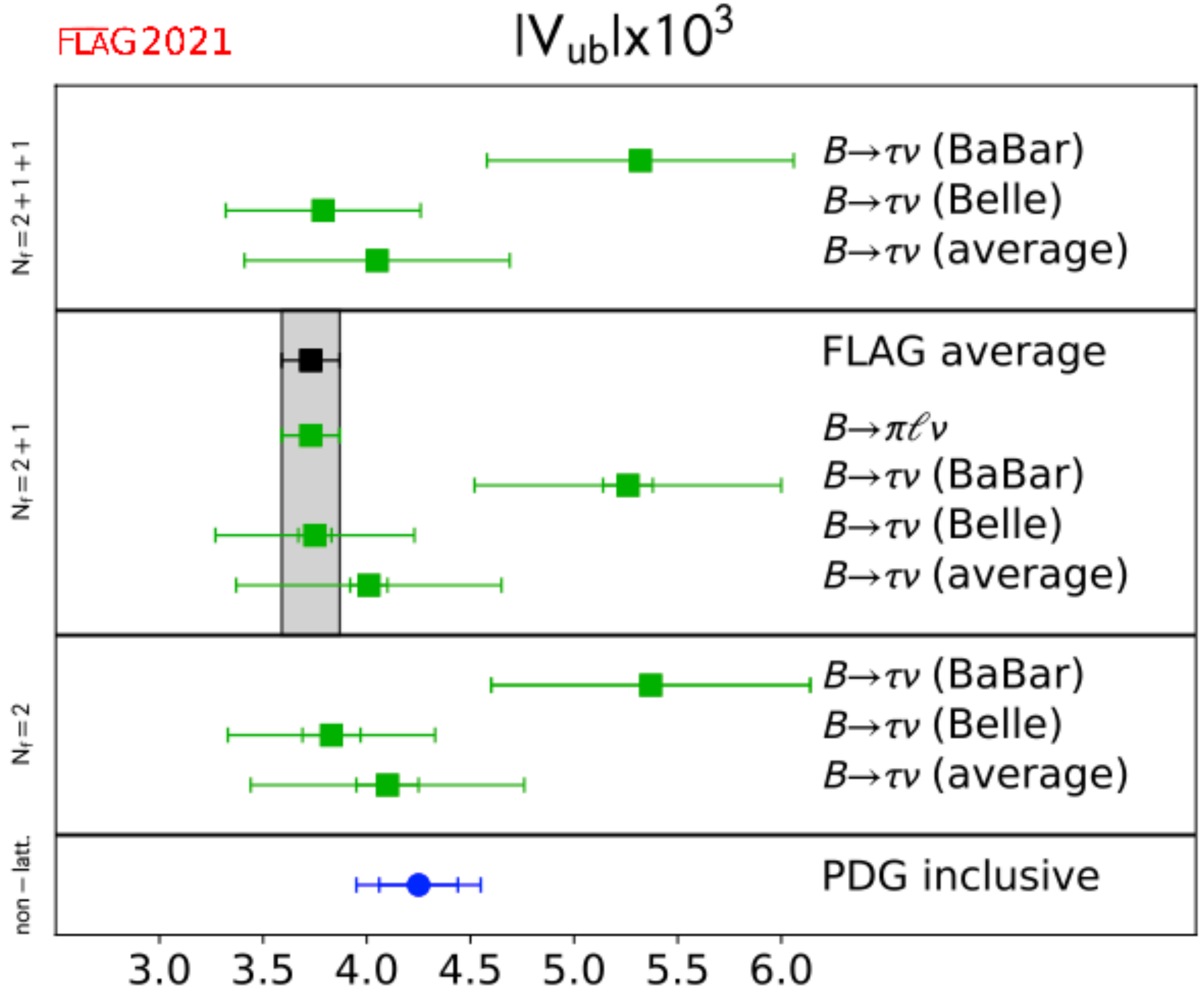
**Flavour physics via the QCD spectral function: inclusive semileptonic $B_{(s)}$ decays on the lattice**



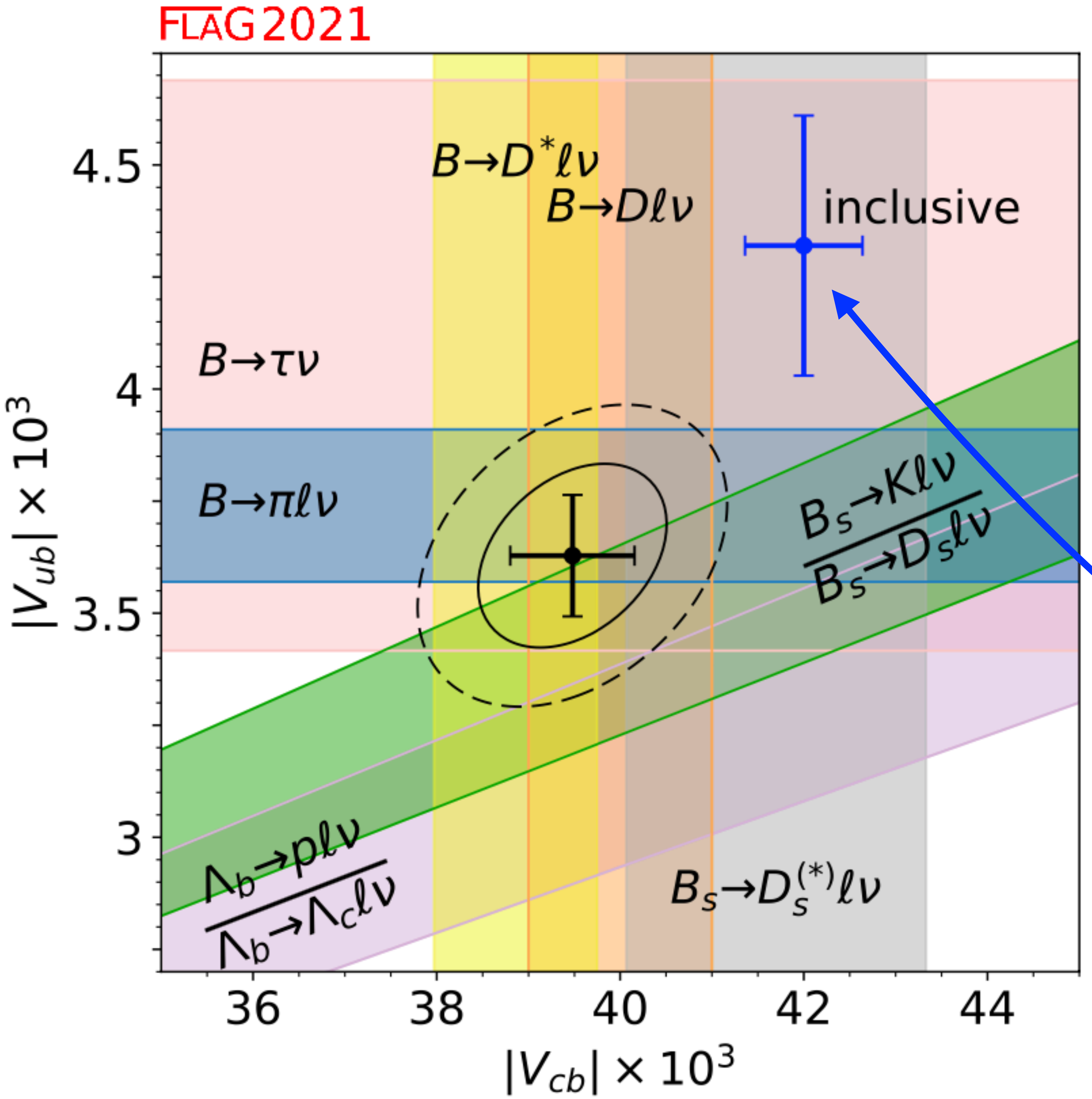

# first: exclusive $B$ decay



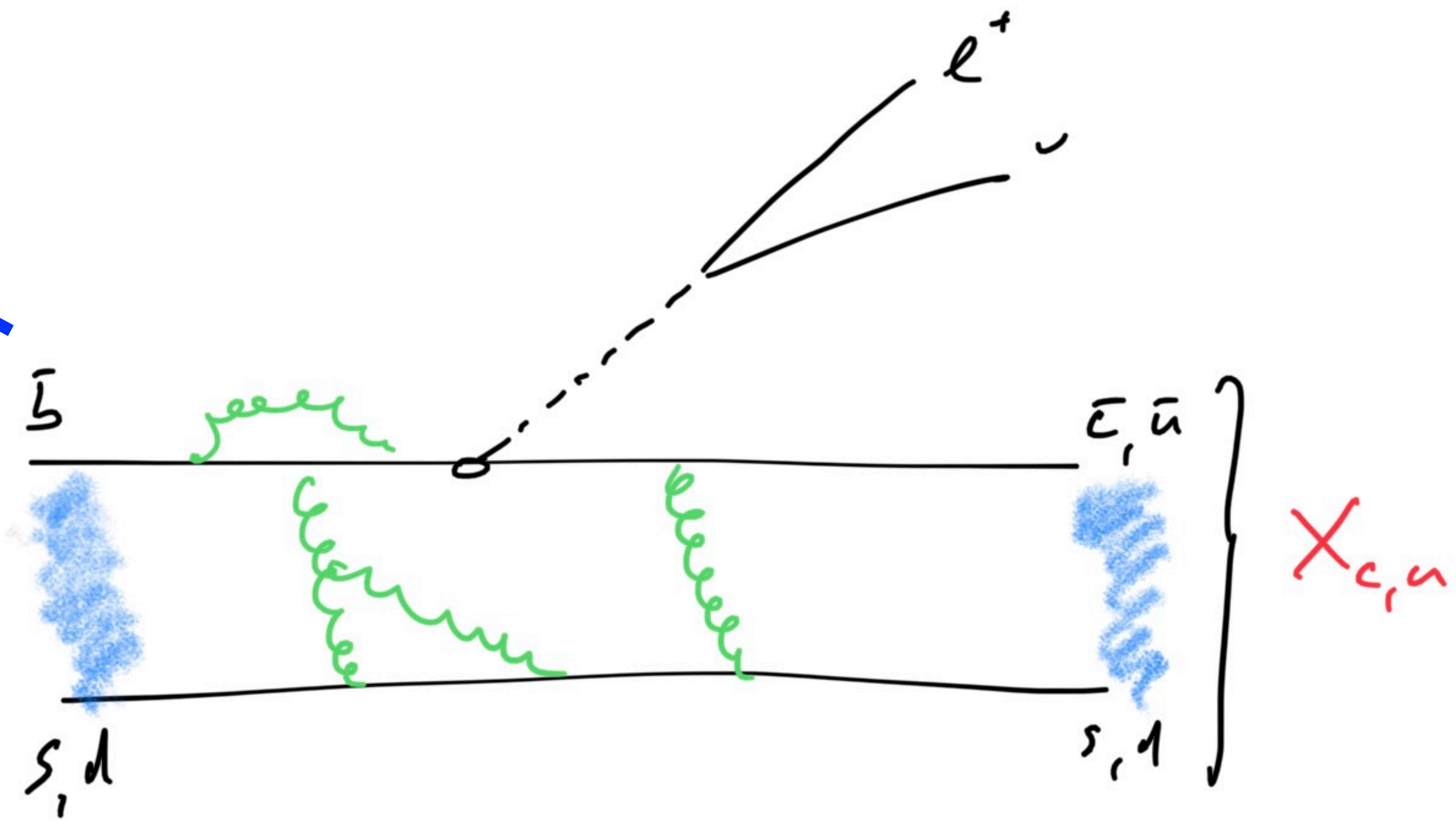



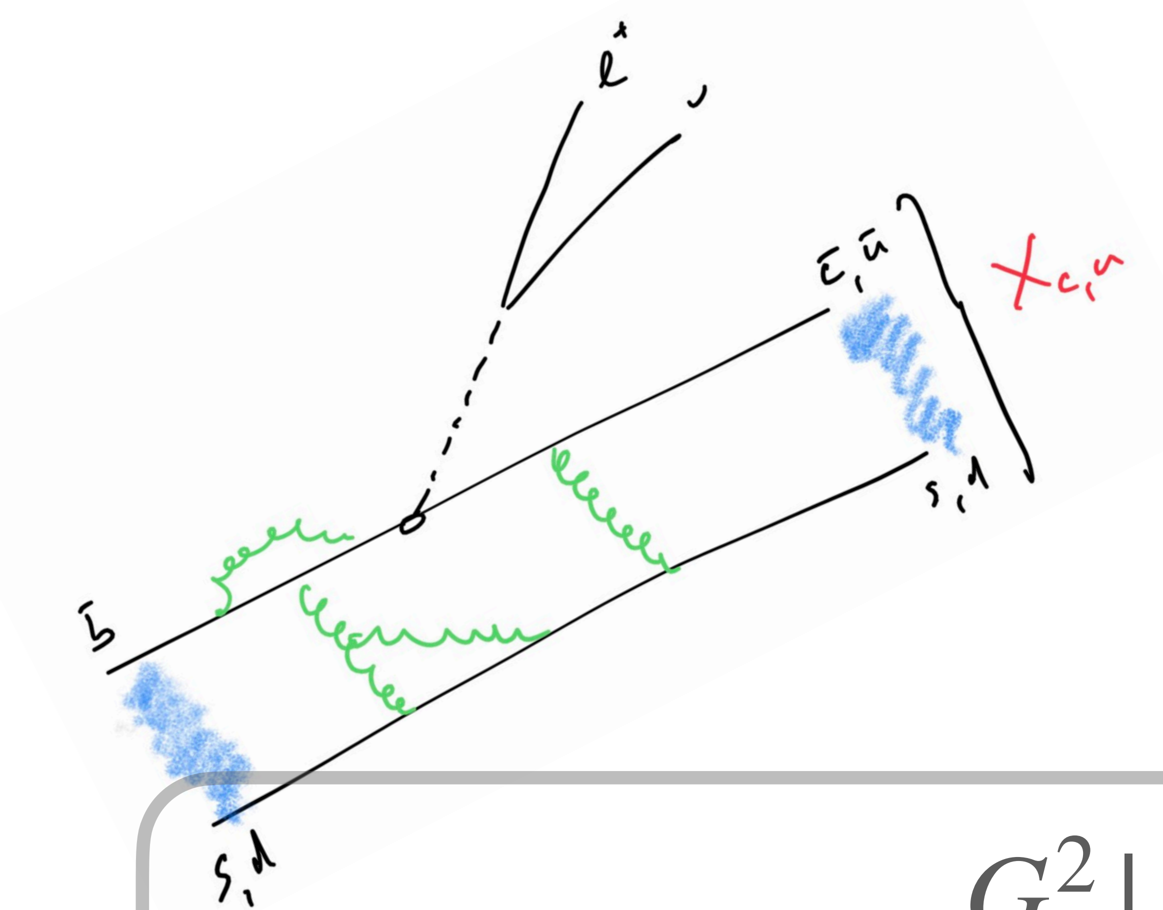
# first: exclusive decay




exclusive decays in lattice QCD well understood
see [FLAG 21 2111.09849]

to date no single complete lattice result for inclusive





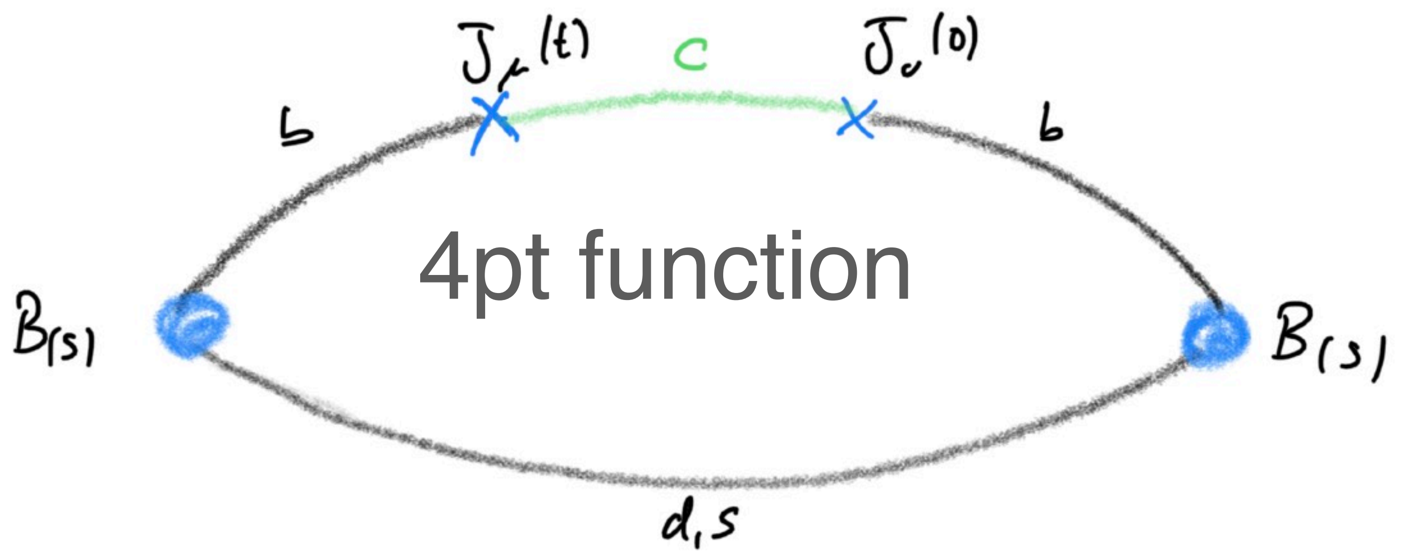
# Inclusive SL decay

$$\boxed{\Gamma(B_s \to X_c l\nu) = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\mathbf{q}^2_{\max}} d\mathbf{q}^2 \sqrt{\mathbf{q}^2} \int_{\omega_0}^{\infty} d\omega\, \boxed{W_{\mu\nu}(\mathbf{q},\omega)} K^{\mu\nu}(\mathbf{q},\omega)}$$

$\omega$ is energy of intermediate state $X_c$
$K^{\mu\nu}$ is known kinematic function

$$\boxed{W_{\mu\nu}} = (2\pi)^3 \delta^{(4)}(p-q-r) \frac{1}{2E_B} \sum_{X_c} \langle B_s(\mathbf{p}) | J_\mu^\dagger(-\mathbf{q}) | X_c(\mathbf{r}) \rangle \langle X_c(\mathbf{r}) | J_\nu(\mathbf{q}) | B_s(\mathbf{p}) \rangle$$

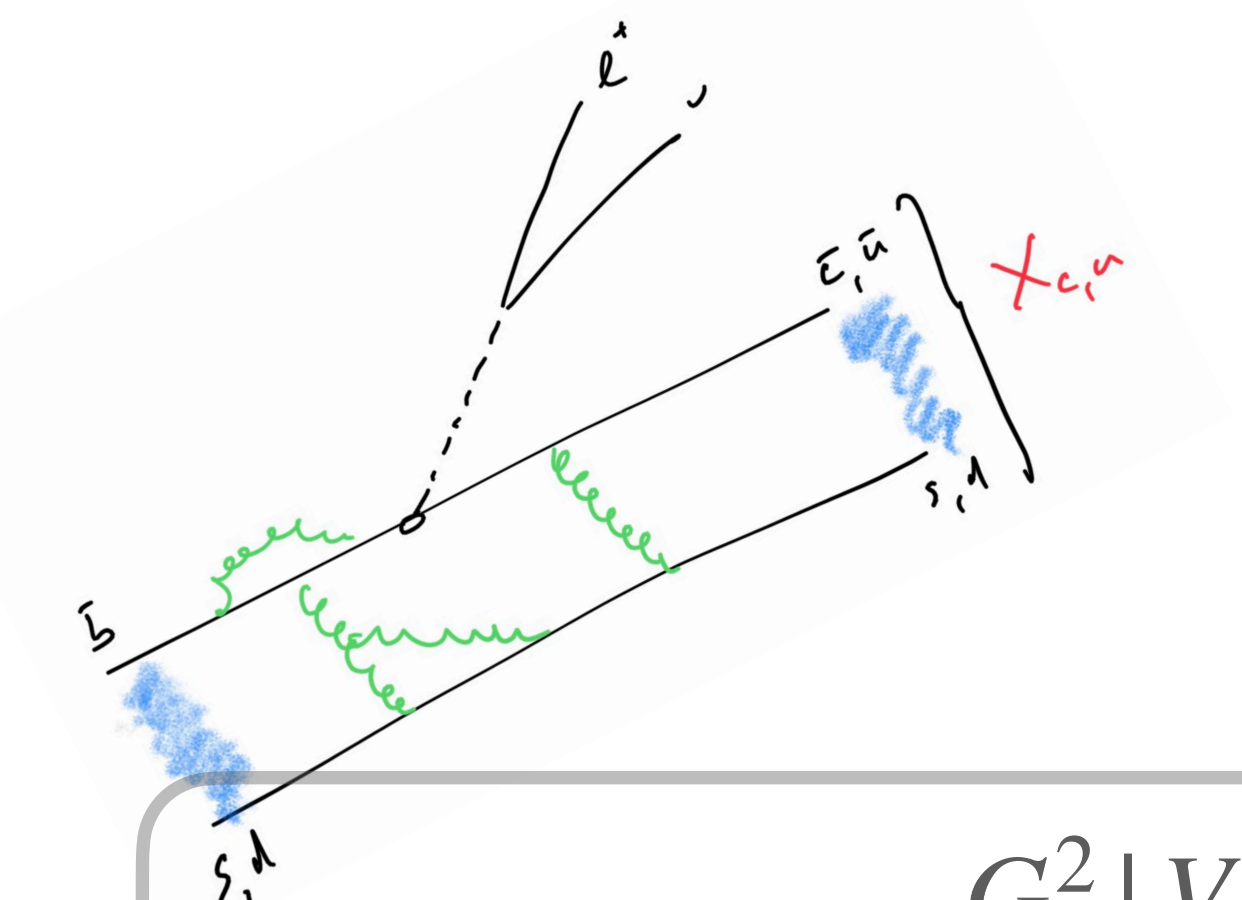
- on the Euclidean lattice it would be impossible to reconstruct all intermediate states by hand there is however an interesting closely related Euclidean 4pt correlation function:




$$C_{\mu\nu}(t) = \langle O_{B_s} J_\mu^\dagger(t) J_\nu(0) O_{B_s} \rangle \to \langle B_s | (J_\mu(0))^\dagger e^{-t\hat{H}} J_\nu(0) | B_s \rangle$$

$$C_{\mu\nu}(t) = \int_0^\infty d\omega\, W_{\mu\nu}(\mathbf{q},\omega)\, e^{-\omega t}$$

Inverse problem: Reconstruct spectral function $W_{\mu\nu}$ from lattice data for $C_{\mu\nu}$



# Inclusive SL decay

Barata, Fredenhagen, CMP 1991
Backus, Gilbert, GJRAS 1968
Hashimoto, PTEP 2017
Gambino, Hashimoto, PRL 2020
Bailas PTEP 2020
Barone et al., 2022
Kellermann et al., 2022

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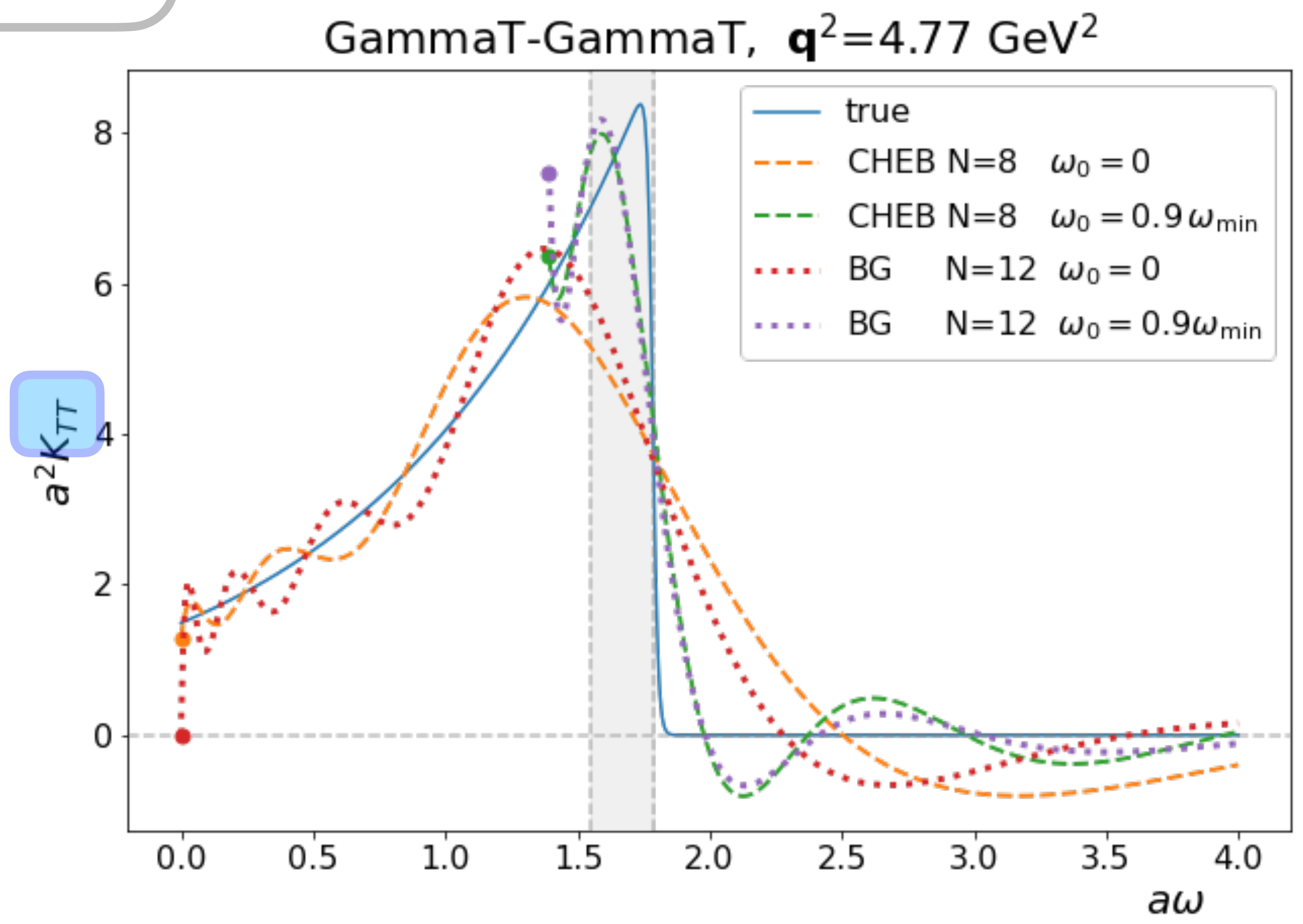
$\omega$ is energy of intermediate state $X_c$
$K^{\mu\nu}$ is known kinematic function

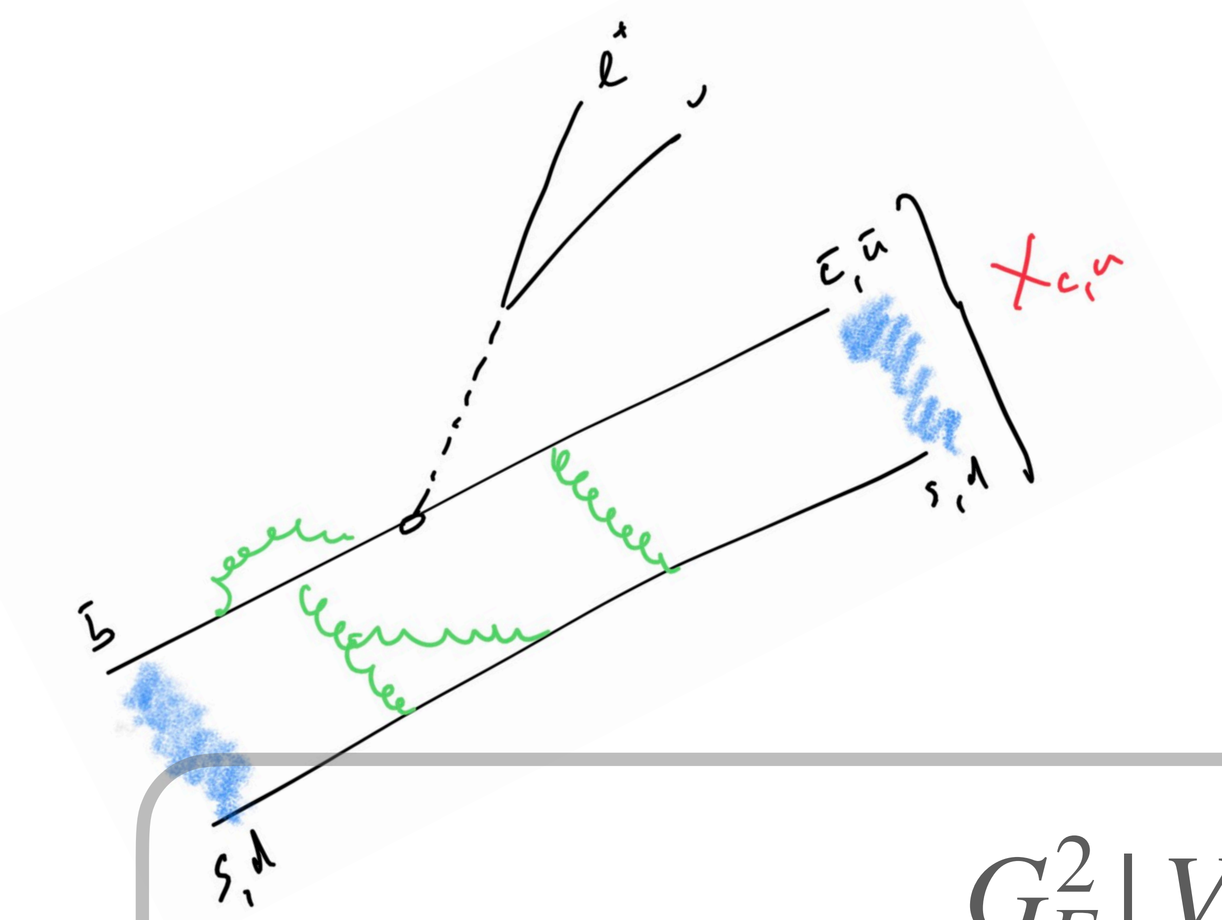
1. analytically compute expansion of kernel

$$K^{\mu\nu} \approx c_{\mu\nu,0} + c_{\mu\nu,1} e^{-\omega} + \ldots + c_{\mu\nu,N} e^{-\omega N}$$

2. combine with linear combinations of lattice-computed

$$C_{\mu\nu}(t) = \int_0^{\infty} d\omega\, W_{\mu\nu}(\mathbf{q},\omega)\, e^{-\omega t}$$





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1. analytically compute expansion of kernel

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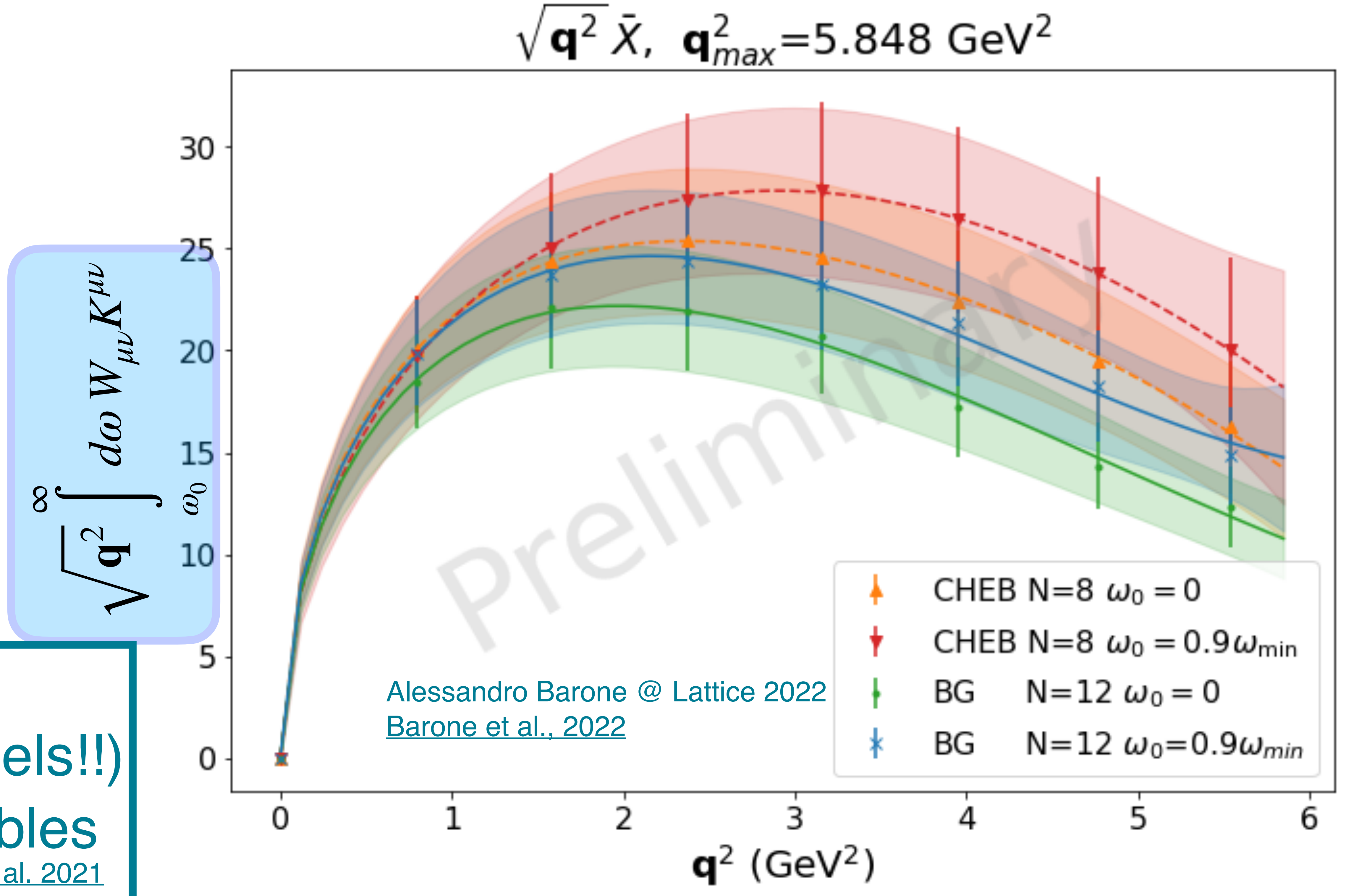
2. combine with linear combinations of lattice-computed

$$C_{\mu\nu}(t) = \int_0^{\infty} d\omega\, W_{\mu\nu}(\mathbf{q},\omega)\, e^{-\omega t} \to \boxed{\sqrt{\mathbf{q}^2} \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu} K^{\mu\nu}}$$




Early days but exploratory results give
inclusive $V_{cb}$ the right ball park — promising (also other channels!!)
Spectral reconstruction might open door to other new observables

Hansen et al. PRD 2019, Bulava et al. 2021

# Summary and outlook

- Lattice QCD crucial input to SM tests
- calculation of a number of quantities mature with good control of systematic effects and small errors
- clear path towards continued improvements, reduction of error for QCD predictions

- There are exciting new developments that will
  - allow to further increase precision (example QCD+QED)
  - extend the set of quantities accessible to lattice computations (example 2nd order Weak processes, spectral-function reconstruction — there are many more, see Lattice 2022)

I am very optimistic that further advances in theoretical physics, algorithms and computing will bring
- increased precision
- wider set of quantities for which lattice can make reliable predictions

This is extremely exciting given the prospect and time scale of FCC