Juan Carlos Vasquez, Amherst College

Resurgence and self-completion in renormalized Gauge Theories

(Work made in collaboration with A. Maiezza. To appear soon)

PASCOS 2023, University of California Irvine, June 2023

Outline

1. Motivation. Statement of the problem

2. Borel-Laplace and Borel-Ecalle resummation

3. Perturbation theory, renormalons and non-linear ODE from RGE

Outline

4. Resurgence of the RGE

5. New fixed points and a self-complete QED

6. Other phenomenological applications

Statement of the Problem

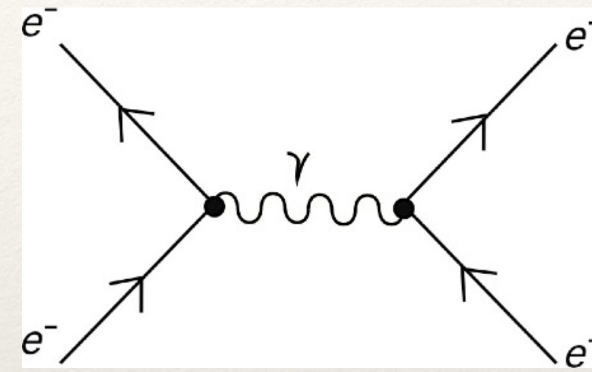
Feynman, 1950

“Feynman” the rules

$\longleftrightarrow$

$+$ renormalization theory

The amplitude



$$\mathscr{A} \propto \sum_n a_n \alpha^n$$

After renormalization: the amplitude is finite at every order in perturbation theory

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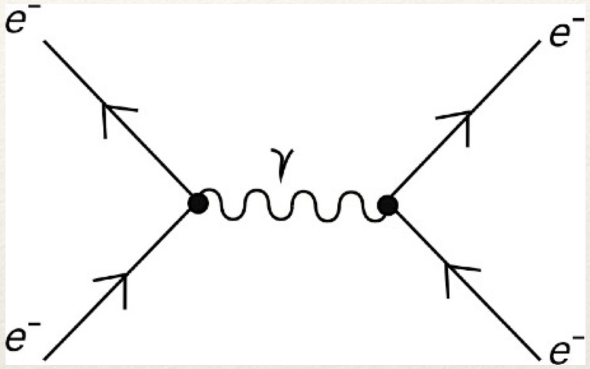

$$\mathscr{A} \propto \sum_{n=0}^{?} a_n \alpha^n$$

After renormalization: the amplitude is finite at every order in perturbation theory

TRUE AS LONG AS n in finite

Statement of the Problem

Feynman, 1950

Results obtained fromFeynman rules
Mathematically not well defined!

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$\longleftrightarrow$

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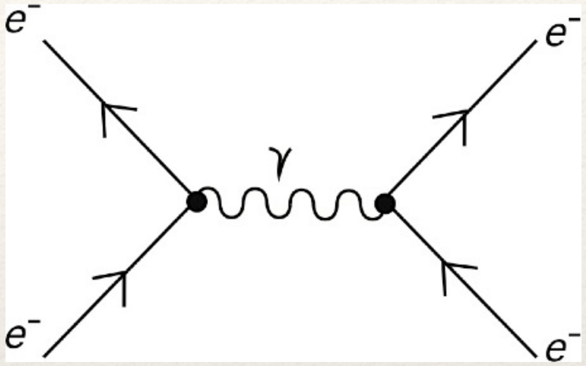

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What happens when $n \to \infty$

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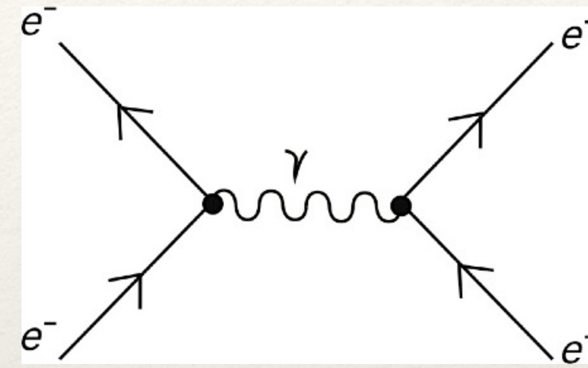

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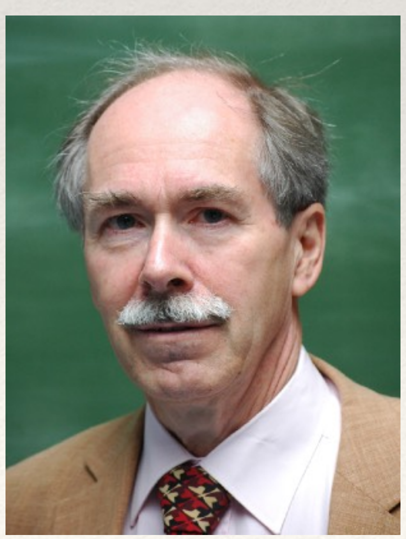
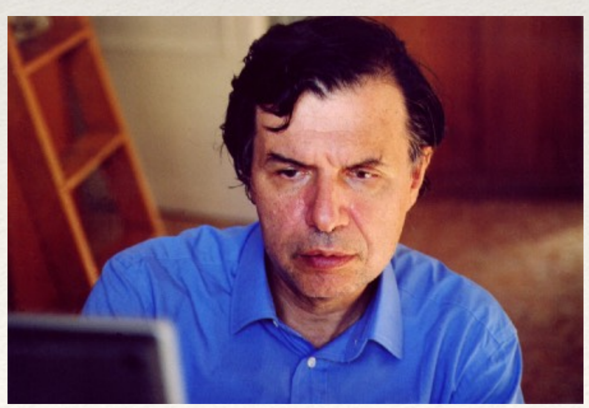
What happens when $n \to \infty$

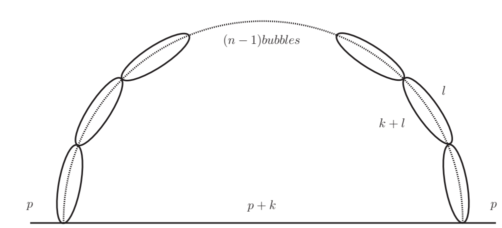
't Hooft, 1979



Parisi, 1978-1980



Asymptotic series! $a_n \propto n!$



$\lambda\phi(x)^4$

$\propto n!\lambda^{n+1}$

Figure: 't Hooft's skeleton diagram.

“renormalons” ['t Hooft '79]

Summation (or resummation)

1. Start from

$$f = \sum_{k=0}^{\infty} a_k x^{k+1}, \quad a_k \propto k!$$

Its Borel transform is ($B(x^{k+1}) = t^k/k!$)

$$\hat{f} = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!},$$

If $\hat{f}$ converges, the Borel sum of f is given by

$$s_\theta(f(x)) = L \circ B(f(x)) = \int_0^{\infty e^{i\theta}} \hat{f}(t)\, e^{-t/x} dt$$

($\theta = 0$, standard Laplace)

1) If $\hat{f}$ has do not have poles in the positive real axis f is Borel sumable

$$\begin{tikzcd}
\boxed{\text{Convergent series}} \arrow[r] \arrow[d] & \boxed{\text{Summation}} \arrow[r] & \boxed{\text{Analytic functions}} \arrow[d] \\
\boxed{\text{Closure under “all” operations}} \arrow[d] & & \boxed{\text{Closure under “all” operations}} \arrow[d] \\
\boxed{\text{Transseries}} \arrow[r] & \boxed{\text{E-B Summation}} \arrow[r] & \boxed{\text{Analyzable functions}}
\end{tikzcd}$$

This is the only known way to close functions under the listed operations.

(i) Algebraic operations: addition, multiplication and their inverses.
(ii) Differentiation and integration.
(iii) Composition and functional inversion.

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

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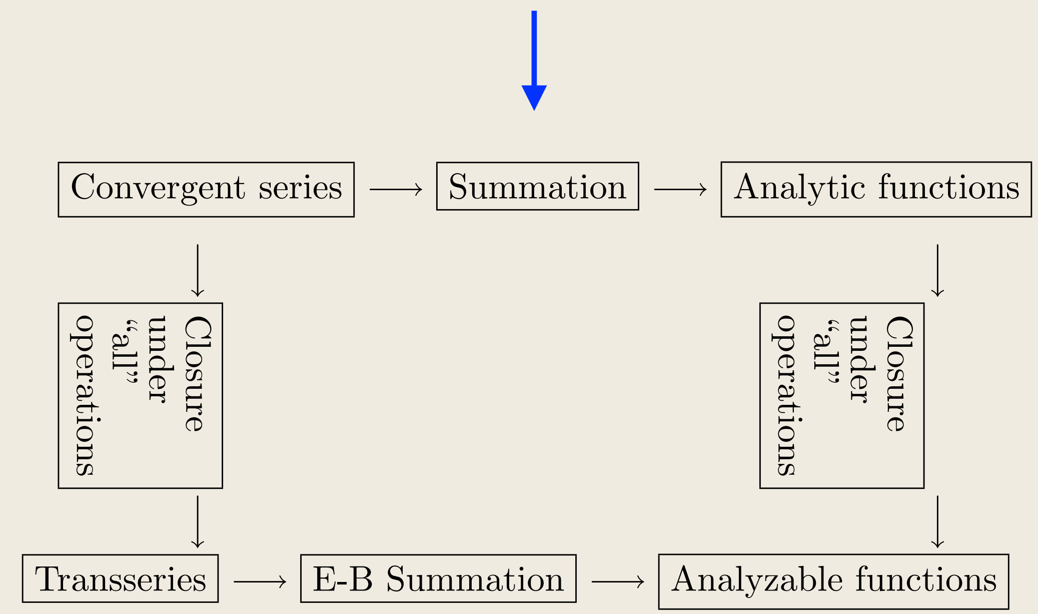
This is the well-known Borel summation



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How do we assign a unique function to the series for f?

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Borel-Ecalle summation

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The Renormalized Green Function in perturbation theory

Consider the renormalized two-point Green function in perturbation theory

$$G^{(2)}_{\mu\nu} = \left(D^{\mu\nu}_{tree}(q^2)\Pi(q^2)\right)^{-1} . \qquad\qquad L = \ln(\mu^2/\mu_0^2)$$

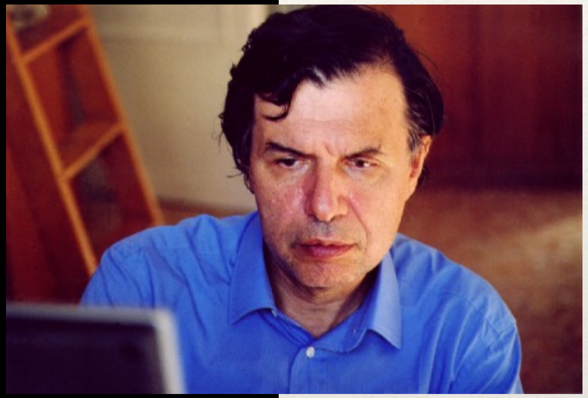
$$\Pi(L) = 1 - \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k .$$

And it satisfies

$$\left[-2\partial_L + \beta(\alpha)\partial_\alpha - 2\gamma(\alpha)\right] \Pi(L) = 0 .$$

The Renormalized Green Function

Parisi, 1978-1980



The description provided by

$$\Pi(L) = 1 - \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k$$

makes sense as long as

$$\mu^2 < \Lambda^2$$

We did a new derivation that do not use the one-loop running but the absolute convergence of the series

Going Beyond Perturbation Theory:
Renormalons, non-linear ODE and Resurgence

Non-linear ODE

Can we extend the validity of the description?

$$G^{(2)}_{\mu\nu} = \left(D^{\mu\nu}_{tree}(q^2)\Pi(q^2)\right)^{-1} . \qquad L = \ln(\mu^2/\mu_0^2)$$

Minimal modification: add an unknown function $R(\alpha)$

$$\Pi(L) = 1 + R(\alpha) - \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k$$

This way the previously derived equations changes and the previous conclusion does not hold!

How do the the new equations look like?
The first equations is now

$$2(R(x)+1)\gamma(x) = 2\gamma_1(x) - \frac{s\beta_1}{2} x^2 \beta(x) R'(x)$$

Non-linear ODE

- Using the results of Refereces

  - A. Maiezza and J. C. Vasquez, *Non-local Lagrangians from Renormalons and Analyzable Functions*, *Annals Phys*. 407 (2019) 78–91, [1902.05847].
  - J. Bersini, A. Maiezza and J. C. Vasquez, *Resurgence of the Renormalization Group Equation*, *Annals Phys*. 415 (2020) 168126, [1910.14507].

$$\frac{dR(\alpha)}{d\alpha} = \frac{2q}{\beta_1\alpha^2}R(\alpha) + \frac{\beta_1(a_0 q + a + s) - \beta_2 q}{\beta_1^2}\frac{R(\alpha)}{\alpha} + a_0\left(\frac{a}{\beta_1} - 1\right) + \mathcal{O}(R(\alpha)^2)$$

$$\gamma(\alpha) = \gamma_1(\alpha) + q\,R(\alpha) + \frac{1}{2}(2s\alpha R(\alpha)) + \mathcal{O}(R^2\,|\,\alpha R),$$

$$\gamma_1(\alpha) = a\alpha + \mathcal{O}(\alpha^2)$$

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ODE in α

$$\frac{dR(\alpha)}{d\alpha} = -\frac{2q}{\beta_1\alpha^2}R(\alpha) + \frac{\beta_1(a_0 q + a + s) - \beta_2\, q}{\beta_1^2}\frac{R(\alpha)}{\alpha} + a_0\left(\frac{a}{\beta_1} - 1\right) + \mathcal{O}(R(\alpha)^2)$$

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Non-linear in $R(\alpha_s)$

$$\gamma_1(\alpha) = a\alpha + \mathcal{O}(\alpha^2)$$

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$-\xi$

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$q = 1$ since Green functions depend on Λ and

$$\Lambda^2 = \mu_0^2 e^{\frac{2}{\beta_1\alpha}}$$

Non linear ODE

The solution is given by

Large order

$$R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\ e^{\frac{n}{\beta_0 \alpha_s}}$$

PT gives $R_0(\alpha_s)$

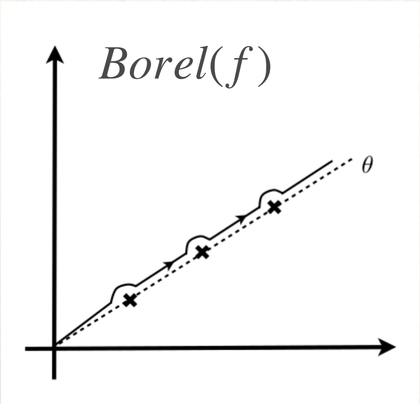
HOW DO WE FIND THE FUNCTIONS $R_n(\alpha_s)$ FOR $n > 0$?

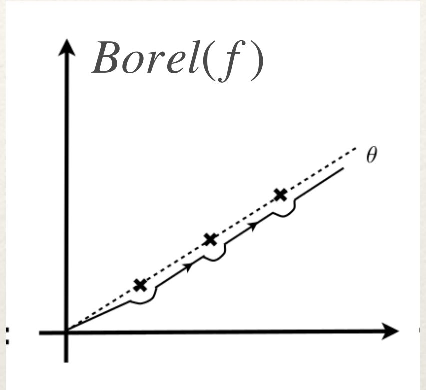
KEY CONCEPT OF "RESURGENCE"

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

A New Mathematical Framework for QFTs: Resurgence

1. Consider the transseries





$$f(x) = \sum_{n=0}^{\infty} f_n(x) e^{-n\lambda/x}$$

2. We are interested in the difference

$$(s_{\theta^-} - s_{\theta^+}) f(x) = \sum_n \left(s_{\theta^-} f_n - s_{\theta^+} f_n \right) \cdot e^{-n\lambda/x}$$

$$s_{\theta^-} = s_{\theta^+} \circ G_\theta = s_{\theta^+} \circ (1 + \mathrm{disc}_\theta)$$

Resurgence and the Alien Derivative

The Stokes Automorphism G_θ has the following structure

$$G_\theta = 1 + \delta_\theta = e^{\log G_\theta} := e^{\dot{\Delta}_\theta}, \text{ where } \quad \dot{\Delta}_\theta = \log G_\theta = \log(1 + \delta_\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n} \delta_\theta^n \,.$$

J. Écalle, Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture

$\dot{\Delta}_\theta$ is the Alien Derivative (it has all the properties of a derivative)

The following property holds

$\left[\dot{\Delta}_\theta, \partial_x\right] = 0$, $\partial_x = \partial/\partial x$ denotes standard derivative

J. Écalle, Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture

Bridge Equation

Consider again

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0 \alpha_s^2} R(\alpha_s) + \frac{\beta_0(a_0 q + a + s) - \beta_1 q}{\beta_0^2} \frac{R(\alpha_s)}{\alpha_s} + a_0 \left(\frac{a}{\beta_0} - 1 \right) + \mathcal{O}(R(\alpha_s)^2)$$

Apply the derivative with respect to the one parameter transseries ($\partial_C \equiv \partial / \partial_C$)

$$\frac{d\partial_C R(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0 \alpha_s^2} \partial_C R(\alpha_s) + \frac{\beta_0(a_0 q + a + s) - \beta_1 q}{\beta_0^2} \frac{\partial_C R(\alpha_s)}{\alpha_s} + \mathcal{O}(\partial_C R(\alpha_s)^2)$$

Compare with

$$\frac{d\dot{\Delta}_\theta R(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0 \alpha_s^2} \dot{\Delta}_\theta R(\alpha_s) + \frac{\beta_0(a_0 q + a + s) - \beta_1 q}{\beta_0^2} \frac{\dot{\Delta}_\theta R(\alpha_s)}{\alpha_s} + \mathcal{O}(\dot{\Delta}_\theta R(\alpha_s)^2)$$

Both $\dot{\Delta}_\theta R(\alpha_s)$ and $\partial_C R(\alpha_s)$

Satisfy the same ODE

then

$\dot{\Delta}_\theta R(\alpha_s) = A_\theta \, \partial_C R(\alpha_s)$ Ecalle Brigde Equation. A_θ Holomorphic constant

A_θ not calculable for renormalons, fit using data!

Resurgence

$\dot{\Delta}_\theta R(\alpha_s) = A_\theta \, \partial_C R(\alpha_s)$ Ecalle Brigde Equation

Plugging $R(\alpha_S) = \sum_{k=0}^{\infty} C^K R_k(\alpha_s)\; e^{\frac{k}{\beta_0 \alpha_s}}$ above and equaling the powers of $C^n \; e^{\frac{n}{\beta_0 \alpha_s}}$ in each side

$\dot{\Delta}_\theta R_n(\alpha_s) = (n+1)\, A_\theta \; e^{\frac{1}{\beta_0 \alpha_s}}\, R_{n+1}(\alpha_s),$ in particular $\dot{\Delta}_\theta R_0(\alpha_s) = \; A_\theta \; e^{\frac{1}{\beta_0 \alpha_s}}\, R_1(\alpha_s)$ and so on …

One can solve the recursion to find

$$R_n(x) = \frac{e^{nx}}{S_0^n} \left(\delta_0 R_0(x) - \sum_{j=1}^{n-1} S_0^j e^{-jx} R_j(x) \right) \quad , n \geq 1 \, .$$ which agree with the results of

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

Self Complete QED

How do we connect the formalism of Alien Calculus with practical calculations in QCD?

A new Kind of fixed point

For QED, one has

$$\beta(\alpha) = \alpha\gamma(\alpha)\,,$$

Which is associated with the non-linear ODE

$$R'(x) = -R(x) + \frac{R(x)(\beta_2 - \beta_1 h)}{\beta_1 x} + \frac{R(x)(\beta_2(\beta_1 h - \beta_2) + \beta_1 R(x) x)}{\beta_1^2 x^2} + \text{analytic terms}\,,$$

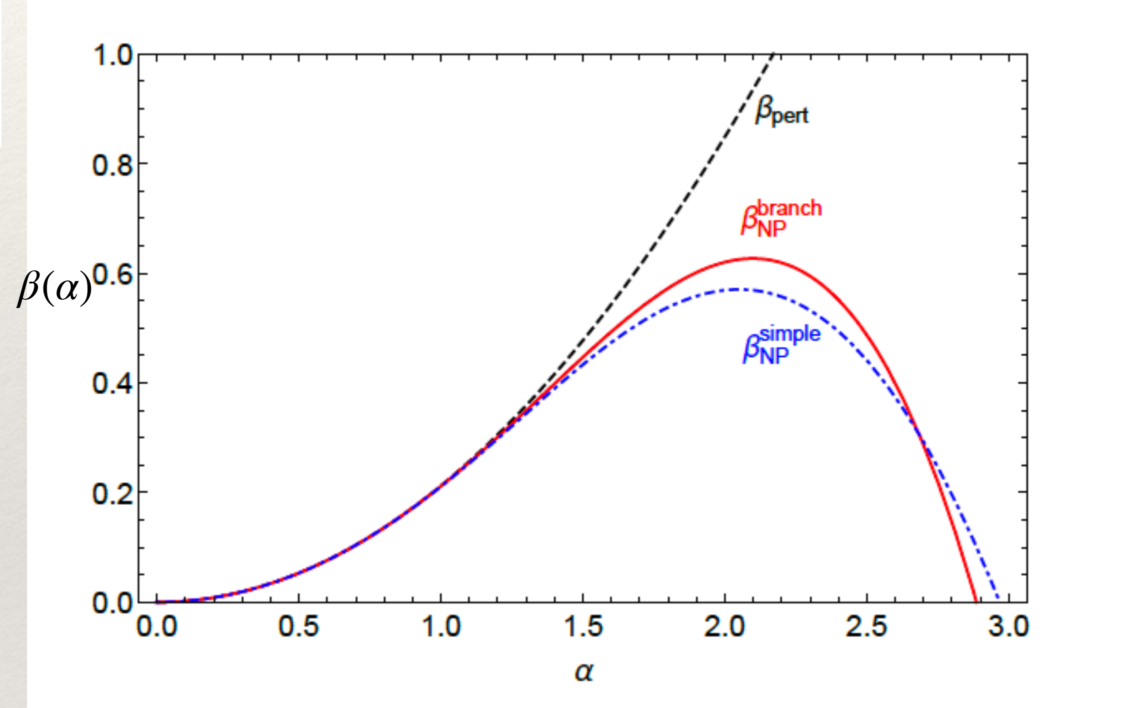
Self Complete QED

Using renormalon Feynman diagrams as an indicator of the analytic structure of the Borel transform

$$R(\alpha_S) = \sum_{k=0}^{\infty} C^K R_k(\alpha_s)\ e^{\frac{k}{\beta_0 \alpha_s}}$$

$$\mathcal{R}_0(z) = \sum_n (-1)^n (z - n\frac{2}{\beta_0})^{-1}$$

$$\beta(\alpha) = \beta_{pert}(\alpha) - \alpha \frac{2\pi C}{1 + e^{\frac{3\pi}{\alpha}}} ,$$



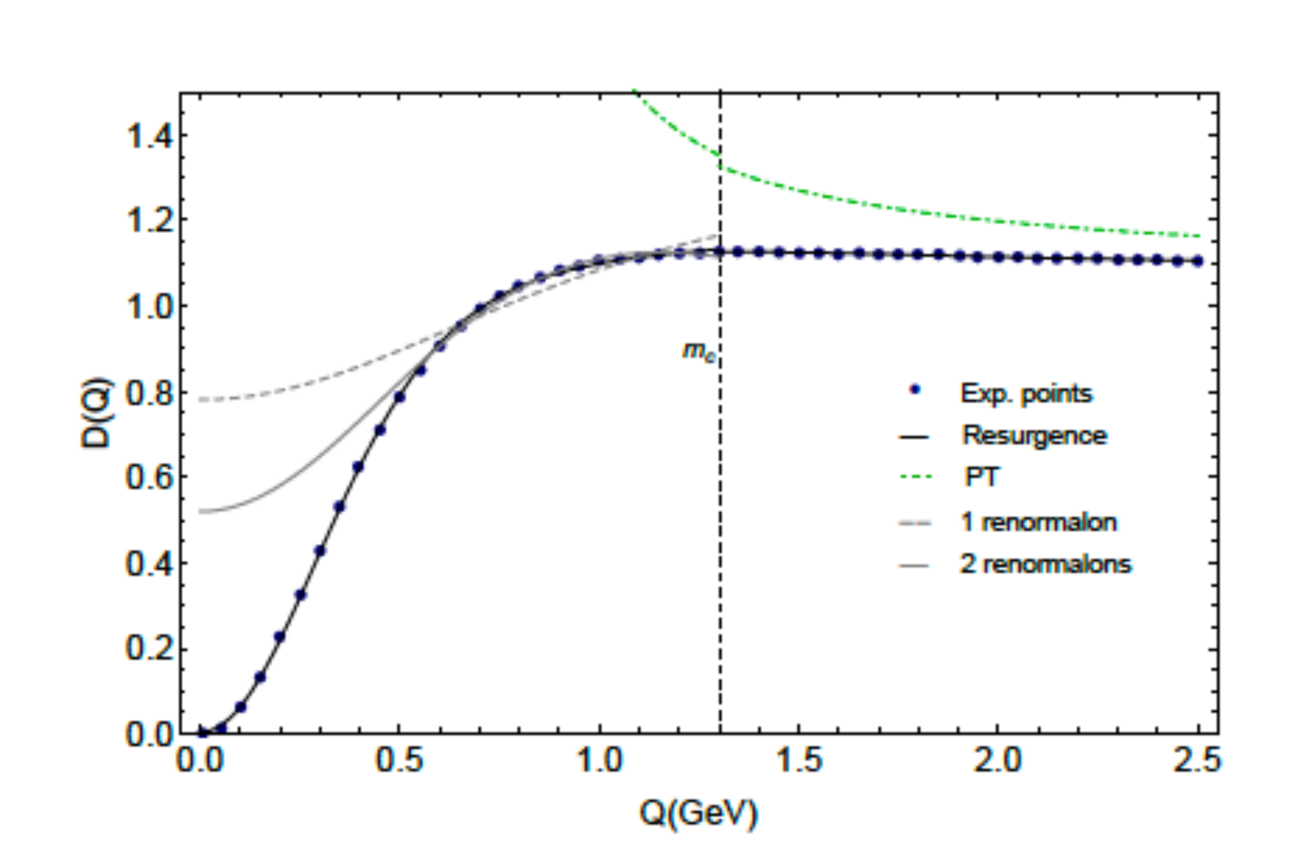
And in this case

$$R^{med}(\alpha) = R_0^{PV}(\alpha) - \frac{2\pi C}{1 + e^{\frac{3\pi}{\alpha}}} ,$$

$$R_0^{PV}(\alpha) = \text{P.V.} \left(\int_0^{\infty} dz e^{-\frac{3\pi z}{\alpha}} \mathcal{R}_0(z) \right) .$$

Applications

- Adler function



ArXiv: 2104.03095 and 2111.06792

$$-i\int d^4x\, \mathrm{e}^{-iqx}\left\langle 0\left|T\left(j_\mu(x)j_\nu(0)\right)\right|0\right\rangle = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi\left(Q\right)\ ,$$

$$D\left(Q\right) = 4\pi^2 Q^2 \frac{\mathrm{d}\Pi\left(Q\right)}{\mathrm{d}Q^2}\ ,$$

$$D_{pert}\left(Q\right) = 1 + \frac{\alpha_s}{\pi}\sum_{n=0}^{\infty}\alpha_s^n\left[d_n\left(-\beta_0\right)^n + \delta_n\right]\ .$$

$$D_{resurg.}(Q) = D_0(Q) - \frac{4\pi}{\beta_0}c_1 e^{\frac{2}{\beta_0\,\alpha_s(Q^2)}} + Ce^{\frac{1}{\beta_0\,\alpha_s(Q^2)}}\left(\frac{1}{\alpha_s(Q^2)}\right)^{a_p} D_1(Q^2)\,,$$

Applications

- Hadronic Width of the Tau Lepton (Caprini, ArXiv: 2304.03504)

The total hadronic branching fraction to the electron branching fraction of the τ lepton is expressed

$$R_\tau = \frac{\Gamma(\tau^- \to \nu_\tau + \text{ hadrons})}{\Gamma(\tau^- \to \nu_\tau e^- \bar{\nu}_e)}$$

$$R_\tau = 3\, S_{\rm EW}(|V_{ud}|^2 + |V_{us}|^2)(1 + \delta^{(0)} + \delta'_{\rm EW} + \delta^D_m),$$

$$\delta^{(0)} = \frac{1}{2\pi i} \oint\limits_{|s|=m_\tau^2} \frac{ds}{s} \left(1 - \frac{s}{m_\tau^2}\right)^3 \left(1 + \frac{s}{m_\tau^2}\right) [D(s) - 1].$$

$$\delta^{(0)}_{resurg} = 0.2629 + 0.0091c.$$

$$\delta^{(0)}_{\rm phen} = 0.1966 \pm 0.0040,$$

$$c = -7.26 \pm 0.44.$$

We first note that the contribution of the perturbative part in (26) to the integral (30) is equal to 0.2629, in perfect agreement with the result given in Fig. 3 of [42].

Conclusions

- Independently of the instatons PT is divergent in some limit
- The latter is related with the Landau pole that we rephrase in a resurgent context
- We show a resurgent non-perturbative completion can be achieved by adding the non-perturbative function $R(\alpha)$
- $R(\alpha)$ satisfy a specific non-linear ODE and must be identified with the Borel-Ecalle resumation of Renormalons

Conclusions

- Resurgence provides a framework that at least allows one to extract from the experiments nonperturbative information from first principles.
- We have shed light on the connections between the theory of the generalized Borel resummation based on the no-linear, ordinary differential equations and the Alien Calculus.
- We have shown the existence of new kind of fixed points that can make QED well-defined in the UV
- There are recent applications to QCD phenomenology and many more applications in the future!

Thank you

Backup Slides

One of the Big Questions

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THE BIG QUESTIONS IN
ELEMENTARY PARTICLE PHYSICS

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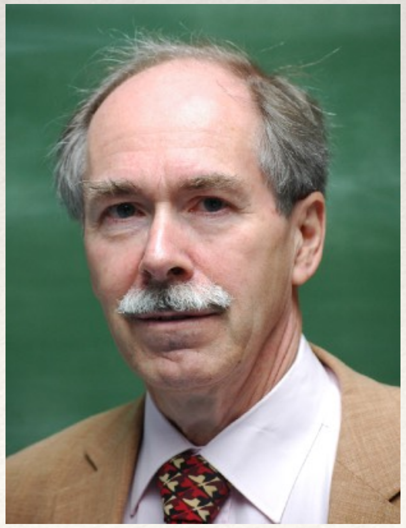

The question *How do we sum the perturbation terms, or is there another way to obtain the exact equations for all interactions?* is correctly posed but it seems to be not so urgent. We can arrange the diagrams in such a way that diagrams calculated using perturbation theory determine with a satisfactory accuracy how the elementary particles will interact under practically all circumstances, as if we *nearly have the 'ultimate theory' at our fingertips.*

But this is not true for many reasons. First, the perturbation expansions are still formally divergent, so that we still do not quite understand what the equations are at the most fundamental level. Secondly, there is one force that can only be taken into account at the most rudimentary level: gravity. The gravitational force cannot be included in an optimal way; we return to this shortly. The third reason for concern is that there appear to be phenomena at a very large distance scale in the universe: dark matter and dark energy. These require extensions of what we know: new particles or new theories or both.

The Renormalized Green Function in perturbation theory

Consider the renormalized two-point Green function in perturbation theory

$$G^{(2)}_{\mu\nu} = \left(D^{\mu\nu}_{tree}(q^2)\Pi(q^2)\right)^{-1} . \qquad L = \ln(\mu^2/\mu_0^2)$$

$$\Pi(L) = 1 - \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k .$$

And it satisfies

$$\left[-2\partial_L + \beta(\alpha)\partial_\alpha - 2\gamma(\alpha)\right] \Pi(L) = 0 .$$

Expanding in powers of L one gets the sets of equation

$$\gamma(\alpha) = \gamma_1(\alpha)$$

$$2\gamma(\alpha)\gamma_k(\alpha) + 2(k+1)\gamma_{k+1}(\alpha) = \beta(\alpha)\gamma_k'(\alpha)$$

. Klaczynski and D. Kreimer ArXiv: [1309.5061

Non linear ODE

$$\frac{dR(\alpha)}{d\alpha} = \frac{2q}{\beta_1 \alpha^2} R(\alpha) + \frac{\beta_1(a_0 q + a + s) - \beta_2\, q}{\beta_1^2} \frac{R(\alpha)}{\alpha} + a_0 \left(\frac{a}{\beta_1} - 1 \right) + \mathcal{O}(R(\alpha)^2) \qquad \Pi(L) = 1 + R(\alpha) - \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k$$

- The solution to the above non-linear equation is

$$R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\, \alpha_s^{k\xi}\, e^{\frac{n}{\beta_0 \alpha_s}} \quad \text{(one parameter transseries)}$$

- The Borel transform of the solution is of the form

$$B(R(g)) \propto \sum_n \frac{1}{\left(z - \frac{n\,q}{\beta_0}\right)^{1+\xi}} \simeq \sum_n \frac{1}{\left(z - \frac{n\,q}{\beta_0}\right)^{2+\mathcal{O}(\beta_1)}}$$

from the bubble-diagrams expression then $q = 1$ and s is such that we get quadratic poles

- The above non-linear differential equation is precisely of the kind studied in

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

The Renormalized Green Function

Absolute convergence for $\Pi(L) = 1 - \sum_{k=1}^{\infty} \gamma_k(\alpha) L^k$ when

$$\left| \frac{\gamma_{k+1}(\alpha) L^{k+1}}{\gamma_k(\alpha) L^k} \right| < 1 \implies -\frac{\gamma_k}{\gamma_{k+1}} < L < \frac{\gamma_k}{\gamma_{k+1}}$$

$$L = \ln(\mu^2/\mu_0^2) \qquad \beta(\alpha) = \beta_1 \alpha^2 + \mathcal{O}(\alpha^3)$$

On the other hand, using the above set of equations one can show

$$\lim_{k\to\infty} \frac{\gamma_k}{\gamma_{k+1}} = \frac{2}{\alpha\beta_1} \implies -\frac{2}{\alpha\beta_1} < \ln(\mu^2/\mu_0^2) < \frac{2}{\alpha\beta_1}$$

or

$\mu^2 < \Lambda^2$ with $\Lambda^2 = \mu_0^2 e^{\frac{2}{\beta_1 \alpha}}$

Non-linear ODE

- It can be shown that

$$\frac{dR(\alpha)}{d\alpha} = \frac{2q}{\beta_1\alpha^2}R(\alpha) + \frac{\beta_1(a_0q + a + s) - \beta_2\, q}{\beta_1^2}\frac{R(\alpha)}{\alpha} + a_0\left(\frac{a}{\beta_1} - 1\right) + \mathcal{O}(R(\alpha)^2)$$

$$\gamma(\alpha) = \gamma_1(\alpha) + q\,R(\alpha) + \frac{1}{2}(2s\alpha R(\alpha)) + \mathcal{O}(R^2\,|\,\alpha R),$$

$$\gamma_1(\alpha) = a\alpha + \mathcal{O}(\alpha^2)$$

$$\gamma_0(\alpha) := 1 + a_0\alpha + \mathcal{O}(\alpha^2)$$

- A. Maiezza and J. C. Vasquez, *Non-local Lagrangians from Renormalons and Analyzable Functions*, *Annals Phys*. 407 (2019) 78–91, [1902.05847].
- J. Bersini, A. Maiezza and J. C. Vasquez, *Resurgence of the Renormalization Group Equation*, *Annals Phys*. 415 (2020) 168126, [1910.14507].

The Renormalized Green Function beyond perturbation theory

Using the Wilson OPE, we know that at the non-perturbative level the green function is of the form

$$G^{(2)}_{\mu\nu} = \left(D^{\mu\nu}_{tree}(q^2)\Pi(q^2)\right)^{-1} . \qquad L = \ln(\mu^2/\mu_0^2)$$

$$\Pi(L) = \sum_{k=0} \times \left[e^{-\frac{1}{(-\beta_0)\alpha(\mu_0)}}\right]^k f_K(\alpha)$$

$$= \sum_{k=0}^{\infty} C_k\left(\alpha(\mu), \ln\frac{\mu^2}{\mu_0^2}\right) \times \frac{1}{\left(\Lambda^2\right)^k} \times \langle \mathcal{O}_k \rangle \qquad \Lambda^2 = \mu_0^2 e^{\frac{2}{\beta_1 \alpha}}$$

In the OPE one has infinite number of C_k

Can resurgence help on this issue? YES!

Bridge Equation

WE CAN FIT A_θ FROM DATA

DIFFICULT TO CALCULATE FOR INSTANTONS

SEE DORIGONI (ARXIV:1411.3585), ANICETO ET. AL.

(ARXIV:1802.10441) REVIEWS.

AND IMPOSIBLE FOR RENORMALONS

T'HOOFT (1979), ZINN-JUSTIN

MAIEZZA, VASQUEZ (2016-PRESENT)

Medianization

One can solve the recursion

$$\dot{\Delta}_\theta R_n(\alpha_s) = (n+1)\, A_\theta\; e^{\frac{1}{\beta_0 \alpha_s}}\, R_{n+1}(\alpha_s)$$

to find

$$R_n(x) = \frac{e^{nx}}{S_0^n}\left(\delta_0 R_0(x) - \sum_{j=1}^{n-1} S_0^j e^{-jx} R_j(x)\right) \quad , n \geq 1\,.$$

which agree with the results of

Taking the Cauchy principal value prescription to deal with the Borel transform of the functions above does not automatically preserved the homorphism structure of the resummation procedure

Medianization

To solve the issue above, medianization is introduced and it is given by

$$S^{med} := S_{0-} \circ G_0^{1/2} = S_{0+} \circ G_0^{-1/2}\,,$$

Such that

$$R^{med}(x) = S^{med} R(x)\,,$$

And since $\left[\dot{\Delta}_\theta, \partial_x\right] = 0$, $R^{med}(x)$ is a solution of the ODE

$$\frac{dR_{med}(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0 \alpha_s^2} R_{med}(\alpha_s) + \frac{\beta_0(a_0 q + a + s) - \beta_1 q}{\beta_0^2} \frac{R_{med}(\alpha_s)}{\alpha_s} + a_0 \left(\frac{a}{\beta_0} - 1 \right) + \mathcal{O}(R(\alpha_s)^2)$$