Off Shell Gluon Amplitudes in the CHY Formalism

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Overview

1. Amplitudes from QFT
2. CHY Formalism
3. Additional Terms to CHY Formulae

Contributions

Many thanks to Dr. Louise Dolan at UNC Chapel Hill for her guidance on this project.

Amplitudes from QFT

Interaction Lagrangian

- Interaction Lagrangian for Gluons

$$\mathcal{L}_I = -g f_{abc} A_\mu^b A_\nu^c \partial^\mu A^{\nu a} - \frac{g^2}{4} f_{abc} f_{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e} \tag{1}$$

- Four-Point Amplitude

$$T^{abcd}_{\mu\nu\rho\sigma}(x_1, x_2, x_3, x_4) = \frac{\langle 0| T A_\mu^a(x_1) A_\nu^b(x_2) A_\rho^c(x_3) A_\sigma^d(x_4) \exp\left(i \int \mathrm{d}^4 x \, \mathcal{L}_I\right) |0\rangle}{\langle 0| \exp\left(i \int \mathrm{d}^4 x \, \mathcal{L}_I\right) |0\rangle} \tag{2}$$

Gauge Dependence Off Shell

- Gluon Propagator in Momentum Space

$$\tilde{D}^{ab}_{\mu\nu}(k) = -\frac{i}{k^2}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2}\right]\delta^{ab} \tag{3}$$

- Gauge Choices
  - Feynman Gauge: $\alpha = 1$
  - Landau Gauge: $\alpha = 0$
- Amplitudes are gauge dependent off shell!
- Will work in Feynman Gauge

4-pt Tree-Level Gluon Scattering (Partial) Amplitude

$$
\begin{aligned}
&\frac{1}{s}\Bigg[\epsilon_3\cdot\epsilon_4\Bigg\{\frac{1}{4}\Bigg(-2(s+t)+(k_1^2+k_2^2+k_3^2+k_4^2)\Bigg)\epsilon_1\cdot\epsilon_2 \\
&\qquad +\epsilon_1\cdot k_2\epsilon_2\cdot k_3-\epsilon_2\cdot k_1\epsilon_1\cdot k_3\Bigg\} \\
&\quad -\epsilon_4\cdot k_3\big(\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_3-\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_3+\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_2\big) \\
&\quad +\epsilon_3\cdot k_4\big(\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_4-\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_4+\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_2\big)\Bigg] \\
&+\frac{1}{t}\Bigg[\epsilon_1\cdot\epsilon_4\Bigg\{\frac{1}{4}\Bigg(-2(s+t)+(k_1^2+k_2^2+k_3^2+k_4^2)\Bigg)\epsilon_2\cdot\epsilon_3 \\
&\qquad +\epsilon_2\cdot k_3\epsilon_3\cdot k_4-\epsilon_3\cdot k_2\epsilon_2\cdot k_4\Bigg\} \\
&\quad -\epsilon_1\cdot k_4\big(\epsilon_2\cdot k_3\epsilon_3\cdot\epsilon_4-\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4+\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3\big) \\
&\quad +\epsilon_4\cdot k_1\big(\epsilon_2\cdot k_3\epsilon_3\cdot\epsilon_1-\epsilon_3\cdot k_2\epsilon_1\cdot\epsilon_2+\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_3\big)\Bigg] \\
&+\frac{1}{2}\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4
\end{aligned}
$$

CHY Formalism

Historical Overview

- *Scattering of Massless Particles in Arbitrary Dimension*; Cachazo, He, Yuan (2013) arXiv:1307.2199
  - Tree-level S-matrix of pure Yang-Mills and gravity theories for N particles
- *The Polynomial Form of the Scattering Equations*; Dolan, Goddard (2014) arXiv:1402.7374
  - Equivalent set of $N-3$ Möbius covariant polynomials, **show uniqueness of scattering equations as representation of Möbius algebra** satisfying $\tilde{h}_m = 0$
- *Off-Shell CHY Amplitudes and Feynman Graphs*; Dolan, Goddard (2019) arXiv:1910.12791
  - Polynomial form of scattering equations for off-shell scalar ϕ^3

Möbius Transformations

$$z \to z' = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{C}, \quad \alpha\delta - \beta\gamma \neq 0 \tag{4}$$

$$\frac{1}{z_i - z_j} \equiv \frac{1}{z_{ij}} \to \frac{(\gamma z_i + \delta)(\gamma z_j + \delta)}{(\alpha\delta - \beta\gamma)} \frac{1}{z_{ij}} \tag{5}$$

$$\mathrm{d}z \to \frac{\alpha\delta - \beta\gamma}{(\gamma z + \delta)^2} \, \mathrm{d}z \tag{6}$$

- Freedom allows us to send three coordinates to fixed points, for $N = 4$ we only integrate over z_3, so send:

$$z_1 \to \infty, \quad z_2 \to 1, \quad z_3 = z, \quad z_4 \to 0 \tag{7}$$

Pfaffian

$$\mathrm{Pf}(A) = \sqrt{\det A} \tag{8}$$

- In CHY, matrix needed for on-shell Yang-Mills is:

$$\Psi_N = \begin{bmatrix} A & C^T \\ C & B \end{bmatrix}, \quad A_{aa} = B_{aa} = 0, \quad C_{aa} = -\sum_{c=1, c\neq a}^{N} \frac{\epsilon_a \cdot k_a}{z_{ac}},$$
$$A_{ab} = \frac{k_a \cdot k_b}{z_{ab}}, \quad B_{ab} = \frac{\epsilon_a \cdot \epsilon_b}{z_{ab}}, \quad C_{ab} = \frac{\epsilon_a \cdot k_b}{z_{ab}},$$
$$a \neq b, \quad 1 \leq a, b \leq N$$

- Define:

$$\mathrm{Pf}'(\Psi_N) = \frac{2(-1)^{i+j}}{z_{ij}} \mathrm{Pf}(\Psi_{N(ij)}), \quad \hat{\Psi}_N = \mathrm{Pf}'(\Psi_N) \prod_{a=1}^{N} (z_a - z_{a+1})$$

- (i, j) means remove i^{th} and j^{th} row/column (result is independent of choice for i, j)

Polynomial Form of Scattering Equations Off Shell

- Define a few sets and quantities:

$$[i,j] = \{a : i \le a \le j\}, \quad 1 \le i \le j < N$$

$$[i,j]^o = \{i-1, i, j, j+1\}^C$$

$$A = \{1, \ldots, N\}$$

$$S \subset A, \quad k_S = \sum_{b \in S} k_b$$

$$\Pi_V^n = \sum_{\substack{i_1 < i_2 < \ldots < i_n \\ i_a \in V}} z_{i_1} z_{i_2} \ldots z_{i_n}$$

- Off-Shell Polynomials:

$$\tilde{h}_m = \sum_{\substack{1 \le i < j < N \\ (i,j) \neq (1,N-1)}} k_{[i,j]}^2 (z_i - z_{i-1})(z_j - z_{j+1}) \Pi_{[i,j]^o}^{m-2} \quad 2 \le m \le N-2$$

Contour Integrals

- Off-shell ϕ^3 amplitudes

$$\mathcal{A}_N^{\phi^3} = \int_{\mathcal{O}} \prod_{m=2}^{N-2} \frac{1}{\tilde{h}_m(z,k)} \prod_{a<b} z_{ab} \prod_{a\in A} \frac{\mathrm{d}z_a}{(z_a - z_{a+1})^2} \Big/ \mathrm{d}\omega \qquad (9)$$

$$\mathrm{d}\omega = \frac{\mathrm{d}z_r\, \mathrm{d}z_s\, \mathrm{d}z_t}{z_{rs} z_{rt} z_{st}}$$

- Contour $\mathcal{O}$ is over solutions to scattering equations $\tilde{h}_m(z,k) = 0$
- To get YM amplitudes, would usually just insert $\hat{\Psi}_N$, won't work here!
- Mismatch from computation will occur in terms included in A matrix

Null Vector of A

- Recall definition of A matrix

$$A_{aa} = 0, \quad A_{ab} = \frac{k_a \cdot k_b}{z_{ab}}$$

- A has null vector $(1, ..., 1)$ (if on-shell CHY scattering equations are satisfied)
- Posit that this should be the case off-shell, replace on-shell scattering equations with off-shell, then find conditions for generic A
- For $N = 4$, we find:

$$A_{13} = f_1 + f_2 - A_{14} - A_{23} - A_{24}, \quad A_{12} = A_{23} + A_{24} - f_2$$
$$A_{34} = -A_{14} - A_{24} - f_4, \quad f_1 + f_2 + f_3 + f_4 = 0$$

New A Matrix

- One solution to set of equations for A_{ij} is:

$$A_{12} = \frac{1}{2}\frac{s_{12}}{z_{12}}, \quad A_{13} = -\frac{1}{2}\frac{s_{12}+s_{23}}{z_{13}}, \quad A_{14} = \frac{1}{2}\frac{s_{23}}{z_{14}},$$
$$A_{23} = \frac{1}{2}\frac{s_{23}}{z_{23}}, \quad A_{24} = -\frac{1}{2}\frac{s_{12}+s_{23}}{z_{24}}, \quad A_{34} = \frac{1}{2}\frac{s_{12}}{z_{34}}$$

4-pt Amplitude Off Shell (1)

$$\mathcal{A}_4^{\text{off-shell}} = \int_{\mathcal{O}} \frac{\hat{\Psi}_4'}{\tilde{h}_2(z,k)} (z_{13} z_{24}) \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_3\,\mathrm{d}z_4}{z_{12} z_{23} z_{34} z_{14}} \Big/ \mathrm{d}\omega \qquad (10)$$

- Choose to integrate over z_3

$$\mathrm{d}\omega = \lim_{z_1 \to \infty} \frac{\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}z_4}{(z_1 - 1)(z_1)}$$

- $\tilde{h}_2(z,k)$ will turn into a simple pole, contour integral is trivial

4-pt Amplitude Off Shell (2)

$$
\begin{aligned}
&A_4^{\mathrm{off-shell}} \\
&= \frac{1}{s}\Bigg[\epsilon_3\cdot\epsilon_4\Bigg(\frac{1}{4}\Bigg(-2(s+t)\Bigg)\epsilon_1\cdot\epsilon_2+\epsilon_1\cdot k_2\epsilon_2\cdot k_3-\epsilon_2\cdot k_1\epsilon_1\cdot k_3\Bigg) \\
&\quad -\epsilon_4\cdot k_3(\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_3-\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_3+\epsilon_3\cdot k_1\epsilon_1\cdot\epsilon_2) \\
&\quad +\epsilon_3\cdot k_4(\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_4-\epsilon_2\cdot k_1\epsilon_1\cdot\epsilon_4+\epsilon_4\cdot k_1\epsilon_1\cdot\epsilon_2)\Bigg] \\
&+\frac{1}{t}\Bigg[\epsilon_1\cdot\epsilon_4\Bigg(\frac{1}{4}\Bigg(-2(s+t)\Bigg)\epsilon_2\cdot\epsilon_3+\epsilon_2\cdot k_3\epsilon_3\cdot k_4-\epsilon_3\cdot k_2\epsilon_2\cdot k_4) \\
&\quad -\epsilon_1\cdot k_4(\epsilon_2\cdot k_3\epsilon_3\cdot\epsilon_4-\epsilon_3\cdot k_2\epsilon_2\cdot\epsilon_4+\epsilon_4\cdot k_2\epsilon_2\cdot\epsilon_3\Big) \\
&\quad +\epsilon_4\cdot k_1(\epsilon_2\cdot k_3\epsilon_3\cdot\epsilon_1-\epsilon_3\cdot k_2\epsilon_1\cdot\epsilon_2+\epsilon_1\cdot k_2\epsilon_2\cdot\epsilon_3)\Bigg] \\
&+\frac{1}{2}\epsilon_1\cdot\epsilon_3\epsilon_2\cdot\epsilon_4
\end{aligned}
\tag{11}
$$

Which Gauge?

- Never chose a gauge for CHY computation, but no α in result?
- Landau gauge would leave $k_i^2 k_j^2 / s_{ij}$ terms in amplitude, not possible to get from CHY in current form
- Maybe giving answers in Feynman gauge? This seems to be closest match, but the off-shell CHY still didn't give correct answer...

Additional Terms to CHY Formulae

Mismatch

- Term that is missing from CHY answer is

$$\frac{1}{4s}\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot \epsilon_4 (k_1^2 + k_2^2 + k_3^2 + k_4^2) + \frac{1}{4t}\epsilon_1 \cdot \epsilon_4 \epsilon_2 \cdot \epsilon_3 (k_1^2 + k_2^2 + k_3^2 + k_4^2)$$

- Issue again seems to lie in A matrix

Additional Terms

- Add to Pfaffian of reduced matrix, $\mathrm{Pf}(\Psi_{4(2,4)})$, new terms

$$\frac{1}{4}\frac{1}{z_{13}}\frac{\epsilon_1\cdot\epsilon_2\epsilon_3\cdot\epsilon_4}{z_{12}z_{34}}(k_1^2+k_2^2+k_3^2+k_4^2)$$
$$+\frac{1}{4}\frac{1}{z_{13}}\frac{\epsilon_2\cdot\epsilon_3\epsilon_1\cdot\epsilon_4}{z_{23}z_{14}}(k_1^2+k_2^2+k_3^2+k_4^2)$$

- Transforms the same was as $\mathrm{Pf}(\Psi_{4(2,4)})$ under Möbius transformations
- Generates missing terms needed to match CHY to QFT answer in Feynman gauge

How to Incorporate these Terms?

- Polarization tensors indicate modification to A needed
- Use BCFW recursion relations instead?
    - Originally formualted on-shell, but there have been papers extending reccurence relations off-shell
    - Could then work on simple $N = 3$ case, use BCFW to build up arbitrary N from there

Questions?