PASCOS 2023

Neutrino Mass and Mixing From Eclectic Flavor Symmetry

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In collaboration with

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Based on **arXiv: 2303.02071** **[JHEP 05(2023)144]**

29th June, 2023

Outline

1. Background
2. Eclectic Flavor Symmetry
3. The EFG $\Omega(1) \cong \Delta(27) \rtimes T'$
4. $\Omega(1)$ invariant lepton masses model
5. Summary and outlook

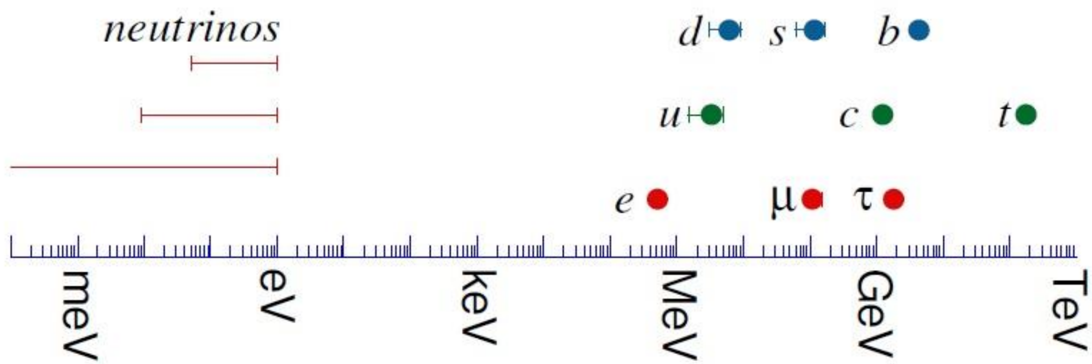
1. Background: Flavor puzzle

(See talk by Raby, Nilles, Omer…)

[F.Feruglio,1503.04071,
Z.Z.Xing, 1909.09610]

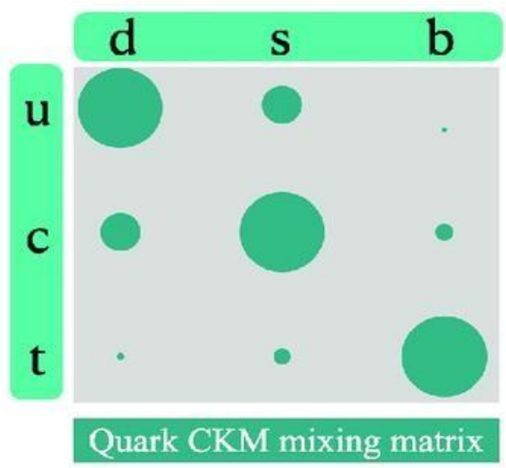
❑ **What is the origin of the masses of leptons & quarks ?**

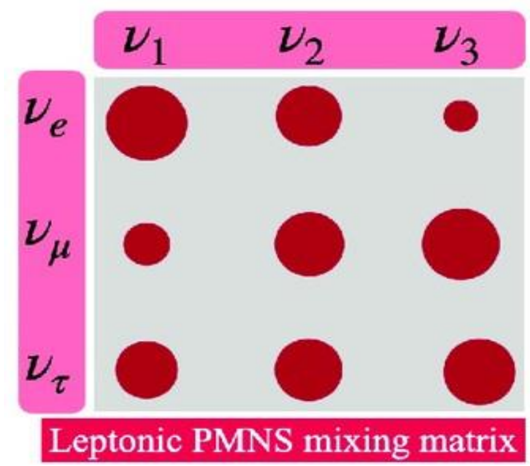
Mass hierarchy:



❑ **How to understand the flavor mixing patterns of leptons & quarks ?**

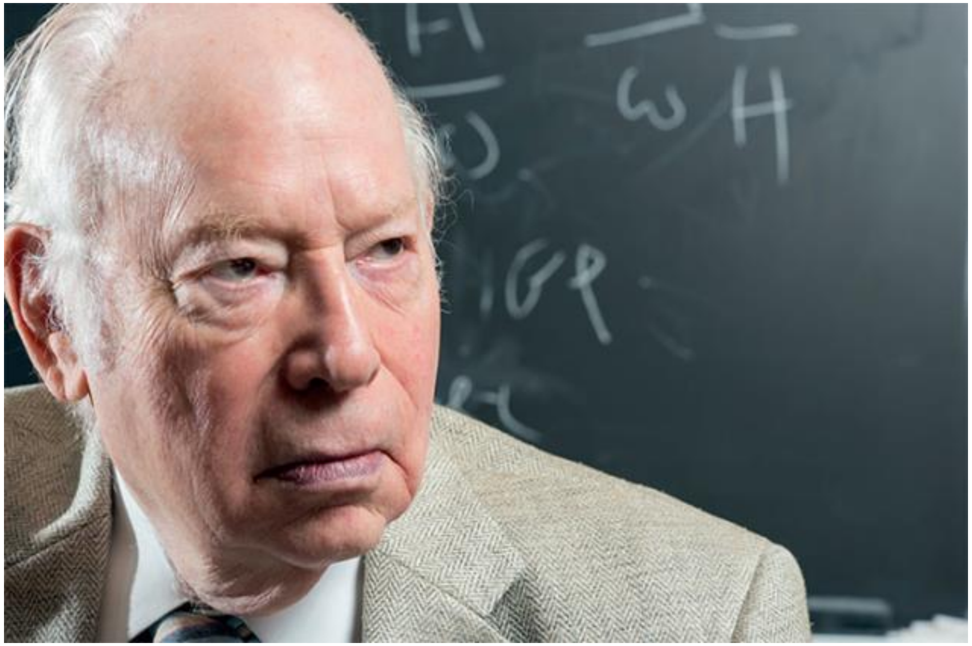
Flavor Mixing:





Asked what single mystery, if he could choose, he would like to see solved in his lifetime, Weinberg doesn't have to think for long: he wants to be able to explain the observed pattern of quark and lepton masses. In the summer of 1972, when the SM was coming together, he set himself the task of figuring it out but couldn't come up with anything. "It was the worst summer of my life! ... And I'm no closer now to answering it than I was in the summer of 1972," he says, still audibly irritated.

---------- From *"Model Physicist, CERN Courier, 13 October 2017"*



- One of S.Weinberg's last published papers:

Models of lepton and quark masses

Steven Weinberg
Phys. Rev. D **101**, 035020 – Published 19 February 2020

Article | References | Citing Articles (9) | PDF | HTML | Export Citation

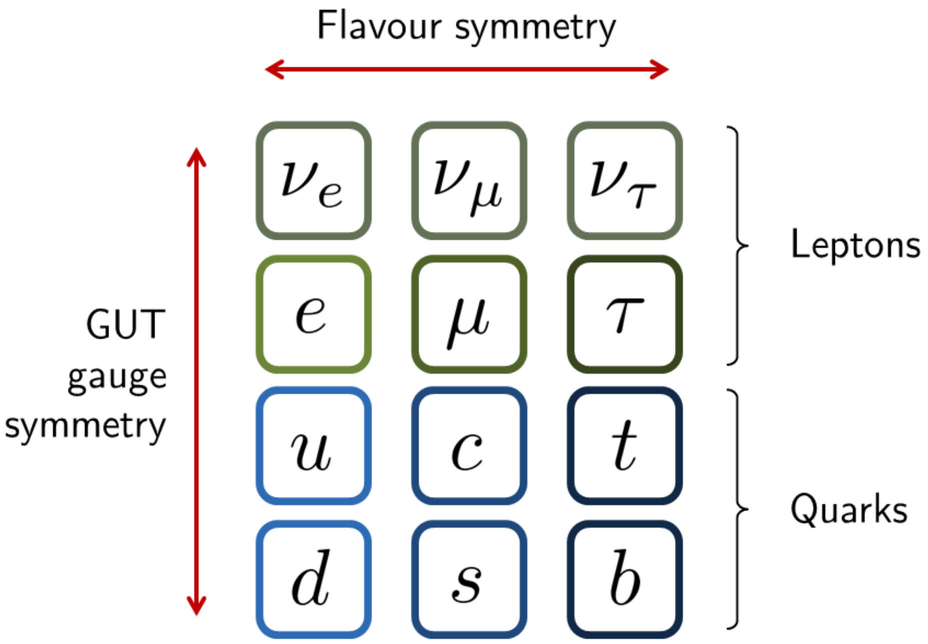
ABSTRACT

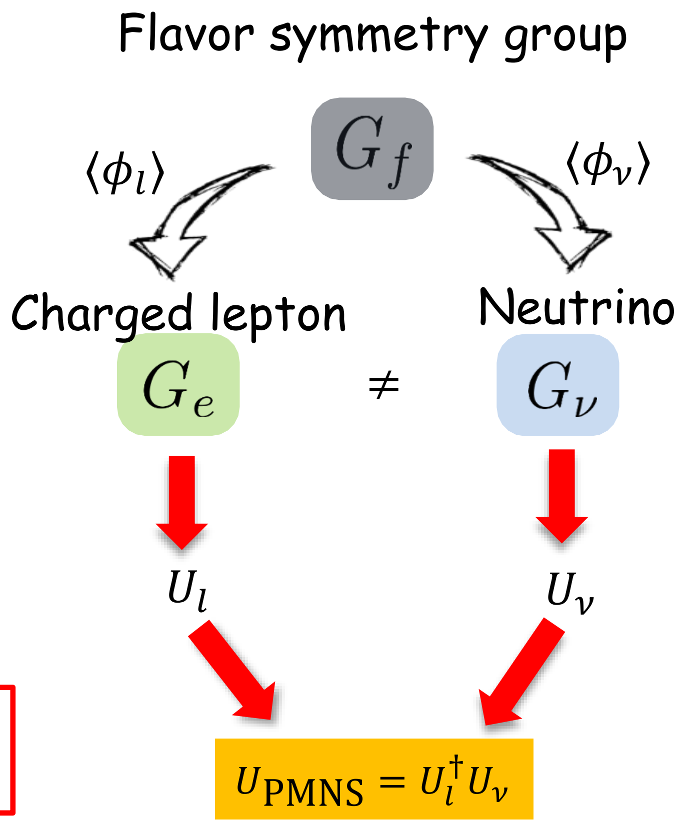
A class of models is considered in which the masses only of the third generation of quarks and leptons arise in the tree approximation, while masses for the second and first generations are produced respectively by one-loop and two-loop radiative corrections. So far, for various reasons, these models are not realistic.

Failed but encouraging

1. Background: Flavor symmetry

(See also talk by Omer)





➤ **Flavor transformation:** $\psi \xrightarrow{g} \rho(g)\psi, \quad g \in G_f\,,$

$$\mathcal{L}_Y \supset -Y_{ij}^l(\langle\phi_l\rangle)\bar{L}_i H e_{Rj} - \frac{1}{2}Y_{ij}^\nu(\langle\phi_\nu\rangle)\overline{L^c}_i H H^T L_j$$

[Feruglio, Romanino, Rev.Mod.Phys.93(2021)
Altarelli and Feruglio, Rev. Mod. Phys. 82, 2701 (2010)…]

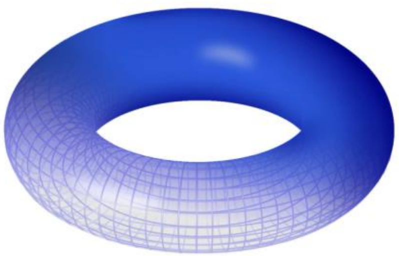
❑ **Drawbacks: Many flavons; Complicated vacuum alignments; Higher dimensional operators ….**

1. Background: Modular symmetry

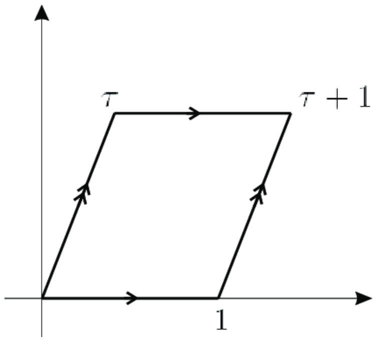
(See also talk by Feruglio, Knapp-Perez …)

[Feruglio, 1706.08749]

- **Torus compactification in string theory leads to Modular Symmetry**



$\cong$



The shape of torus is characterized by complex modulus: $\tau=\omega_2/\omega_1$, $\mathrm{Im}(\tau)>0$

- **Modular (flavor) transformation:**

$$\tau \xrightarrow{\gamma} \gamma\tau \equiv \frac{a\tau+b}{c\tau+d},$$

$$\psi \xrightarrow{\gamma} (c\tau+d)^{-k_\psi}\rho(\gamma)\psi, \quad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}),$$

modular weight (integers)

Irreps of **finite modular groups** [Liu,Ding,2112,14761]
Or equivalently, irreps of SL(2,Z) with finite images

Finite modular groups: $\Gamma_N: \Gamma_2=S_3,\ \Gamma_3=A_4,\ \Gamma_4=S_4,\ \ \Gamma_5=A_5$, … [Feruglio, 1706.08749]
$\Gamma'_N: \Gamma'_2=S_3, \Gamma'_3=T', \Gamma'_4=S'_4,\ \ \Gamma'_5=A'_5$, … [Liu,Ding,1907.01488]

1. Background: Modular invariant SUSY

[Ferrara et al, 1989; Feruglio, 1706.08749]

- $\mathcal{N}$=1 global supersymmetry theory with modular symmetry:

  - The action: $\mathcal{S} = \int d^4x d^2\theta d^2\bar{\theta} \mathcal{K}(\psi_I, \bar{\psi}_I; \tau, \bar{\tau}) + \int d^4x d^2\theta \mathcal{W}(\psi_I, \tau) + \mathrm{h.c.}$

  - (Minimal) Kähler potential: $\mathcal{K} = -h \ln(-i\tau + i\bar{\tau}) + \sum_n (-i\tau + i\bar{\tau})^{-k_n} |\psi_n|^2$

  - Superpotential: $\mathcal{W} = \sum_n Y_{I_1 I_2 \ldots I_n}(\tau) \psi_{I_1} \psi_{I_2} \cdots \psi_{I_n}$

[Feruglio, 1706.08749]

- **Modular invariance requires Yukawa couplings to be Modular Forms!**

$$Y_{I_1 I_2 \ldots I_n}(\tau) \to Y_{I_1 I_2 \ldots I_n}(\gamma\tau) = (c\tau + d)^k \rho_Y(\gamma) Y_{I_1 I_2 \ldots I_n}(\tau)$$

$$\begin{cases} k_Y = k_{I_1} + k_{I_2} + \cdots + k_{I_n} \\ \rho_Y \otimes \rho_{I_1} \otimes \cdots \otimes \rho_{I_n} \supset 1 \end{cases}$$

Remarks:

- Only one flavon: modulus τ . All higher dimensional operators.
- Model building in bottom-up approach depends on: k_I, ρ_I
- For a given k_Y, ρ_Y , the modular forms space is finite-dimensional

➡ Only a finite number of possible Yukawa couplings !

N	$\dim \mathcal{M}_k(\Gamma(N))$	Γ_N (Γ'_N)	Modular forms multiplets			
			$k=1$	$k=2$	$k=3$	$k\geq 4$
2	$k/2+1$ ($k\in$ even)	S_3 (S_3)	$-$	$Y^{(2)}_{\mathbf{2}}$	$-$	...
3	$k+1$	A_4 (T')	$Y^{(1)}_{\mathbf{2}}$	$Y^{(2)}_{\mathbf{3}}$	$Y^{(3)}_{\mathbf{2}}$, $Y^{(3)}_{\mathbf{2}''}$	...
4	$2k+1$	S_4 (S'_4)	$Y^{(1)}_{\hat{\mathbf{3}}'}$	$Y^{(2)}_{\mathbf{2}}$, $Y^{(2)}_{\mathbf{3}}$	$Y^{(3)}_{\hat{\mathbf{1}}'}$, $Y^{(3)}_{\hat{\mathbf{3}}}$, $Y^{(3)}_{\hat{\mathbf{3}}'}$	...
5	$5k+1$	A_5 (A'_5)	$Y^{(1)}_{\mathbf{6}}$	$Y^{(2)}_{\mathbf{3}}$, $Y^{(2)}_{\mathbf{3}'}$, $Y^{(2)}_{\mathbf{5}}$	$Y^{(3)}_{\mathbf{4}'}$, $Y^{(3)}_{\mathbf{6}I}$, $Y^{(3)}_{\mathbf{6}II}$	...

[Kobayashi, Tanaka, and Tatsuishi 2018; Feruglio 2017; Penedo and Petcov 2019; Novichkov et al. 2019; Ding, King, and Liu 2019b; Liu and Ding 2019; Liu, Yao, and Ding 2021; Novichkov, Penedo, and Petcov 2021; Wang, Yu, and Zhou 2021; Yao, Liu, and Ding 2021 ...]

Drawback: The Kähler potential is not under control !

[Chen, Ramos-Sanchez, Ratz 1909.06910]

Most general Kahler potential:

$$\mathcal{K} \supset \sum_{\psi_n} \sum_{k\geq 1} (-i\tau + i\bar{\tau})^{-k+k_n} \sum_a \kappa_a^{(k)} \left[Y^{(k)}(\tau) \otimes \bar{Y}^{(k)}(\tau) \otimes \psi_n \otimes \bar{\psi}_n \right]_{1,a}$$

Additional terms affect the prediction!

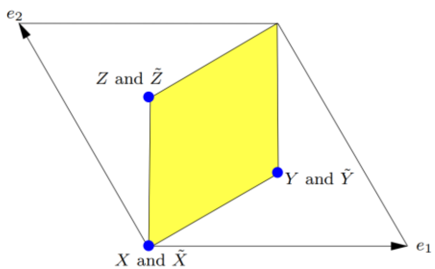
2. Eclectic Flavor Symmetry

(See also talk by Nilles, Ramos-Sanchez …)

[Baur, Nilles, Trautner, Vaudrevange, 1901.03251]

[Nilles, Ramos-Sanchez, Vaudrevange, 2001.01736]

- Outer automorphisms of Narain space group = Eclectic Flavor symmetry



$$\mathrm{Out}\,(\mathcal{S}) = (\Sigma,\ T)$$

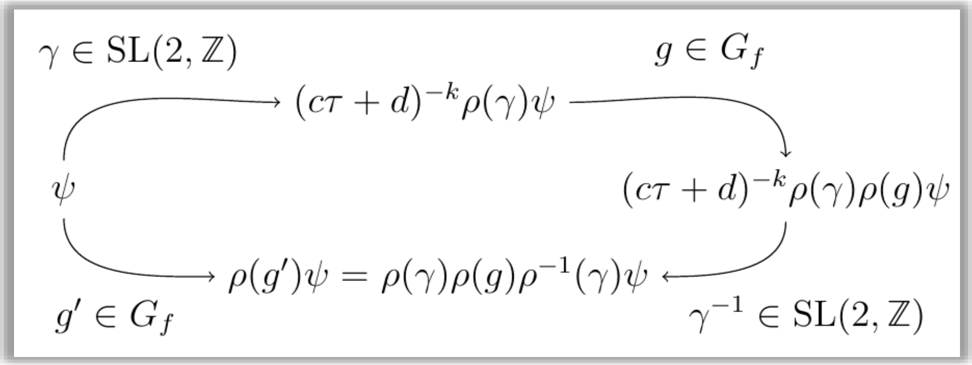
- Rotations $(\Sigma_i, 0)$ ⇒ Modular symmetry: $O(2,2,\mathbb{Z})$
- Translations $(1, T_i)$ ⇒ Traditional flavor symmetry: $\Delta(54)$

- **EFG** is a combination of traditional flavor & finite modular group:

TFT: $\begin{cases} \tau \xrightarrow{g} \tau \\ \psi_I \xrightarrow{g} \rho_I(g)\psi_I \end{cases}$

MT: $\begin{cases} \tau \xrightarrow{\gamma} \dfrac{a\tau + b}{c\tau + d} \\ \psi_I \xrightarrow{\gamma} (c\tau + d)^{-k_I}\rho_I(\gamma)\psi_I \end{cases}$

⇒



➢ **Consistency conditions:** $\rho(\gamma)\rho(g)\rho^{-1}(\gamma) = \rho(u_\gamma(g))$ $\quad u_\gamma : G_f \to G_f$

➡ **EFG unifies traditional flavor & modular (and gCP) symmetry**

$$G_{ecl} \cong G_f \rtimes \Gamma'_N\,(G_f \rtimes \Gamma_N)\,.$$

❑ Advantages of EFG:

- There are very few candidates for self-consistently eclectic flavor groups G_{ecl} when u_γ nontrival: $G_f = \mathbb{Z}_3 \times \mathbb{Z}_3,\ \Delta(27),\ \Delta(54)$ ...
- In general, Yukawa couplings are functions of flavons & modulus: $Y_{ijk}(\phi_i, \tau)$, which are highly constrained by EFG.
- Kähler potential is under control due to the traditional flavor symmetry!
- EFG has a natural UV completion —— Heterotic string on orbifold

❑ Two EFG models have been established so far:

1) Based on $\boldsymbol{G_{ecl} = A_4 \times \Gamma_3}$ [Chen, Knapp-Perez,Ramos-Hamud, Ramos-Sanchez, Ratz 2108.02240]
2) Based on $\boldsymbol{G_{ecl} = \Omega(2) = \Delta(54) \cup T' \cup \mathbb{Z}_9^R}$ [Baur,Nilles,Ramos-Sanchez,Vaudrevange 2207.10677]

➡ There is currently no bottom-up minimal model based on **smaller** $\mathbf{G_{ecl}}$ with **non-direct product** structure !

➡ We choose $\mathbf{G_{ecl} = \Omega(1) \cong \Delta(27) \rtimes T'}$ $\quad |\boldsymbol{\Omega(1)}| = \mathbf{648}$

3. The EFG $\Omega(1) \cong \Delta(27) \rtimes T'$

- **Strategy**: Constructing $\mathbf{\Omega(1)}$ from $\mathbf{\Delta(27)}$

  - The multiplication rules of $\Delta(27)$: $A^3 = B^3 = (AB)^3 = (AB^2)^3 = 1$

  - The two outer automorphisms u_S, u_T form finite modular group T'

$$u_S(A) = B^2 A, \quad u_S(B) = B^2 A^2; \qquad u_T(A) = BA, \quad u_T(B) = B;$$

➡ $(u_S)^4 = (u_T)^3 = (u_S u_T)^3 = 1, \; (u_S)^2 u_T = u_T (u_S)^2$ Multiplication rules of T'

  - Solving the following consistency conditions

$$\begin{aligned} \rho_{\boldsymbol{r}}(S)\,\rho_{\boldsymbol{r}}(A)\,\rho_{\boldsymbol{r}}^{-1}(S) &= \rho_{\boldsymbol{r}}(B^2 A), & \rho_{\boldsymbol{r}}(S)\,\rho_{\boldsymbol{r}}(B)\,\rho_{\boldsymbol{r}}^{-1}(S) &= \rho_{\boldsymbol{r}}(B^2\,A^2)\,, \\ \rho_{\boldsymbol{r}}(T)\,\rho_{\boldsymbol{r}}(A)\,\rho_{\boldsymbol{r}}^{-1}(T) &= \rho_{\boldsymbol{r}}(BA), & \rho_{\boldsymbol{r}}(T)\,\rho_{\boldsymbol{r}}(B)\,\rho_{\boldsymbol{r}}^{-1}(T) &= \rho_{\boldsymbol{r}}(B)\,, \end{aligned}$$

Where $\boldsymbol{r}$ is generally the reducible representation for $\Delta(27)$ and T', but ultimately correspond to the irreducible representation of $\Omega(1)$.

➡ $\Omega(1) \cong \Delta(27) \rtimes T' = \langle\, \rho(S), \rho(T), \rho(A), \rho(B) \,\rangle$

4. $\Omega(1)$ invariant lepton masses model

- Field contents and their transformation properties:

Flavons: ϕ, φ, χ, ξ — Modular forms: $Y_{\boldsymbol{r}}^{(k_Y)}$

Fields	L	E^c	H_u	H_d	ϕ	φ	χ	ξ	$Y_{\boldsymbol{r}}^{(k_Y)}$
$\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$	$(\mathbf{2},-\frac{1}{2})$	$(\mathbf{1},1)$	$(\mathbf{2},\frac{1}{2})$	$(\mathbf{2},-\frac{1}{2})$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$	$(\mathbf{1},0)$
$\Delta(27)$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$
$\Gamma'_3\cong T'$	$\mathbf{3}_0$	$\mathbf{3}_0$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}_1$	$\mathbf{3}_0$	$\mathbf{3}_1$	$\mathbf{1}$	$\boldsymbol{r}$
modular weight	0	0	0	0	5	5	7	-1	k_Y
Z_2	1	-1	1	1	-1	1	1	1	1
Z_3	ω	ω^2	1	1	1	ω	ω	1	1

- Kähler potential :

$$\mathcal{K} = \underbrace{\mathcal{K}_{\mathrm{LO}}}_{\propto 1} + \mathcal{K}_{\mathrm{NLO}} + \mathcal{K}_{\mathrm{NNLO}} + \ldots$$

Off-diagonal → Correction ~ $\frac{\langle\Phi\rangle^2}{\Lambda^2}$

$$\mathcal{K}_{\mathrm{NLO}} = \sum_{m,n,\boldsymbol{r_1},\boldsymbol{r_2},s} \frac{1}{\Lambda^2}(-i\tau+i\bar{\tau})^{-k_\psi-k_\Theta+m}\left(Y_{\boldsymbol{r_1}}^{(m)\dagger}Y_{\boldsymbol{r_2}}^{(n)}\psi^\dagger\psi\Theta^\dagger\Phi\right)_{(\mathbf{1}_{0,0},\mathbf{1}),s} + \text{h.c.}$$

Ignore!

4. $\Omega(1)$ invariant lepton masses model

- Field contents and their transformation properties:

Flavons | Modular forms

Fields	L	E^c	H_u	H_d	ϕ	φ	χ	ξ	$Y_{\boldsymbol{r}}^{(k_Y)}$
$SU(2)_L \times U(1)_Y$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 1)$	$(\mathbf{2}, \frac{1}{2})$	$(\mathbf{2}, -\frac{1}{2})$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$	$(\mathbf{1}, 0)$
$\Delta(27)$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{3}$	$\mathbf{1}_{0,0}$	$\mathbf{1}_{0,0}$
$\Gamma'_3 \cong T'$	$\mathbf{3}_0$	$\mathbf{3}_0$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{3}_1$	$\mathbf{3}_0$	$\mathbf{3}_1$	$\mathbf{1}$	$\boldsymbol{r}$
modular weight	0	0	0	0	5	5	7	-1	k_Y
Z_2	1	-1	1	1	-1	1	1	1	1
Z_3	ω	ω^2	1	1	1	ω	ω	1	1

- Superpotential：

$$\begin{aligned}\mathcal{W} &= \frac{\alpha}{\Lambda}\left(E^c L\phi Y_{\mathbf{2}'}^{(5)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_d + \frac{\beta}{\Lambda^2}\left(E^c L\xi\phi Y_{\mathbf{1}}^{(4)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_d \\ &\quad + \frac{g_1}{2\Lambda^2}\left(LL\varphi Y_{\mathbf{2}''}^{(5)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_u H_u + \frac{g_2}{2\Lambda^2}\left(LL\chi Y_{\mathbf{2}'}^{(7)}\right)_{(\mathbf{1}_{0,0},\mathbf{1})} H_u H_u\,.\end{aligned}$$

Symmetry breaking

[M.Leurer, Y.Nir, N.Seiberg 1992]

❖ No exact (nontrival) flavor symmetries are preserved at low energy!

➡ EFG must be fully broken by VEVs of flavons & modulus:

VEVs of flavons:

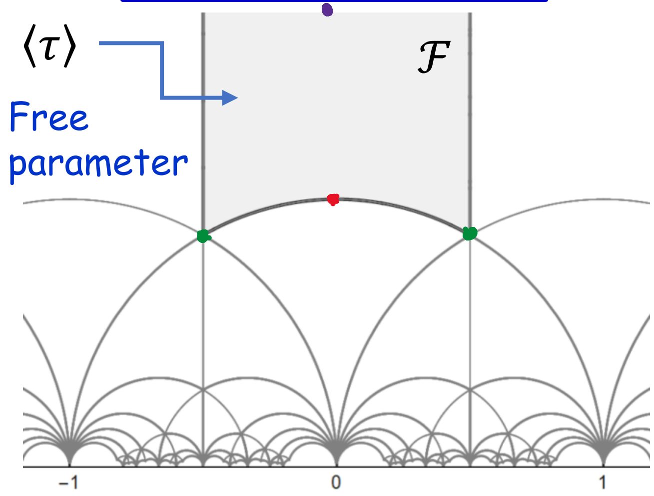
$$\langle\phi\rangle = (\omega^2, 1, 1)^T v_\phi : \qquad \Omega(1) \xrightarrow{\langle\phi\rangle} Z_3^{A^2BA^2},$$
$$\langle\varphi\rangle = (0, 0, 1)^T v_\varphi : \qquad \Omega(1) \xrightarrow{\langle\varphi\rangle} Z_3^{ABA^2},$$
$$\langle\chi\rangle = (1, 0, 0)^T v_\chi : \qquad \Omega(1) \xrightarrow{\langle\chi\rangle} Z_3^{B}.$$

$\Omega(1) \cong \Delta(27) \rtimes T'$

$\langle\phi\rangle$ → Charged lepton: $Z_3^{A^2BA^2}$

$\langle\varphi\rangle, \langle\chi\rangle$ → Neutrino: 1

VEV of modulus:



$$\Omega(1) \xrightarrow{\langle\tau\rangle} \Delta(27)$$

$$\Omega(1) \xrightarrow{\langle\phi\rangle,\langle\varphi\rangle,\langle\chi\rangle,\langle\tau\rangle} 1$$

EFG completely broken!

■ Lepton mass matrices:

$$m_l = \frac{\alpha v_\phi v_d}{\Lambda}\begin{pmatrix} \sqrt{2}\omega^2 Y^{(5)}_{\mathbf{2'},1} & \omega Y^{(5)}_{\mathbf{2'},2} & \omega Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} & Y^{(5)}_{\mathbf{2'},2} \\ \omega Y^{(5)}_{\mathbf{2'},2} & Y^{(5)}_{\mathbf{2'},2} & \sqrt{2} Y^{(5)}_{\mathbf{2'},1} \end{pmatrix} + \frac{i\beta Y^{(4)}_{\mathbf{1}} v_\xi v_\phi v_d}{\Lambda^2}\begin{pmatrix} 0 & \omega & -\omega \\ -\omega & 0 & 1 \\ \omega & -1 & 0 \end{pmatrix},$$

$$m_\nu = \frac{g_1 v_\varphi v_u^2}{\Lambda^2}\begin{pmatrix} 0 & \omega Y^{(5)}_{\mathbf{2''},2} & 0 \\ \omega Y^{(5)}_{\mathbf{2''},2} & 0 & 0 \\ 0 & 0 & \sqrt{2} Y^{(5)}_{\mathbf{2''},1} \end{pmatrix} + \frac{g_2 v_\chi v_u^2}{\Lambda^2}\begin{pmatrix} \sqrt{2} Y^{(7)}_{\mathbf{2'},1} & 0 & 0 \\ 0 & 0 & \omega Y^{(7)}_{\mathbf{2'},2} \\ 0 & \omega Y^{(7)}_{\mathbf{2'},2} & 0 \end{pmatrix},$$

Unitary rotation matrix:

$$U_l^\dagger m_l^\dagger m_l U_l = \mathrm{diag}\left(m_e^2, m_\mu^2, m_\tau^2\right) \Rightarrow U_l = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & \omega^2 & \omega \\ \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 \end{pmatrix}.$$

Charged lepton masses:

$$m_e = \left|\sqrt{2} Y^{(5)}_{\mathbf{2'},1} - Y^{(5)}_{\mathbf{2'},2} - \frac{\sqrt{6}\beta v_\xi Y^{(4)}_{\mathbf{1}}}{\alpha\Lambda}\right| \frac{\alpha v_\phi v_d}{\Lambda},$$

$$m_\mu = \left|\sqrt{2} Y^{(5)}_{\mathbf{2'},1} - Y^{(5)}_{\mathbf{2'},2} + \frac{\sqrt{6}\beta v_\xi Y^{(4)}_{\mathbf{1}}}{\alpha\Lambda}\right| \frac{\alpha v_\phi v_d}{\Lambda},$$

$$m_\tau = \left|\sqrt{2} Y^{(5)}_{\mathbf{2'},1} + 2 Y^{(5)}_{\mathbf{2'},2}\right| \frac{\alpha v_\phi v_d}{\Lambda}.$$

Numerical fitting and prediction

- Best-fit values of the free input parameters:

$$\langle\tau\rangle = 0.00177 + 1.120i,$$

Close to the critical point i

$$|\beta v_\xi/(\alpha\Lambda)| = 0.0480, \quad \arg(\beta v_\xi/(\alpha\Lambda)) = 1.04\pi,$$
$$|g_2 v_\chi/(g_1 v_\varphi)| = 0.9787, \quad \arg(g_2 v_\chi/(g_1 v_\varphi)) = 1.005\pi,$$
$$\alpha v_\xi v_\phi v_d/\Lambda^2 = 262.5\mathrm{MeV}, \quad g_1 v_\varphi v_u^2/\Lambda^2 = 5.401\mathrm{meV}.$$

8 input parameters ⟶ **6** masses + **3** mixing angles + **3** CP phases

- The predictions for various flavor observables:

$$\sin^2\theta_{13} = 0.02251, \quad \sin^2\theta_{12} = 0.3284, \quad \sin^2\theta_{23} = 0.4954, \quad \delta_{CP} = 1.434\pi,$$
$$\alpha_{21} = 0.961\pi, \quad \alpha_{31} = 0.926\pi, \quad m_1 = 15.13\mathrm{meV}, \quad m_2 = 17.40\mathrm{meV},$$
$$m_3 = 52.31\mathrm{meV}, \quad \sum_{i=1}^{3} m_i = 84.84\mathrm{meV}, \quad m_{\beta\beta} = 5.619\mathrm{meV},$$
$$m_e = 0.511\mathrm{MeV}, \quad m_\mu = 106.5\mathrm{MeV}, \quad m_\tau = 1.803\mathrm{GeV}.$$

Almost all flavor observables are within the 3σ regions !

$\mu-\tau$ reflection symmetry

In the charged lepton diagonal basis: $\omega = e^{2\pi i/3}$

$$
\begin{aligned}
m'_\nu &= U_l^T m_\nu U_l \\
&= \frac{g_1 v_\varphi v_u^2}{\Lambda^2}\begin{pmatrix}
\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}+2Y^{(5)}_{\mathbf{2}'',2} & \omega\left(\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}-Y^{(5)}_{\mathbf{2}'',2}\right) & \omega^2\left(\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}-Y^{(5)}_{\mathbf{2}'',2}\right) \\
\omega\left(\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}-Y^{(5)}_{\mathbf{2}'',2}\right) & \omega^2\left(\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}+2Y^{(5)}_{\mathbf{2}'',2}\right) & \sqrt{2}Y^{(5)}_{\mathbf{2}'',1}-Y^{(5)}_{\mathbf{2}'',2} \\
\omega^2\left(\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}-Y^{(5)}_{\mathbf{2}'',2}\right) & \sqrt{2}Y^{(5)}_{\mathbf{2}'',1}-Y^{(5)}_{\mathbf{2}'',2} & \omega\left(\sqrt{2}Y^{(5)}_{\mathbf{2}'',1}+2Y^{(5)}_{\mathbf{2}'',2}\right)
\end{pmatrix} \\
&+\frac{g_2 v_\chi v_u^2}{\Lambda^2}\begin{pmatrix}
\sqrt{2}Y^{(7)}_{\mathbf{2}',1}+2Y^{(7)}_{\mathbf{2}',2} & \omega^2\left(\sqrt{2}Y^{(7)}_{\mathbf{2}',1}-Y^{(7)}_{\mathbf{2}',2}\right) & \omega\left(\sqrt{2}Y^{(7)}_{\mathbf{2}',1}-Y^{(7)}_{\mathbf{2}',2}\right) \\
\omega^2\left(\sqrt{2}Y^{(7)}_{\mathbf{2}',1}-Y^{(7)}_{\mathbf{2}',2}\right) & \omega\left(\sqrt{2}Y^{(7)}_{\mathbf{2}',1}+2Y^{(7)}_{\mathbf{2}',2}\right) & \sqrt{2}Y^{(7)}_{\mathbf{2}',1}-Y^{(7)}_{\mathbf{2}',2} \\
\omega\left(\sqrt{2}Y^{(7)}_{\mathbf{2}',1}-Y^{(7)}_{\mathbf{2}',2}\right) & \sqrt{2}Y^{(7)}_{\mathbf{2}',1}-Y^{(7)}_{\mathbf{2}',2} & \omega^2\left(\sqrt{2}Y^{(7)}_{\mathbf{2}',1}+2Y^{(7)}_{\mathbf{2}',2}\right)
\end{pmatrix}.
\end{aligned}
$$

gCP + $\Re\tau = 0$:

Input parameters: $8 \to 5$

$$P_{\nu\tau}^T\, m'_\nu\, P_{\nu\tau} = (m'_\nu)^*, \qquad P_{\nu\tau} = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

[W.Grimus,L.Lavoura, 2003]

$\mu-\tau$ reflection symmetry: → $\theta_{23} = \pi/4, \quad \delta_{CP} = \pm\pi/2, \quad \alpha_{21}, \alpha_{31} = 0, \pi$

Atmospheric mixing angle and Dirac CP phase are predicted to be maximal !

Numerical fitting and prediction

- Best-fit values of the free input parameters:

$$\langle\tau\rangle = 1.120i,$$
$$\beta v_{\xi}/(\alpha\Lambda) = -0.0484, \quad g_2 v_{\chi}/(g_1 v_{\varphi}) = -0.981,$$
$$\alpha v_d v_{\xi} v_{\phi}/\Lambda^2 = 263.0\text{MeV}, \quad g_1 v_u^2 v_{\varphi}/\Lambda^2 = 5.409\text{meV}.$$

5 input parameters ⟶ 6 masses + 3 mixing angles + 3 CP phases

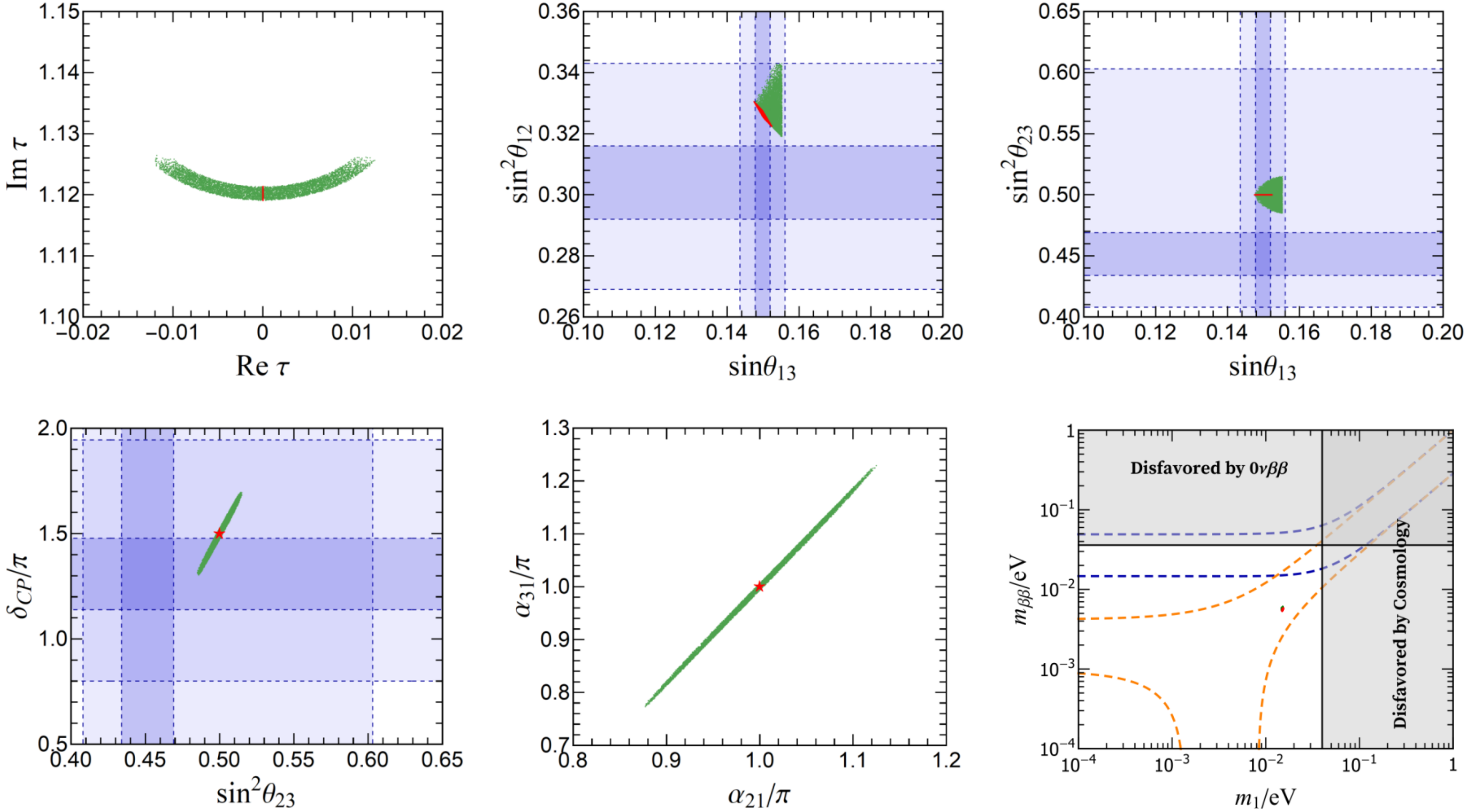
- The predictions for various flavor observables:

$$\sin^2\theta_{13} = 0.02238, \quad \sin^2\theta_{12} = 0.3266, \quad \sin^2\theta_{23} = 0.5, \quad \delta_{CP} = 1.5\pi,$$
$$\alpha_{21} = \pi, \quad \alpha_{31} = \pi, \quad m_1 = 15.18\text{meV}, \quad m_2 = 17.44\text{meV},$$
$$m_3 = 52.43\text{meV}, \quad \sum_i m_i = 85.05\text{meV}, \quad m_{\beta\beta} = 5.595\text{meV},$$
$$m_e = 0.511\text{MeV}, \quad m_{\mu} = 106.5\text{MeV}, \quad m_{\tau} = 1.807\text{GeV}.$$

Almost all flavor observables are within the 3σ regions!

❑ The correlations among mixing parameters:

Green: Allowed region for 8 para
Red: Allowed region for 5 para
Blue: Bounds of experiments



- The predicted mixing parameters are within very narrow ranges.
- The predicted $m_{1,2,3}$, θ_{23} , δ_{CP} and the effective mass $m_{\beta\beta}$ in 0ν2β decay can be tested in future experiments.

5. Summary and outlook

- **We develop the formalism of eclectic flavor symmetry of $\Omega(1)$ from the bottom-up perspective.**
- **A comprehensive analysis of superpotential and Kähler potential of models based on EFG $\Omega(1)$.** (Not included in this talk)
- **A $\Omega(1)$ invariant lepton masses model with 8(5) parameters.**
- **Correction from Kähler potential are suppressed by $\langle\Phi\rangle^2/\Lambda^2$.**
- **$\mu-\tau$ reflection symmetry for neutrino mass matrix is preserved.**
- How to dynamically explain these flavons & modulus VEVs?
- More insights from the top-down approach?

Thanks for your attention

Backup

❑ Modular forms multiplets for $\Gamma_3' \cong T'$:

[Liu,Ding,1907.01488]

Dedekind eta function:

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$$

with $q = e^{2\pi i\tau}$

$$k=1: \quad Y_{\mathbf{2''}}^{(1)}(\tau) = \begin{pmatrix} 3\frac{\eta^3(3\tau)}{\eta(\tau)} + \frac{\eta^3(\tau/3)}{\eta(\tau)} \\ 3\sqrt{2}\frac{\eta^3(3\tau)}{\eta(\tau)} \end{pmatrix}$$

$$k=2: \quad Y_{\mathbf{3}}^{(2)}(\tau) = \begin{pmatrix} -\sqrt{2}Y_{\mathbf{2''},1}^{(1)}Y_{\mathbf{2''},1}^{(1)} \\ Y_{\mathbf{2''},1}^{(1)}Y_{\mathbf{2''},2}^{(1)} + Y_{\mathbf{2''},2}^{(1)}Y_{\mathbf{2''},1}^{(1)} \\ \sqrt{2}Y_{\mathbf{2''},2}^{(1)}Y_{\mathbf{2''},2}^{(1)} \end{pmatrix}$$

$$k=3: \quad \dots.$$

❑ Inhomogeneous(Homogeneous) finite modular group:

$$\Gamma_N \equiv \mathrm{SL}(2,\mathbb{Z})/\pm\Gamma(N) = \{S,T|S^2 = (ST)^3 = T^N = 1\}$$

$$\Gamma_N' \equiv \mathrm{SL}(2,\mathbb{Z})/\Gamma(N) = \{S,T|S^4 = (ST)^3 = T^N = 1,\ S^2T = TS^2\}$$

<table>
<tr><th colspan="2">nature of symmetry</th><th>outer automorphism of Narain space group</th><th colspan="4">flavor groups</th></tr>
<tr><td rowspan="5">eclectic</td><td rowspan="2">modular</td><td>rotation S $\in$ SL$(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_4$</td><td colspan="2" rowspan="2">T'</td><td rowspan="5">$\Omega(1)$</td></tr>
<tr><td>rotation T $\in$ SL$(2,\mathbb{Z})_T$</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td rowspan="3">traditional flavor</td><td>translation A</td><td>$\mathbb{Z}_3$</td><td rowspan="2">$\Delta(27)$</td><td rowspan="3">$\Delta(54)$</td></tr>
<tr><td>translation B</td><td>$\mathbb{Z}_3$</td></tr>
<tr><td>rotation C $= \mathrm{S}^2$ $\in$ SL$(2,\mathbb{Z})_T$</td><td colspan="2">$\mathbb{Z}_2^R$</td></tr>
</table>

<table>
<tr><th>flavor group $\mathcal{G}_{\mathrm{fl}}$</th><th>GAP ID</th><th>Aut$(\mathcal{G}_{\mathrm{fl}})$</th><th colspan="2">finite modular groups</th><th>eclectic flavor group</th></tr>
<tr><td rowspan="2">Q_8</td><td rowspan="2">[8, 4]</td><td rowspan="2">S_4</td><td>without $\mathcal{CP}$</td><td>S_3</td><td>GL$(2,3)$</td></tr>
<tr><td>with $\mathcal{CP}$</td><td>–</td><td>–</td></tr>
<tr><td rowspan="2">$\mathbb{Z}_3\times\mathbb{Z}_3$</td><td rowspan="2">[9, 2]</td><td rowspan="2">GL$(2,3)$</td><td>without $\mathcal{CP}$</td><td>S_3</td><td>$\Delta(54)$</td></tr>
<tr><td>with $\mathcal{CP}$</td><td>$S_3\times\mathbb{Z}_2$</td><td>[108, 17]</td></tr>
<tr><td rowspan="2">A_4</td><td rowspan="2">[12, 3]</td><td rowspan="2">S_4</td><td>without $\mathcal{CP}$</td><td>S_3
S_4</td><td>S_4
S_4</td></tr>
<tr><td>with $\mathcal{CP}$</td><td>–</td><td>–</td></tr>
<tr><td rowspan="2">T'</td><td rowspan="2">[24, 3]</td><td rowspan="2">S_4</td><td>without $\mathcal{CP}$</td><td>S_3</td><td>GL$(2,3)$</td></tr>
<tr><td>with $\mathcal{CP}$</td><td>–</td><td>–</td></tr>
<tr><td rowspan="2">$\Delta(27)$</td><td rowspan="2">[27, 3]</td><td rowspan="2">[432, 734]</td><td>without $\mathcal{CP}$</td><td>S_3
T'</td><td>$\Delta(54)$
$\Omega(1)$</td></tr>
<tr><td>with $\mathcal{CP}$</td><td>$S_3\times\mathbb{Z}_2$
GL$(2,3)$</td><td>[108, 17]
[1296, 2891]</td></tr>
<tr><td rowspan="2">$\Delta(54)$</td><td rowspan="2">[54, 8]</td><td rowspan="2">[432, 734]</td><td>without $\mathcal{CP}$</td><td>T'</td><td>$\Omega(1)$</td></tr>
<tr><td>with $\mathcal{CP}$</td><td>GL$(2,3)$</td><td>[1296, 2891]</td></tr>
</table>

❑ Including (generalized) CP symmetry: [Baur, Nilles, Trautner, Vaudrevange, 1901.03251]

- Generalized CP transformation:

$$\begin{cases} \tau \xrightarrow{K_*} -\tau^* \\ \psi_I(t,\vec{x}) \xrightarrow{K_*} \rho_I(K_*)\psi_I^*(t,-\vec{x}) \end{cases} \qquad K_* = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- The consistency between the gCP and modular & traditional flavor

$$\begin{cases} \rho(K_*)\rho^*(\gamma)\rho^{-1}(K_*) = \rho\left(u_{K_*}(\gamma)\right) \text{ with } u_{K_*}(\gamma) = K_*\gamma K_*^{-1} \\ \rho(K_*)\rho^*(g)\rho^{-1}(K_*) = \rho\left(u_{K_*}(g)\right) \end{cases}$$

[Novichkov et al, 1905.11970; Ding et al, 2102.06716]

[Feruglio et al, JHEP 07 (2013) 027; Holthausen et al, JHEP 04 (2013) 122; Chen et al, Nucl.Phys.B 883 (2014) 267-305...]

- The automorphisms of G_f satisfy [Hans Peter Nilles et al, 2001.01736]

$$(u_S)^{N_s} = (u_T)^N = (u_S \circ u_T)^3 = 1, \qquad (u_S)^2 \circ u_T = u_T \circ (u_S)^2 ,$$
$$(u_{K_*})^2 = 1, \qquad u_{K_*} \circ u_S \circ u_{K_*} = u_S^{-1}, \qquad u_{K_*} \circ u_T \circ u_{K_*} = u_T^{-1} ,$$

➡ $G_{ecl} = \langle \rho(S), \rho(T), \rho(g), \rho(K_*) \rangle = G_{flavor} \cup G_{modular} \cup CP$

■ As an outs, u_γ generally maps one irrep of $\Delta(27)$ to another:

$$u_\gamma: \quad \boldsymbol{r_i} \longmapsto \boldsymbol{r_j}$$

u_S: $\mathbf{1}_{0,1} \to \mathbf{1}_{2,2} \to \mathbf{1}_{0,2} \to \mathbf{1}_{1,1} \to \mathbf{1}_{0,1}$; $\quad \mathbf{1}_{1,0} \to \mathbf{1}_{1,2} \to \mathbf{1}_{2,0} \to \mathbf{1}_{2,1} \to \mathbf{1}_{1,0}$,

u_T: $\mathbf{1}_{0,1} \to \mathbf{1}_{1,1} \to \mathbf{1}_{2,1} \to \mathbf{1}_{0,1}$; $\quad \mathbf{1}_{0,2} \to \mathbf{1}_{2,2} \to \mathbf{1}_{1,2} \to \mathbf{1}_{0,2}$;

	E	A^2B^2	A^2B	A	AB^2	AB	A^2	B^2	B	BAB^2A^2	ABA^2B^2
	$1C_1$	$3C_3^{(1)}$	$3C_3^{(2)}$	$3C_3^{(3)}$	$3C_3^{(4)}$	$3C_3^{(5)}$	$3C_3^{(6)}$	$3C_3^{(7)}$	$3C_3^{(8)}$	$1C_3^{(1)}$	$1C_3^{(2)}$
$\mathbf{1}_{0,0}$	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_{0,1}$	1	ω^2	ω	1	ω^2	ω	1	ω^2	ω	1	1
$\mathbf{1}_{0,2}$	1	ω	ω^2	1	ω	ω^2	1	ω	ω^2	1	1
$\mathbf{1}_{1,0}$	1	ω^2	ω^2	ω	ω	ω	ω^2	1	1	1	1
$\mathbf{1}_{1,1}$	1	ω	1	ω	1	ω^2	ω^2	ω^2	ω	1	1
$\mathbf{1}_{1,2}$	1	1	ω	ω	ω^2	1	ω^2	ω	ω^2	1	1
$\mathbf{1}_{2,0}$	1	ω	ω	ω^2	ω^2	ω^2	ω	1	1	1	1
$\mathbf{1}_{2,1}$	1	1	ω^2	ω^2	ω	1	ω	ω^2	ω	1	1
$\mathbf{1}_{2,2}$	1	ω^2	1	ω^2	1	ω	ω	ω	ω^2	1	1
$\mathbf{3}$	3	0	0	0	0	0	0	0	0	3ω	$3\omega^2$
$\bar{\mathbf{3}}$	3	0	0	0	0	0	0	0	0	$3\omega^2$	3ω

Outer automorphisms are symmetries of character table !

➢ Irreps matrices for T' :

$\boldsymbol{r = 1_{0,0}}$:

$$\rho_{\mathbf{1}^k}(S) = 1\ , \qquad \rho_{\mathbf{1}^k}(T) = \omega^k\ ,$$

$\boldsymbol{r = 3}$:

$$\rho_{\mathbf{3}_k}(S) = \frac{i}{\sqrt{3}}\begin{pmatrix} \omega^2 & \omega & \omega \\ \omega & \omega^2 & \omega \\ \omega^2 & \omega^2 & 1 \end{pmatrix}\ , \qquad \rho_{\mathbf{3}_k}(T) = \omega^k \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix}\ ,$$

$$\mathbf{3}_k = \mathbf{1}^{[k+1]} \oplus \mathbf{2}^{[k+2]}\ .$$

$\boldsymbol{r = \bar{3}}$:

$$\bar{\mathbf{3}}_k = \mathbf{1}^{[2-k]} \oplus \mathbf{2}^{[1-k]}\ .$$

$\boldsymbol{r = 8}$:

$$\rho_{\mathbf{8}_k}(S) = \begin{pmatrix} 0&0&0&0&0&0&0&1 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&1&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix}, \rho_{\mathbf{8}_k}(T) = \omega^k \begin{pmatrix} 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix},$$

$$\mathbf{8}_k = \mathbf{1}^k \oplus \mathbf{2}^{[k+1]} \oplus \mathbf{2}^{[k+2]} \oplus \mathbf{3},$$

- Irreps matrices for $\Delta(27)$:

$$
\begin{aligned}
\mathbf{1}_{r,s} &: \quad \rho_{\mathbf{1}_{r,s}}(A) = \omega^r , \qquad \rho_{\mathbf{1}_{r,s}}(B) = \omega^s , \quad \text{with} \quad r,s = 0,1,2 , \\
\mathbf{3} &: \quad \rho_{\mathbf{3}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \qquad \rho_{\mathbf{3}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} , \\
\bar{\mathbf{3}} &: \quad \rho_{\bar{\mathbf{3}}}(A) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} , \qquad \rho_{\bar{\mathbf{3}}}(B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} .
\end{aligned}
$$

- The correspondence between (reducible/irreducible) representations:

$\Delta(27)$ **reps**:	$\mathbf{1}_{0,0}$,	$\mathbf{1}_{0,1} \oplus \cdots \oplus \mathbf{1}_{2,2}, = \mathbf{8}$,	$\mathbf{3}$,	$\overline{\mathbf{3}}$
$\times$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
T' **reps**:	$\mathbf{1}^i$,	$\mathbf{1}^i \oplus \mathbf{2}^{[i+1]} \oplus \mathbf{2}^{[i+2]} \oplus \mathbf{3} = \mathbf{8}_i$,	$\mathbf{1}^{[i+1]} \oplus \mathbf{2}^{[i+2]} = \mathbf{3}_i$,	$\mathbf{1}^{[2-i]} \oplus \mathbf{2}^{[1-i]} = \overline{\mathbf{3}}_i$
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
$\Omega(1)$ **irreps**:	$\mathbf{1}_i$,	$\mathbf{8}_i$,	$\mathbf{3}_i$,	$\overline{\mathbf{3}}_i$, ...