Probing the Local Dark Matter Halo with Neutrino Oscillations

Andrey Shkerin

FTPI, University of Minnesota

with Tony Gherghetta

2305.06441

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Motivation

- **Dark matter exists.**

  Ultra-light scalar or vector bosons are well-motivated dark matter candidates.

  These include QCD axion and dark photon.

  Particle Data Group, Prog. Theor. Exp. Phys. 2022, 083C01 (2022)

  They can form a coherently oscillating background.

  astro-ph/0003365, 1105.2812, 1610.08297, 1907.06243

  Interaction of this background with Standard Model fields leads, e.g., to the variation of fundamental constants of Nature.

  ⇒ New possibilities for dark matter searches, e.g., in atomic clock experiments.

  1710.01833

- **Ultra-light dark matter particles can form dense compact objects.**

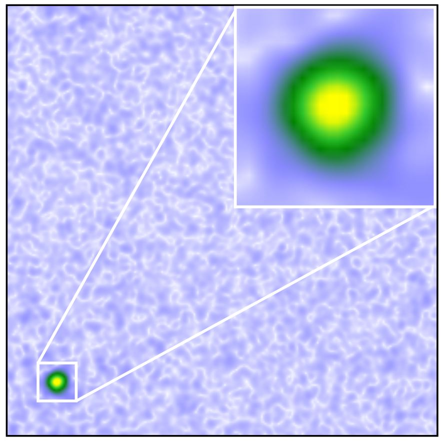
  hep-ph/9303313, 1406.6586, 1610.08297, 1804.05857, 1804.09647, 1809.07673, 1809.09241, 1906.01348, 2207.04057, 2304.13054

  They are called **Boson** / **Axion** / **Proca stars** and represent coherent / solitonic / classical lumps of dark matter.

  They are bound by **self-gravity** and can form by the Jeans mechanism.

  ⇒ They can be classically stable (if in isolation).

  Self-interaction and interaction with the surrounding medium are also important.



A boson star. Taken from 1804.05857.

Motivation

- A solitonic configuration may also be sustained by an **external potential.**

  Suppose that dark matter particles are trapped in a potential created by some astrophysical body. Then they can form a soliton ("**local halo**").

  The Earth or the Sun could have such a halo.

1902.08212, 1912.04295

2306.12477

  The halo can be sustained solely by the **gravitational potential** of the host body.

- Inside the compact object $\rho_{\mathrm{DM,local}} \gg \rho_{\mathrm{DM,average}}$

  $\Rightarrow$ Effects from coupling to the Standard Model fields are enhanced.

  This is intriguing: the presence of the local halo could greatly enhance the sensitivity of terrestrial experiments to new physics.

1902.08212, 1906.06193, 1912.04295, 2103.03783, 2201.02042, 2305.01785

Local scalar halo

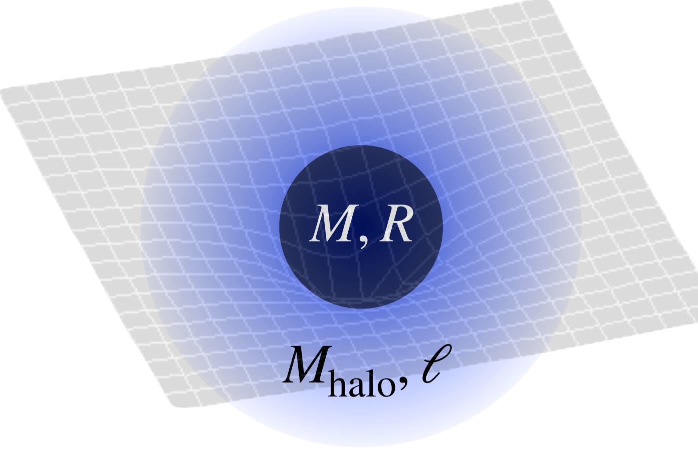
The energy of the bound state is $E_0 \ll mc^2$

- Consider a local, **nonrelativistic** halo, made of **scalar** particles of mass m, which is the bound state in the **gravitational potential** $\Phi(r)$ of the host body.

$$ds^2 = -N(r)c^2dt^2 + \frac{dr^2}{N(r)} + r^2d\Omega^2, \qquad N(r) = 1 + \frac{2\Phi(r)}{c^2}$$

We do not study the halo formation $\Rightarrow$ the **halo mass** $M_{\rm halo}$ is a free parameter (subject to experimental constraints and $M_{\rm halo} \ll M$).

0808.0899



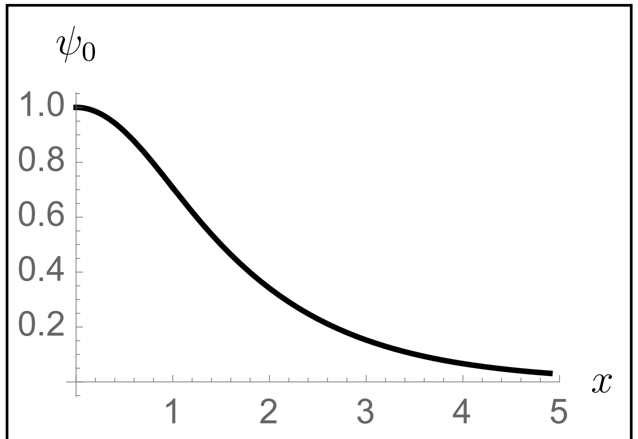
The **halo size** ℓ is fixed by m and the parameters of the host body: its mass M and radius R.

- Then finding the halo profile reduces to solving the Schroedinger equation:

$$\varphi(r,t) = \sqrt{\frac{2c}{m}}\left(\Psi(r,t)e^{-imc^2t} + c.c.\right), \quad \Psi(r,t) = \psi(r)e^{-iEt}$$

$$x = r/R\,, \quad \mathscr{M} = Gm^2MR\,, \quad \mathscr{E} = EmR^2$$

$$\boxed{-\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi}{dx}\right) + 2(\mathscr{M}\tilde{\Phi} - \mathscr{E})\psi = 0}$$



Local scalar halo at $\mathscr{M} = 1$.

where $\tilde{\Phi}$ is the rescaled grav. potential.

The halo is characterised by the parameter $\mathscr{M}$.
We are interested in the ground state $\psi_0(x)$.

Earth halo

- Take the Earth as the host body. From the properties of $\psi_0(x)$ it follows that

$$\ell \sim R_\oplus \left(\frac{10^{-9}\,\mathrm{eV}}{m}\right)^2 \qquad m \ll 10^{-9}\,\mathrm{eV}$$

If $\ell > R_\oplus$, the halo extends beyond the Earth's surface
$\Rightarrow$ can be probed in terrestrial and near-orbit experiments — "**big halo**"

$$\ell \sim R_\oplus \left(\frac{10^{-9}\,\mathrm{eV}}{m}\right)^{1/2} \qquad m \gg 10^{-9}\,\mathrm{eV}$$

If $\ell < R_\oplus$, the halo is in the Earth's interior

$\Rightarrow$ much harder to probe — "**small (interior) halo**"

Neutrino oscillations as a probe of the local halo

- Assume that the particles comprising the halo couple to neutrinos.
  Then one can look for the halo in the neutrino oscillation data.

  Observational consequences of possible interactions between ultra light dark matter and neutrinos have been extensively studied in various terrestrial, astrophysical and cosmological setups.

1608.01307, 1705.06740, 1705.09455, 1804.05117, 1803.01773, 1809.01111, 1908.02278, 2007.03590, 2107.10865, 2205.03749, 2212.05073, 2301.04152

  What can we add?

- If $\ell \gtrsim R_\oplus$ $\Rightarrow$ Enhanced homogeneous oscillating background $\Rightarrow$ Stronger constraints on the couplings

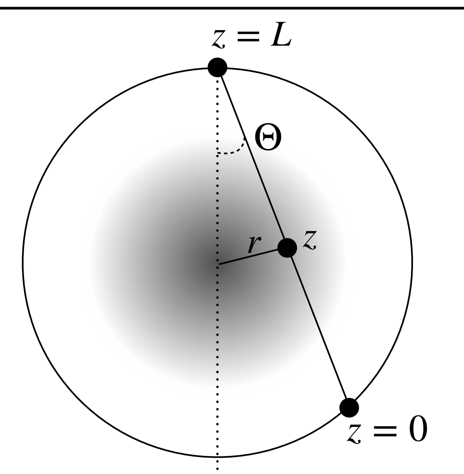
  In reactor / accelerator neutrino experiments

  $m \sim 10^{-10} - 10^{-9}$ eV, "**big halo**"

- If $\ell \ll R_\oplus$ $\Rightarrow$ Probe the small-size, interior halo inaccessible by other means; resolve its spatial profile

  With atmospheric neutrinos flying through the Earth

  $m \gg 10^{-9}$, "**small (interior) halo**"



The path of a neutrino propagating through a halo core located inside the Earth.

- This is at the cost of the hypothesis that the halo exists; the constraints are functions of $M_{\rm halo}$.

(Scalar) dark matter - neutrino interaction

- Adopt the plane-wave treatment of neutrino oscillations.

  Neglect effects of neutrino decoherence and dispersion.

  The evolution equation for the ultrarelativistic neutrino wavefunction:

$$\boxed{i\frac{d\nu_a}{dz} = H_{ab}\nu_b}$$

vacuum mass-squared differences

z-dependent eigenvalues

$$H = \frac{1}{2E}U_0\mathbf{diag}(0,\Delta m^2_{0,21},\ldots)U_0^\dagger + \Delta H = \frac{1}{2E}U\mathbf{diag}(0,\Delta m^2_{21},\ldots)U^\dagger$$

vacuum neutrino mixing matrix

diagonalises the full Hamiltonian

- The correction ΔH comes from the neutrino interacting with the background halo configuration

$$\varphi(r,t) = f(r)\cos(mt+\delta)\ , \quad f(r) \sim \psi_0(r)\ , \quad f_0 \equiv f(0)$$

the halo phase at the moment of neutrino production

(Scalar) dark matter - neutrino interaction

- Consider the following scalar-neutrino interaction terms:

complex, symmetric

$$\mathscr{L}_{4,\text{int}} = -\, y h_{ab} \varphi \bar{\psi}_{La} \psi^C_{Lb} + h.c.$$

small coupling

shifts the neutrino mass $\longrightarrow$

$$\Delta H_4 = \frac{y}{E}\varphi(h^\dagger m_\nu + m_\nu^\dagger h) + \frac{2y^2}{E}\varphi^2 h^\dagger h$$

where m_ν is the neutrino mass matrix in the flavour basis

See, e.g., R. Mohapatra, G. Senjanovic, Z. Phys. C 17, 53 (1983)

Hermitian

$$\mathscr{L}_{5,\text{int}} = -\, \frac{g_{ab}}{\Lambda_5}\, \partial_\mu \varphi\, \bar{\psi}_{La} \gamma^\mu \psi_{Lb}$$

large suppression scale

shifts the neutrino four-momentum $\longrightarrow$

$$\Delta H_5 = \frac{m}{\Lambda_5} g\varphi$$

since $|\partial_0 \varphi| \sim m f_0$ and $|\nabla \varphi| \sim f_0/\ell$,
and for our halo we have $m\ell \gtrsim 10^4$

See, e.g., 2107.14018

Big halo

Adiabatic regime

$$P_{ab}(L) = \left| \sum_i U_{ai}(0) e^{-\frac{i}{2E}\int_0^L dz\, m_i^2(z)} U^\star_{bi}(L) \right|^2 , \qquad \langle P_{aa}\rangle_\delta = \frac{1}{2\pi}\int_0^{2\pi} d\delta\, P_{aa}$$

— Survival probability

Perturbation theory

Introduce the following parameters: $\beta_4 = \dfrac{y\sum m_\nu}{2E}$, $\beta_5 = \dfrac{m}{2\Lambda_5}$, $\beta = \beta_4$ or β_5

$$\epsilon \equiv \frac{\beta f_0}{m} \sim \left(\frac{\beta}{10^{-22}}\right)\left(\frac{m}{10^{-10}\,\mathrm{eV}}\right)\left(\frac{M_{\mathrm{halo}}}{10^{15}\,\mathrm{kg}}\right)^{1/2}$$

— Expansion parameter, depends on the halo mass

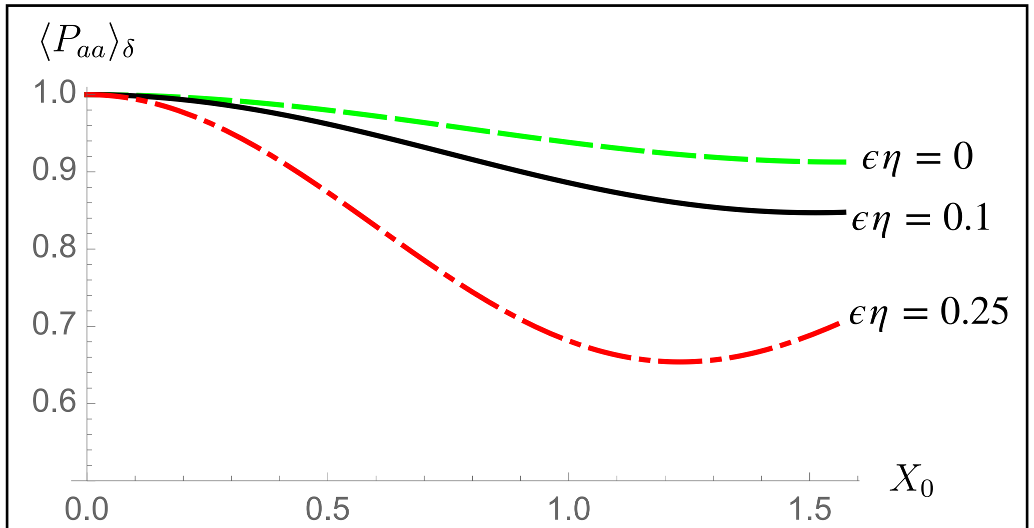
$$\eta \equiv \frac{mE}{\Delta m_0^2} = \left(\frac{2.5\times 10^{-3}\,\mathrm{eV}^2}{\Delta m_0^2}\right)\left(\frac{m}{10^{-10}\,\mathrm{eV}}\right)\left(\frac{E}{25\,\mathrm{MeV}}\right)$$

— Number of halo oscillations per one neutrino oscillation

Then $\langle P_{aa}\rangle_\delta = P_{0,aa} + (\epsilon\eta)^2 \langle P_{2,aa}\rangle_\delta$

- If $\eta \ll 1$, this is like the usual MSW effect; P.T. works until $\epsilon \sim \eta^{-1} \gg 1$.
- If $\eta \gg 1$, the neutrino propagates in the wildly oscillating background; P.T. works until $\epsilon \sim \eta^{-1}$?

Not really: **the adiabatic approximation breaks down** at $\epsilon \sim \eta^{-2} \ll \eta^{-1}$.



Here we take the derivative scalar-neutrino coupling, $\eta = 0.1$, $X_0 = \pi L / L_0^{\mathrm{osc}}$, $\sin^2 2\theta_0 = 0.087$, $g_{11} = 0.5$, $g_{12} = i$, $g_{22} = 0$.

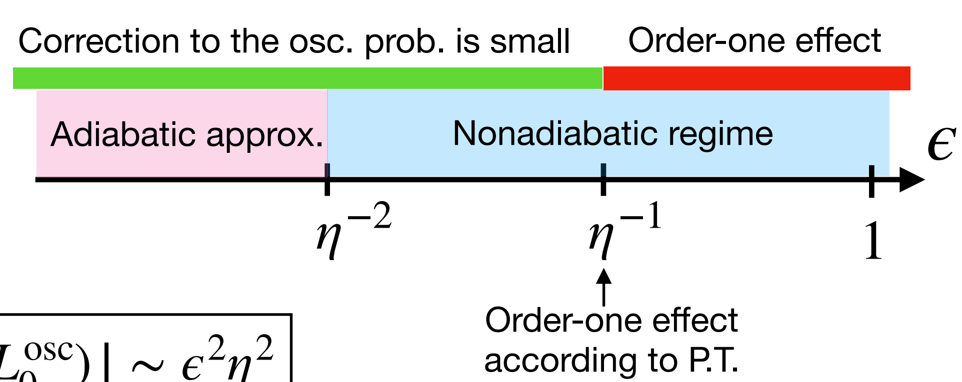
Nonadiabatic regime in big halo

- If $\eta \gg 1$, what is the scale at which the correction becomes sizeable?
  What is the shape of the modified oscillation probability curve?

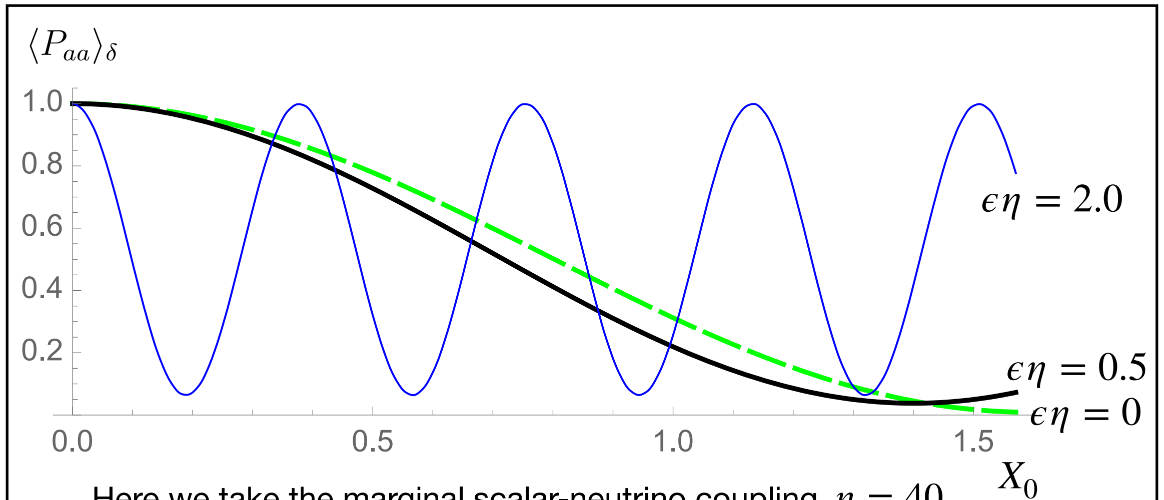
  The situation is different for the different types of scalar-neutrino interaction.

  Let $\nu_i(z) = (1 + \Delta\nu_i(z))e^{-i\frac{m_i^2 z}{2E}}$ be the vacuum neutrino mass eigenstates.

ΔH_4



$$\boxed{|\Delta\nu_i(L_0^{\rm osc})| \sim \epsilon^2\eta^2}$$



Here we take the marginal scalar-neutrino coupling, $\eta = 40$,
$X_0 = \pi L/L_0^{\rm osc}$, $\sin^2 2\theta_0 = 0.99$, $\Delta m_0^2 = 2.5\cdot 10^{-3}$ eV^2,
$\sum m_\nu = 0.1$ eV, $h_{11} = 0.5$, $h_{12} = i$, $h_{22} = 0$.

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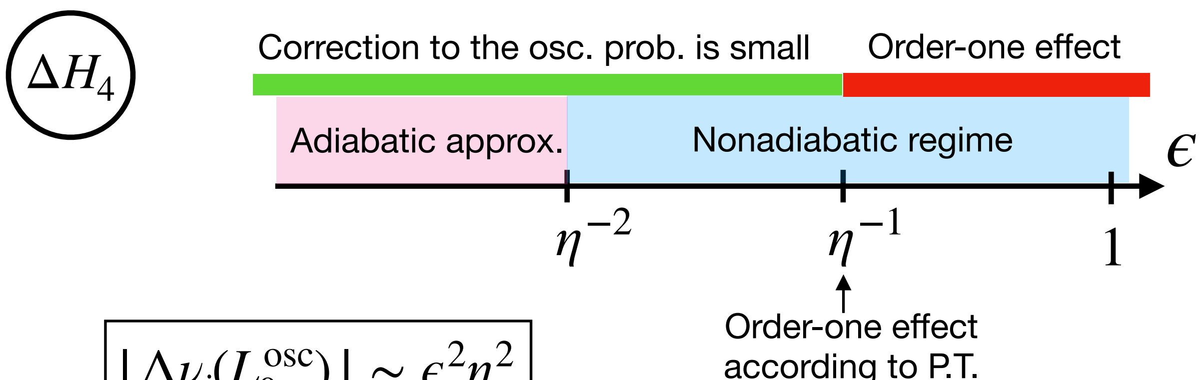
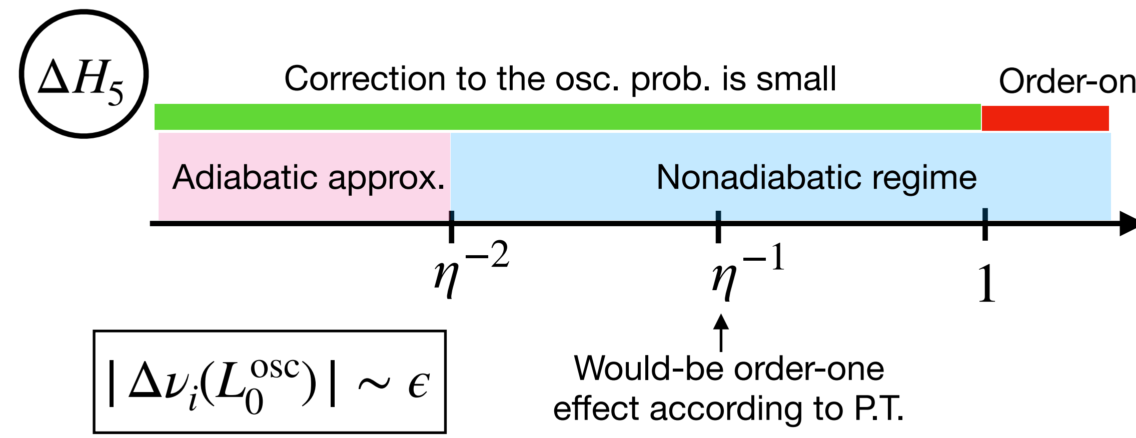
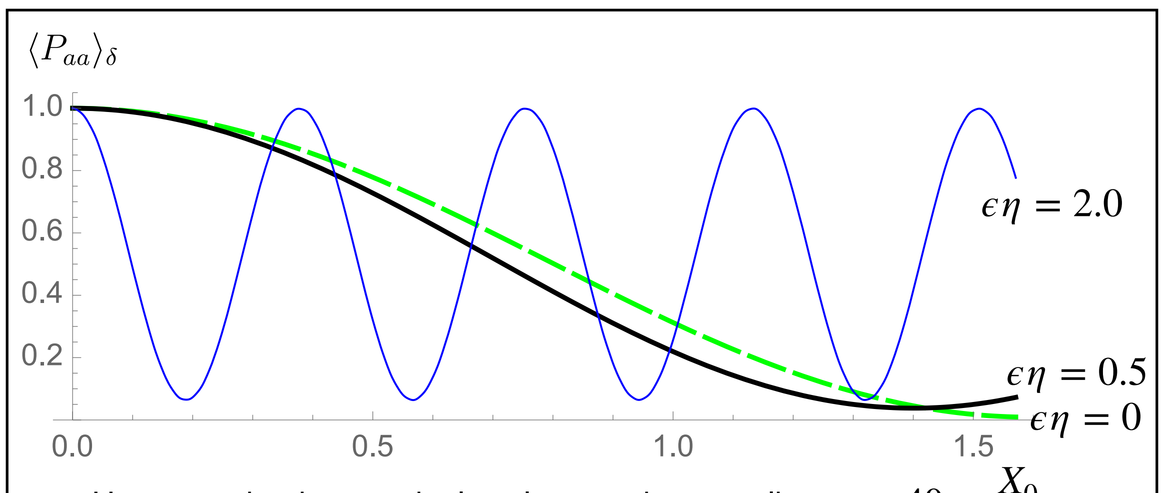


$$\boxed{|\Delta\nu_i(L_0^{\rm osc})| \sim \epsilon^2\eta^2}$$



$$\boxed{|\Delta\nu_i(L_0^{\rm osc})| \sim \epsilon}$$



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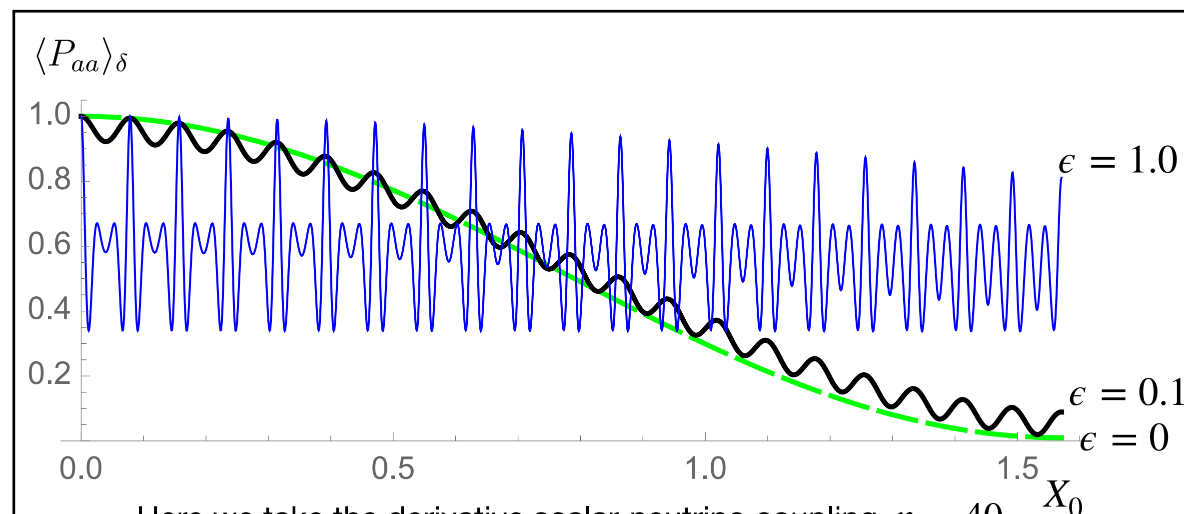
- Resonance effects can **suppress** the deviation from vacuum oscillations.



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Nonadiabatic regime in big halo

Summary

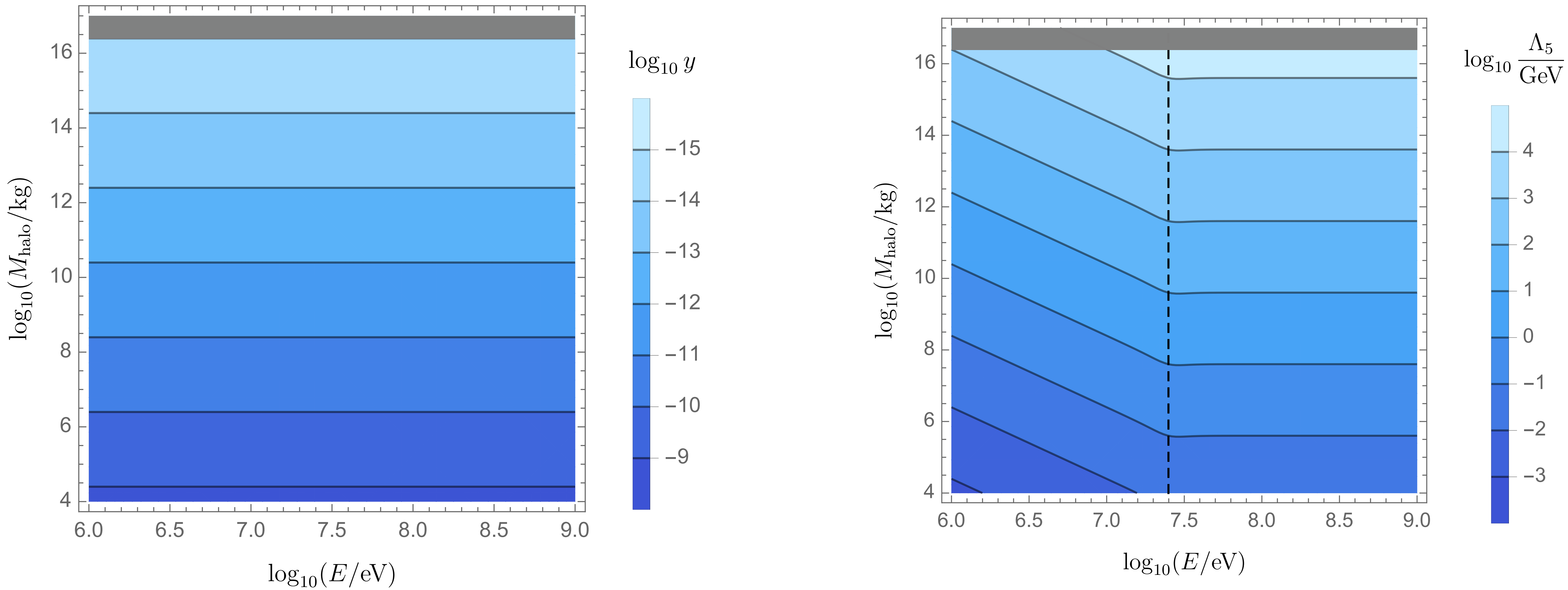
- The correction due to the halo can be small and at the same time be dominated by nonadiabatic effects.

- The correction gives rise to interesting features in the oscillation curve.

- The magnitude of the correction is essentially energy-independent.

Big scalar halo: results

- The above results can be rephrased in terms of the neutrino energy, the halo mass, and the scalar-neutrino coupling parameter, y or Λ_5.



Values of the parameters at which the relative deviation from the vacuum oscillation probability reaches 0.1.
The grey shaded region depicts the experimentally excluded values of $M_{\rm halo}$, the dashed line is $\eta = 1$.
We take $m = 10^{-10}$ eV and $\Delta m_0^2 = 3.5 \cdot 10^{-3}$ eV^2.

Probing the interior halo

- The previous analysis gives us the qualitative understanding of what happens in the case of small halo ($m \gtrsim 10^{-9}$ eV).

  We have in mind atmospheric neutrinos of $E \gtrsim 1$ GeV traversing the Earth.

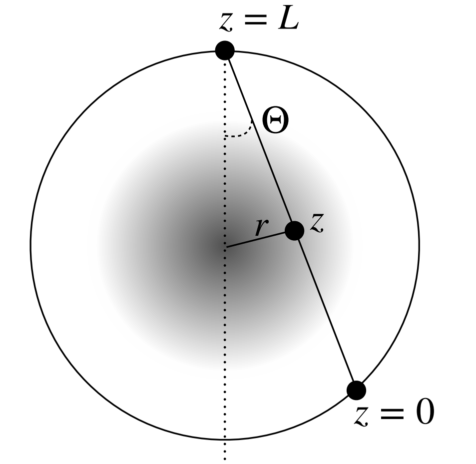
  The relevant parameters are

$$\epsilon \sim \left(\frac{\beta}{10^{-23}}\right)\left(\frac{10^{-9}\,\mathrm{eV}}{m}\right)^{5/4}\left(\frac{M_{\rm halo}}{10^{15}\,\mathrm{kg}}\right)^{1/2}$$

$$\eta = 400\left(\frac{2.5\times10^{-3}\,\mathrm{eV}^2}{\Delta m_0^2}\right)\left(\frac{m}{10^{-9}\,\mathrm{eV}}\right)\left(\frac{E}{1\,\mathrm{GeV}}\right)$$

  Recall that $\beta \sim f(0)$ — the amplitude of the halo at its centre.

  Clearly, $\eta \gg 1$.



The path of a neutrino propagating through a halo core located inside the Earth.

- Perturbation theory is straightforward, but is limited to $\epsilon \lesssim \eta^{-2} \ll 1$.

  $\Rightarrow$ Visible distortions in the oscillation probability arise due to the nonadiabatic effects.

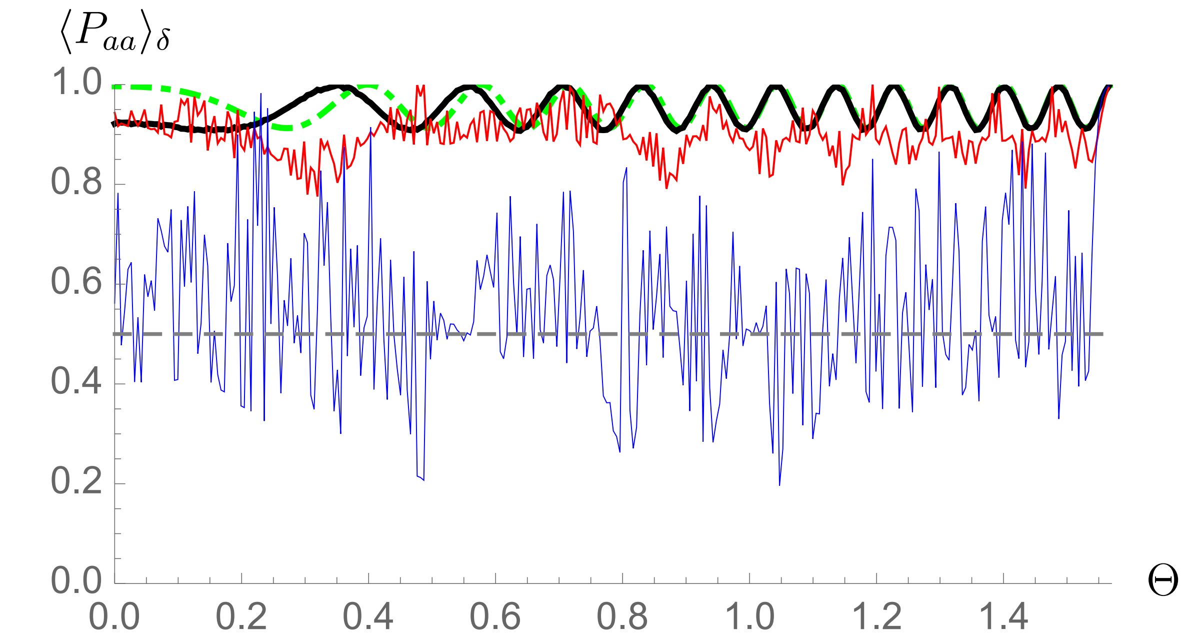
  They become order-one when $f(0)$ is such that $\epsilon \sim \eta^{-1}$ (for dim-4 int.) or $\epsilon \sim 1$ (for dim-5 int.)

- For small halos, the magnitude of the effect depends on the angle Θ of the incoming neutrino.

Probing the interior halo

- One computes numerically the neutrino wavefunction as it travels through the halo.

  Here is the typical result for the survival probability after traversing the Earth (neglecting the MSW effect):



Here we take the derivative scalar-neutrino coupling, $m = 3 \times 10^{-9}$ eV, $E = 1$ GeV, $\Delta m_0^2 = 2.5 \times 10^{-3}$ eV^2, $\sin^2 2\theta_0 = 0.087$. The step is $\Delta\Theta = 0.005$.

vacuum oscillations

$\epsilon = 0.1$ — the effect is only visible at small Θ, due to the fact that $\ell / L_0^{\mathrm{osc}} \sim 10$ at this value of m.

$\epsilon = 0.5$ — the effect is visible at all Θ.

$\epsilon = 2.0$ — the survival probability tends to $1/2$ (grey dashed line).

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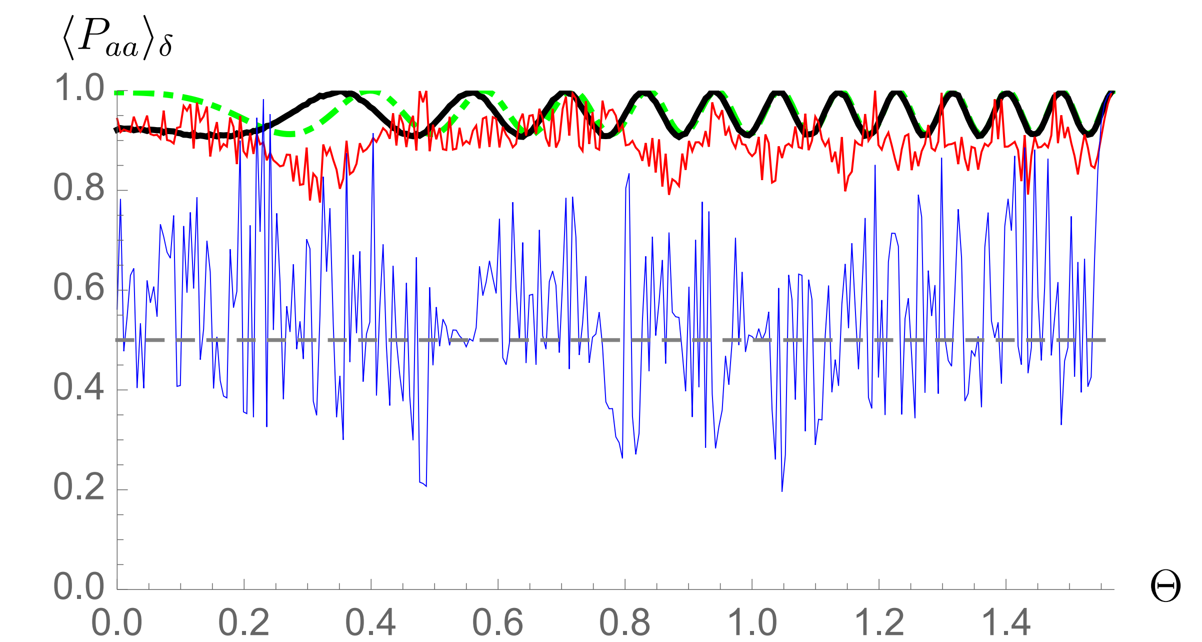

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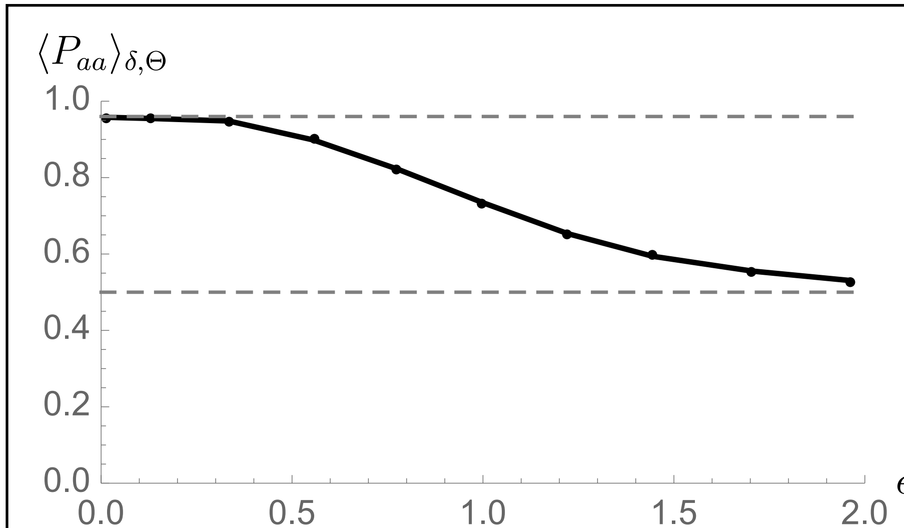
- What happens at $m \gg 10^{-9}$ eV corresponding to $\ell \ll R_\oplus$?

  The sensitivity goes down due to the limited angular resolution $\Theta_{\rm res}$ of a detector. We can replace:

$$\epsilon \mapsto \epsilon_{\rm eff} = \Theta_{\rm res}^{-1} \int_0^{\Theta_{\rm res}} d\Theta \, \epsilon(\Theta) \, , \qquad \epsilon(\Theta) = \frac{\beta}{m} f(R_\oplus \sin\Theta)$$

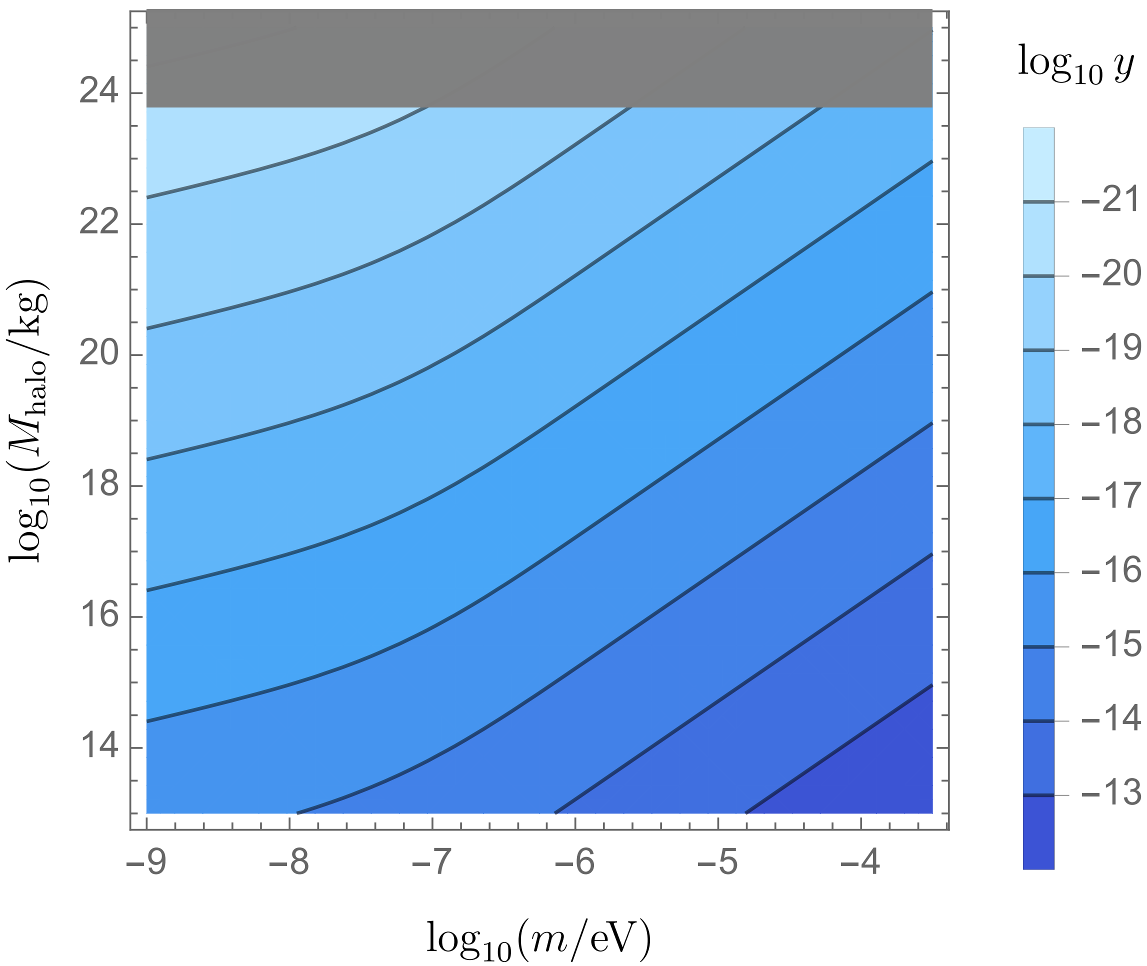
  The effect is still there, if the halo amplitude is big enough.

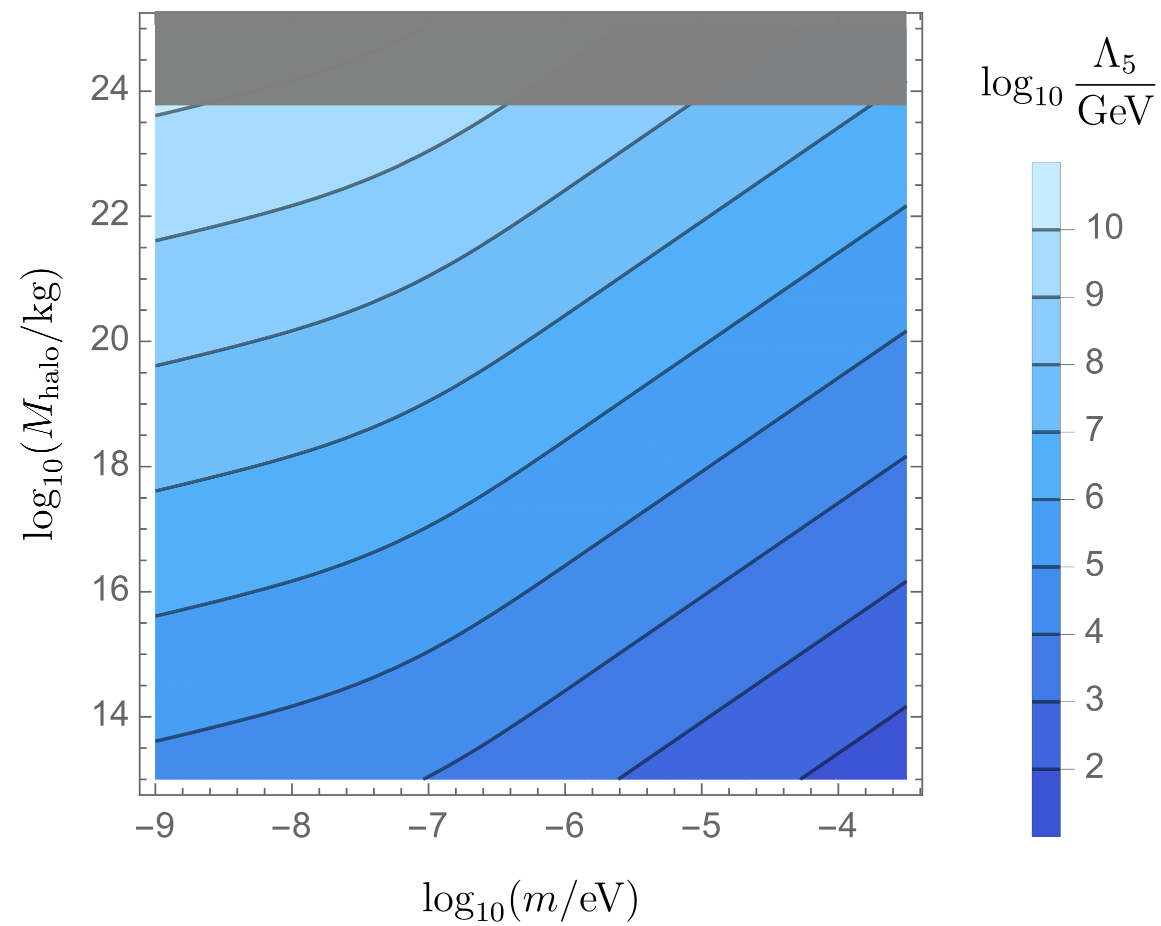


The survival probability averaged over the nadir angle Θ of the incoming neutrino, $-\pi/2 \leqslant \Theta \leqslant \pi/2$. The parameters are the same as on the previous plot.

Probing the interior halo: results

- The above results can be rephrased in terms of the dark matter particle mass, the halo mass, and the scalar-neutrino coupling parameter, y or Λ_5.





Values of the parameters at which the relative deviation from the vacuum oscillation probability reaches 0.1, for $m \gtrsim 10^{-9}$ eV. The grey shaded region depicts the values of $M_{\rm halo} > 0.1 M_\oplus$. We take $\Delta m_0^2 = 3.5 \cdot 10^{-3}$ eV^2, $E = 1$ GeV, and $\Theta_{\rm res} = 30°$.

Outlook

- We worked in the approximate 2-flavour oscillation scheme.
  It is interesting to do the full 3-flavour analysis and to include the MSW effect.

- We repeated the analysis for the (radially-polarised) local vector halo coupled to the neutrino current.
  It is interesting to consider other types of polarisations, since this would introduce additional directional dependence.

- Possible dark matter - nucleon/electron interactions may affect the structure of the halo.

- It is important to understand how reliable the local halo hypothesis is.

2306.12477

It is intriguing that a local dark matter halo could exist surrounding the Earth (or the Sun).

Possible interactions with neutrinos provide a novel way to search for the dark matter particle in neutrino oscillation experiments.

Thank you!

Local vector halo

- It is interesting to repeat the analysis for the local halo made of vector particles A_μ.

  New effects are expected due to the polarisation.

- Consider **radially-polarised,** spherically-symmetric configurations described by the ansatz:

A. Yu. Loginov, Phys.Rev.D 91 (2015) 10, 105028

$$A_t(r,t) = cu(r)\cos\omega t\ , \quad A_r(r,t) = v(r)\sin\omega t$$

In the external gravitational background, the equations of motion are

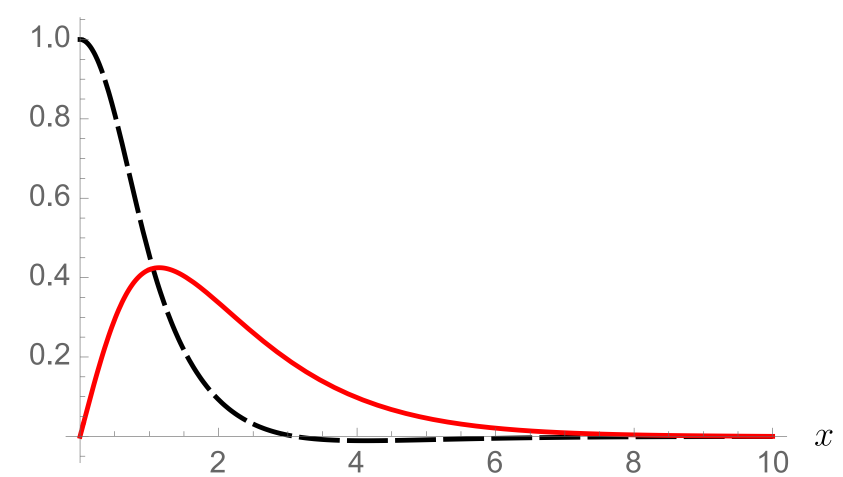
1508.05395 (this is similar to Proca stars, but in our case $N(r)$ is not dynamical)

$$\omega v(r) - cu'(r) = \frac{m^2c^4}{\omega}N(r)v(r)$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2(cu'(r)-\omega v(r))\right) = m^2c^3\frac{u(r)}{N(r)}$$

Make them dimensionless, take the nonrel. limit:

$$\boxed{-\frac{1}{x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left(x^2\frac{\mathrm{d}u}{\mathrm{d}x}\right) + \underbrace{\boxed{\frac{\mathcal{M}\tilde{\Phi}'}{\mathcal{M}\tilde{\Phi}-\mathcal{E}}\frac{\mathrm{d}u}{\mathrm{d}x}}}_{\text{“friction” term}} + 2(\mathcal{M}\tilde{\Phi}-\mathcal{E})u = 0} \qquad v = \frac{mR_\oplus}{2(\mathcal{E}-\mathcal{M}\tilde{\Phi})}\frac{\mathrm{d}u}{\mathrm{d}x}$$



Nonrelativistic vector soliton (ground state) at $\mathscr{M} = 2$. The black dashed line depicts the normalised function $u_0(x)$, the red line depicts $v_0(x)/(mR_\oplus)$.

Choose the ground state $u_0(x), v_0(x)$.

Probing the vector halo

- Consider the vector-neutrino coupling of the form

$$\blacktriangleright\ \mathscr{L}_{V,\mathrm{int}} = -\underbrace{y_V}_{\text{small coupling}} \overbrace{\kappa_{ab}}^{\text{Hermitian}} A_\mu \bar{\psi}_{La} \gamma^\mu \psi_{Lb}$$

See, e.g., 2212.05073, 1909.12845, 2005.01515

Like the derivative scalar-neutrino interaction, $\mathscr{L}_{V,\mathrm{int}}$ shifts the neutrino four-momentum.

Unlike the scalar case, now the spatial component of the current dominates for the range of m we are interested in: $|v_0| \gg |u_0|$ for 10^{-12} eV $\lesssim m \lesssim 10^{-3}$ eV.

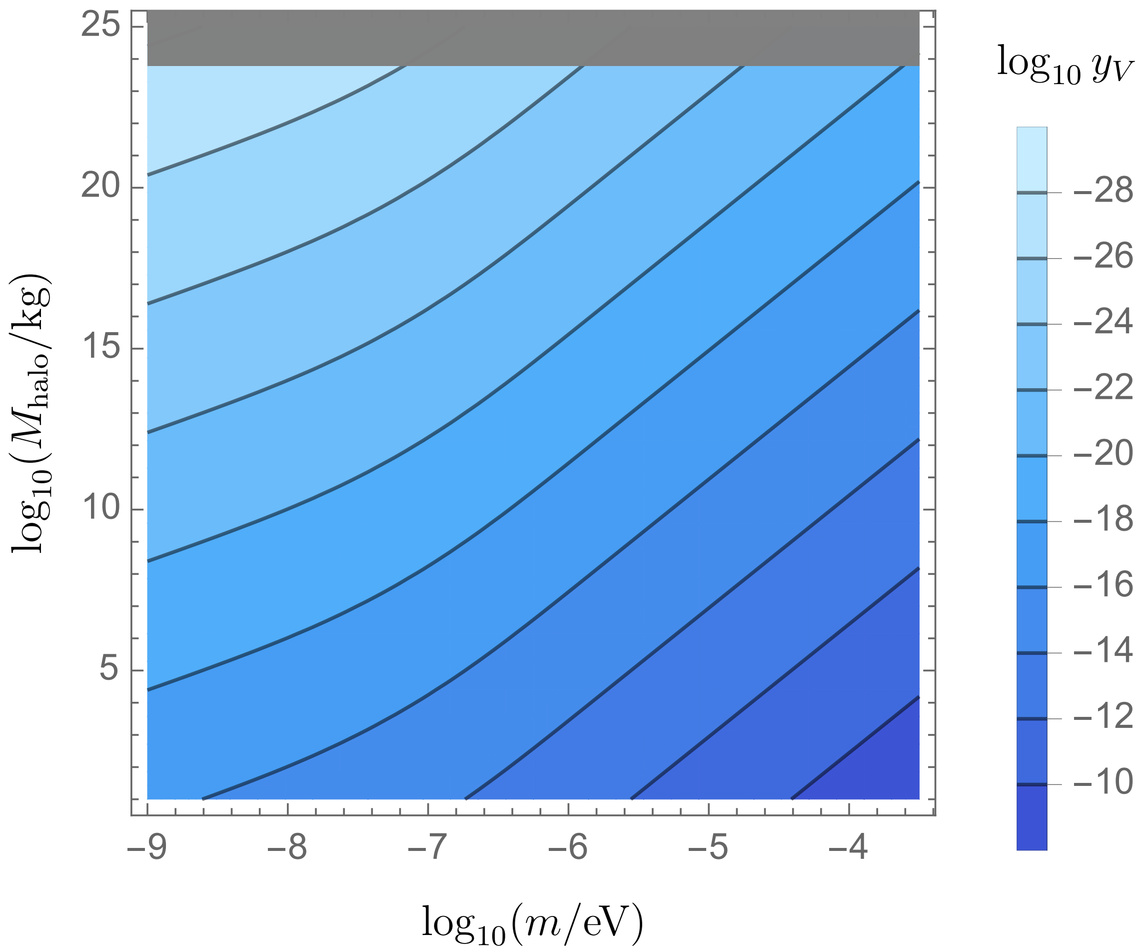
If $y_V |v_0| \ll \Delta m_0^2 / E$, then $\boxed{\Delta H_V = y_V \vec{n} \cdot \vec{A} \kappa}$

$\Rightarrow$ For the big halo, experiments with $L \ll R_\oplus$ conducted on the surface are less advantageous than in the scalar case.

For the small halo, the analysis is similar to the scalar case.

Probing the small vector halo: results

- The result for the vector halo resembles the result for the scalar halo and the derivative scalar-neutrino coupling, with $\Lambda_5 \mapsto m/(2y_V)$.



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More on perturbation theory in the big halo

Survival probability to the 2nd order in β: $\langle P_{aa}\rangle_\delta = P_{0,aa} + (\epsilon\eta)^2\langle P_{2,aa}\rangle_\delta$

where $P_{0,aa} = 1 - \sin^2 2\theta_0 \sin^2 X_0$, $X_0 = \pi L / L_0^{\rm osc}$, and

$$
\begin{aligned}
\langle P_{aa}\rangle_\delta &= 1 - \sin^2 2\theta_0 \sin^2 X_0 \\
&- 2\epsilon^2 \Big(A_{m1}^2 \sin^2 2\theta_0 \cos(2X_0) \sin^2(2\eta X_0) \\
&\quad + \eta A_{\theta 1} A_{m1} \sin 4\theta_0 \sin(2X_0) \sin(4\eta X_0) \\
&\quad + 2\eta^2 \Big[X_0 A_{m2} \sin^2 2\theta_0 \sin(2X_0) \\
&\quad + 2A_{\theta 1}^2 \big(\cos 4\theta_0 \sin^2 X_0 \cos^2(2\eta X_0) \\
&\quad + \cos^2 X_0 \sin^2(2\eta X_0)\big) + A_{\theta 2} \sin 4\theta_0 \sin^2 X_0 \Big] \Big)
\end{aligned}
$$

More on perturbation theory in the big halo

Coefficients in the expression for $\langle P_{aa}(L)\rangle_\delta$

— For the 5-dim derivative interaction:

$$
\begin{aligned}
\mathcal{A}_{\theta 1} &= 2\,\mathrm{Re}(g_{12})\cos 2\theta_0 - (g_{22}-g_{11})\sin 2\theta_0 \ , \\
\mathcal{A}_{m1} &= 2\,\mathrm{Re}(g_{12})\sin 2\theta_0 + (g_{22}-g_{11})\cos 2\theta_0 \ , \\
\mathcal{A}_{\theta 2} &= -2\mathcal{A}_{\theta 1}\mathcal{A}_{m1} + 4\,\mathrm{Im}^2(g_{12})\cot 2\theta_0 \ , \\
\mathcal{A}_{m2} &= -\mathcal{A}_{m1}^2 + 4|g_{12}|^2 + (g_{22}-g_{11})^2 \ .
\end{aligned}
$$

— For the 4-dim interaction:

$$
\begin{aligned}
\mathcal{A}_{\theta 1} &= \frac{2}{\sum m_\nu}\Big(\mathrm{Re}\big[(m_{11}+m_{22})h_{12}^* + m_{12}^*(h_{11}+h_{22})\big]\cos 2\theta_0 - \mathrm{Re}\big[m_{22}h_{22}^* - m_{11}h_{11}^*\big]\sin 2\theta_0\Big) \, , \\
\mathcal{A}_{m1} &= \frac{2}{\sum m_\nu}\Big(\mathrm{Re}\big[(m_{11}+m_{22})h_{12}^* + m_{12}^*(h_{11}+h_{22})\big]\sin 2\theta_0 + \mathrm{Re}\big[m_{22}h_{22}^* - m_{11}h_{11}^*\big]\cos 2\theta_0\Big) \ , \\
\mathcal{A}_{\theta 2} &= -2\mathcal{A}_{\theta 1}\mathcal{A}_{m1} + 4\,\mathrm{Im}^2\big[m_{12}h_{11}^* - m_{11}h_{12}^* + m_{22}h_{12}^* - m_{12}h_{22}^*\big]\cot 2\theta_0 \\
&\quad + \frac{2\Delta m_0^2}{(\sum m_\nu)^2}\Big(2\,\mathrm{Re}[(h_{11}+h_{22})h_{12}^*]\cos 2\theta_0 + (|h_{11}|^2 - |h_{22}|^2)\sin 2\theta_0\Big) \, , \\
\mathcal{A}_{m2} &= -\mathcal{A}_{m1}^2 + \frac{2}{(\sum m_\nu)^2}\Big\{\Delta m_0^2\Big((|h_{22}|^2 - |h_{11}|^2)\cos 2\theta_0 + 2\,\mathrm{Re}\big[(h_{11}+h_{22})h_{12}^*\big]\sin 2\theta_0\Big) \\
&+ \mathrm{Re}\Big[(m_{11}h_{11}^* - m_{22}h_{22}^*)^2 - 2m_{11}m_{22}^*h_{11}^*h_{22} + 4\left(m_{11}m_{22}h_{12}^{*2} + m_{11}m_{12}h_{11}^*h_{12}^* + m_{12}^2h_{11}^*h_{22}^* + m_{12}m_{22}h_{12}^*h_{22}^*\right.\\
&\left. + m_{12}m_{11}^*h_{12}h_{22}^* + m_{12}m_{22}^*h_{12}h_{11}^*\right)\Big] + 2|m_{12}|^2(|h_{11}|^2 + |h_{22}|^2) + 2(|m_{11}|^2 + |m_{22}|^2)|h_{12}|^2 + |h_{11}|^2|m_{11}|^2 + |h_{22}|^2|m_{22}|^2\Big\}
\end{aligned}
$$