Probing String-Modified Gravity in Neutron Stars

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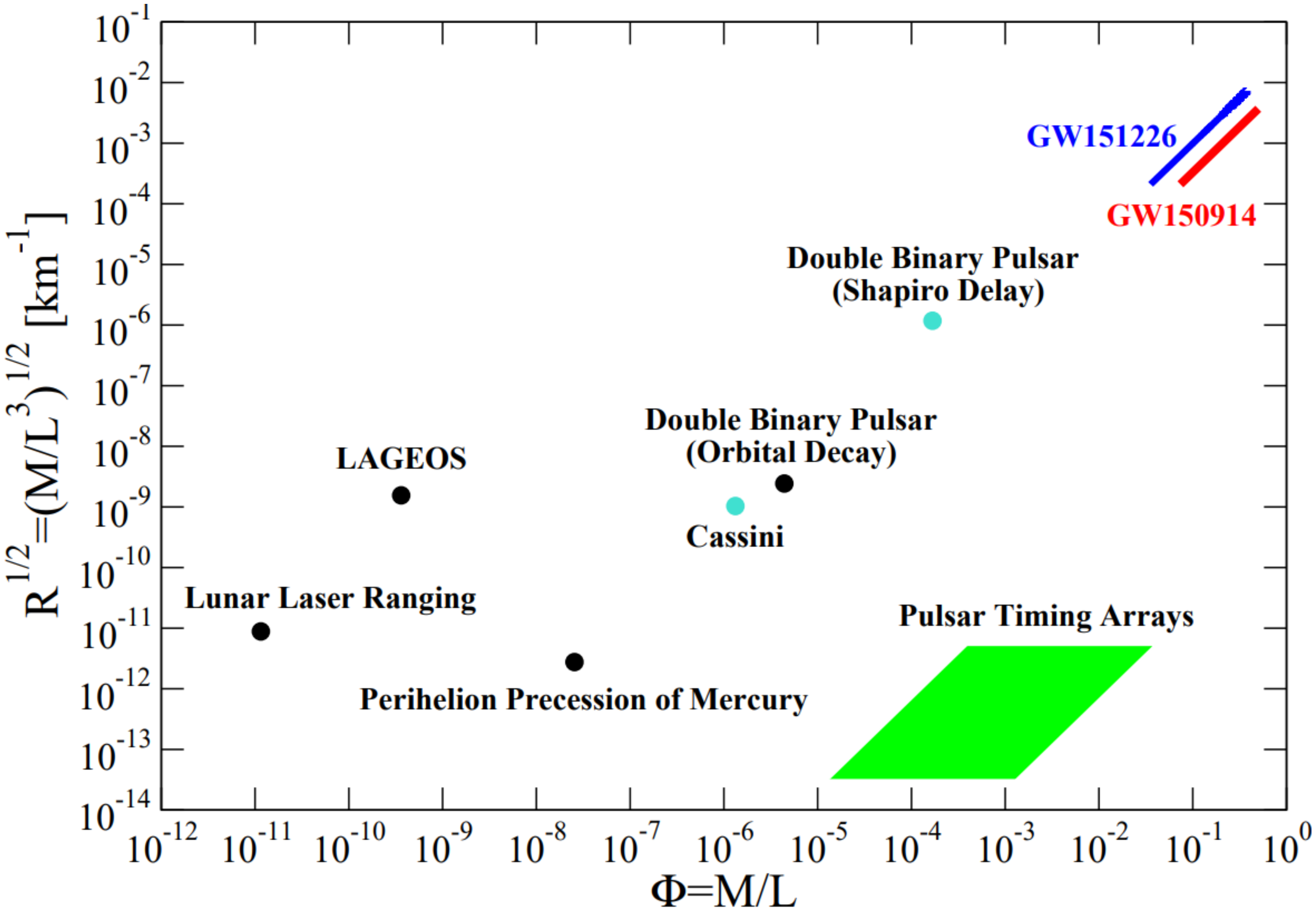
OUTLINE

- Background
  - Strong gravity regime
  - Binary NS systems
  - TOV equations
- Scalar-tensor theories
- Field Equations
- Ongoing work
- Summary

STRONG GRAVITY REGIME

Yagi, Yunes and Pretorius, 2016 PRD (ArXiv:1603.08955)



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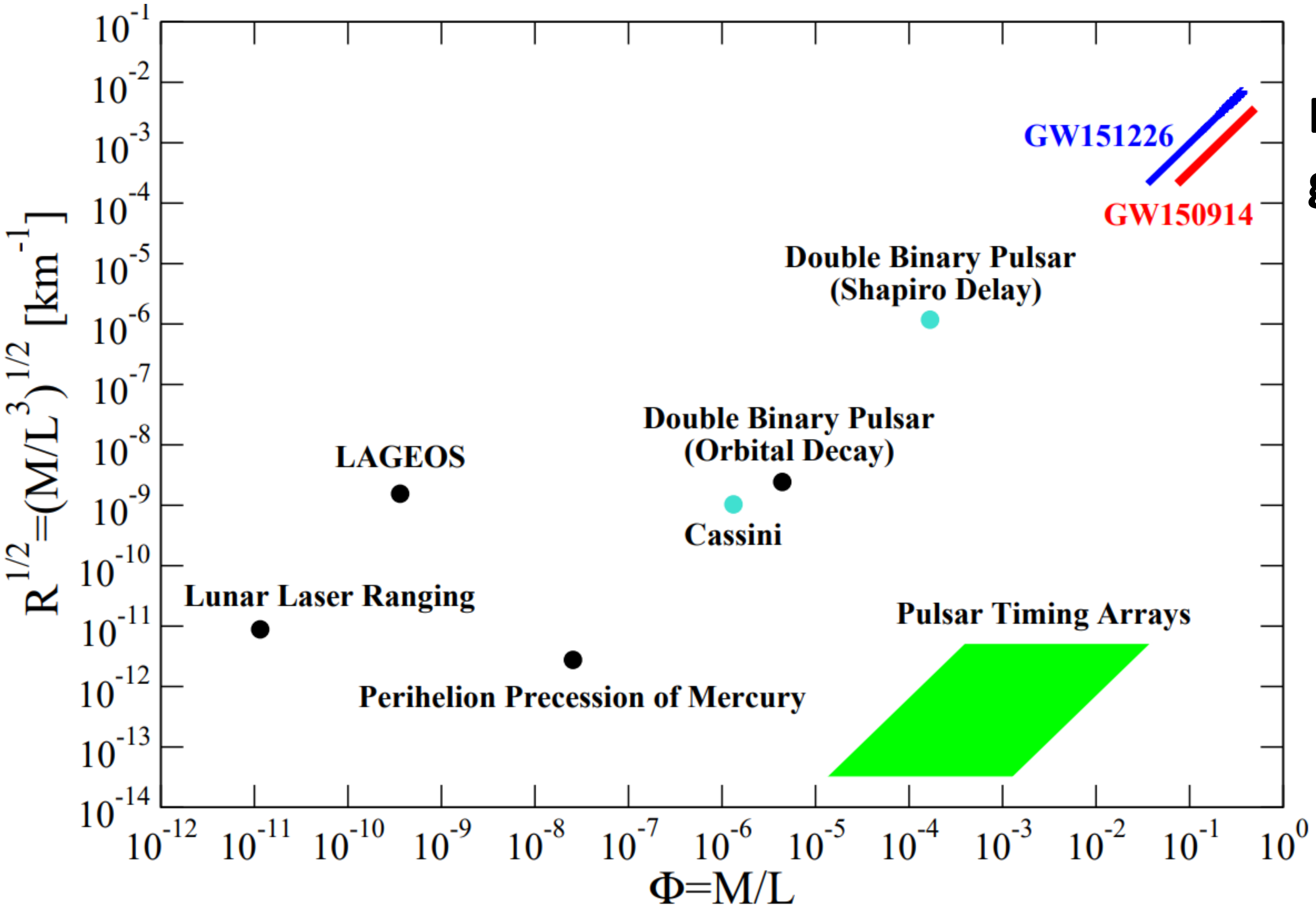
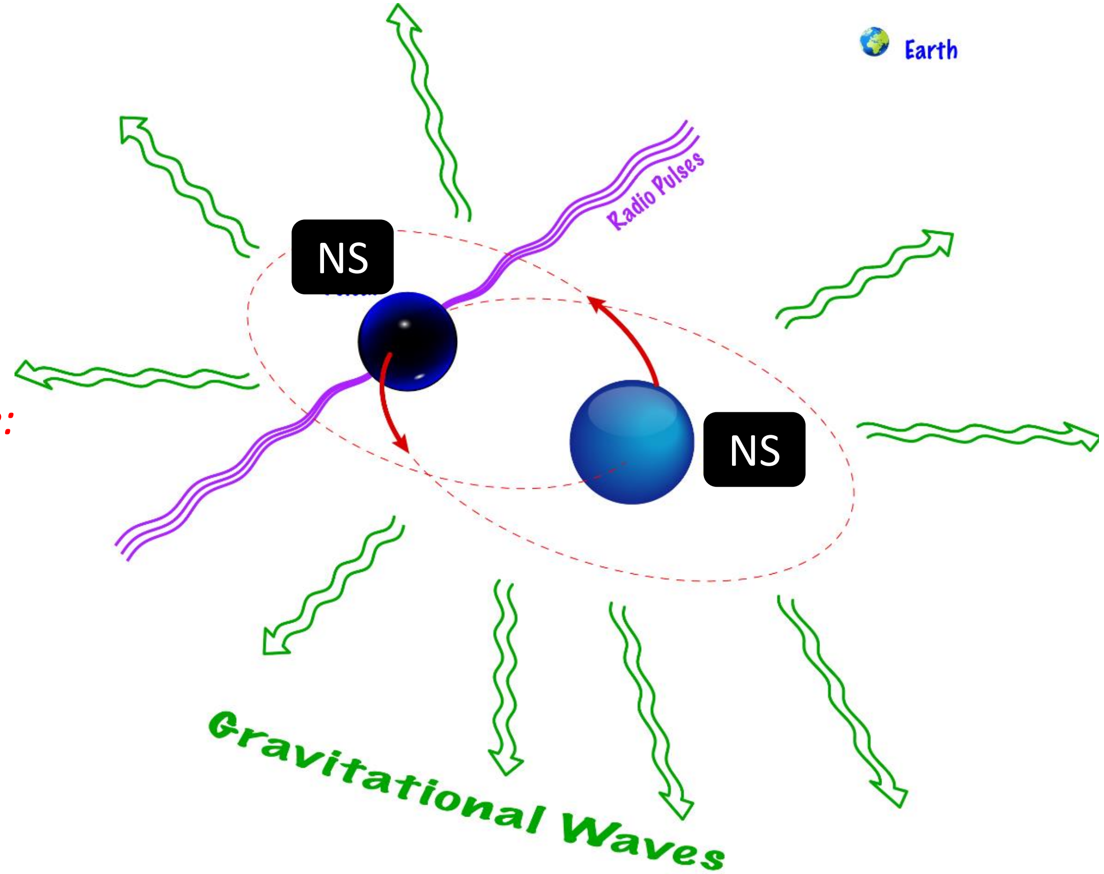
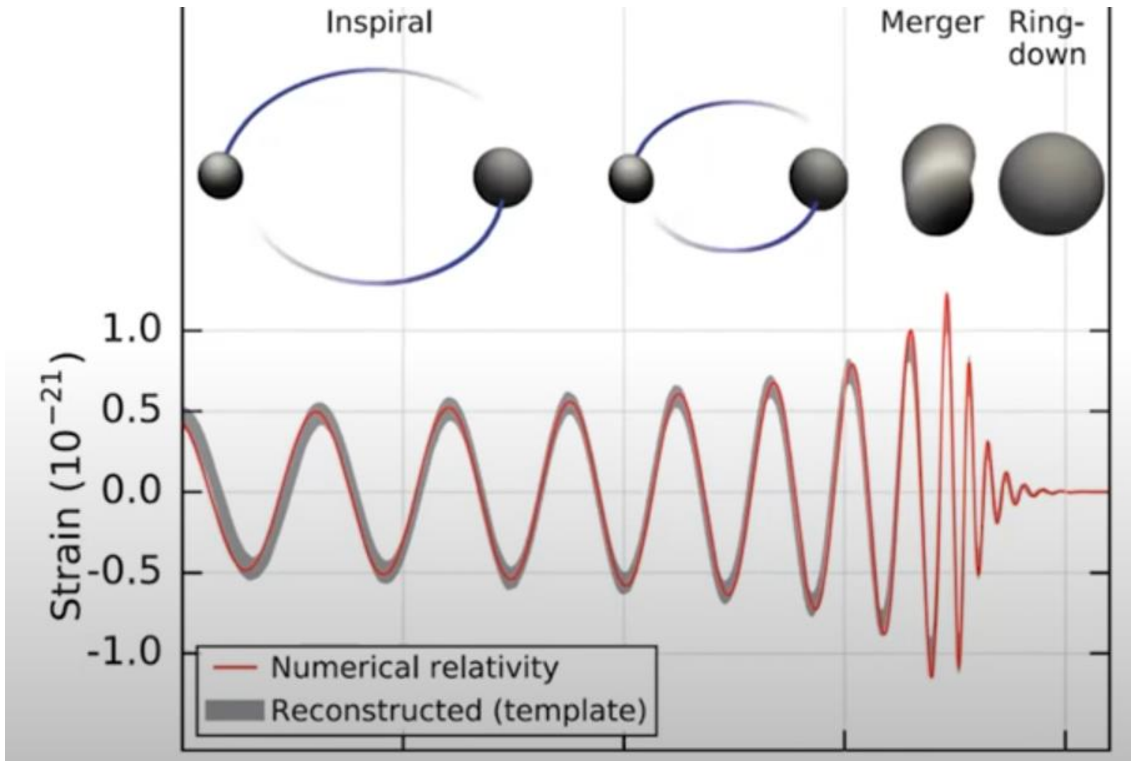


Direct probe of gravity

BINARY NS SYSTEMS



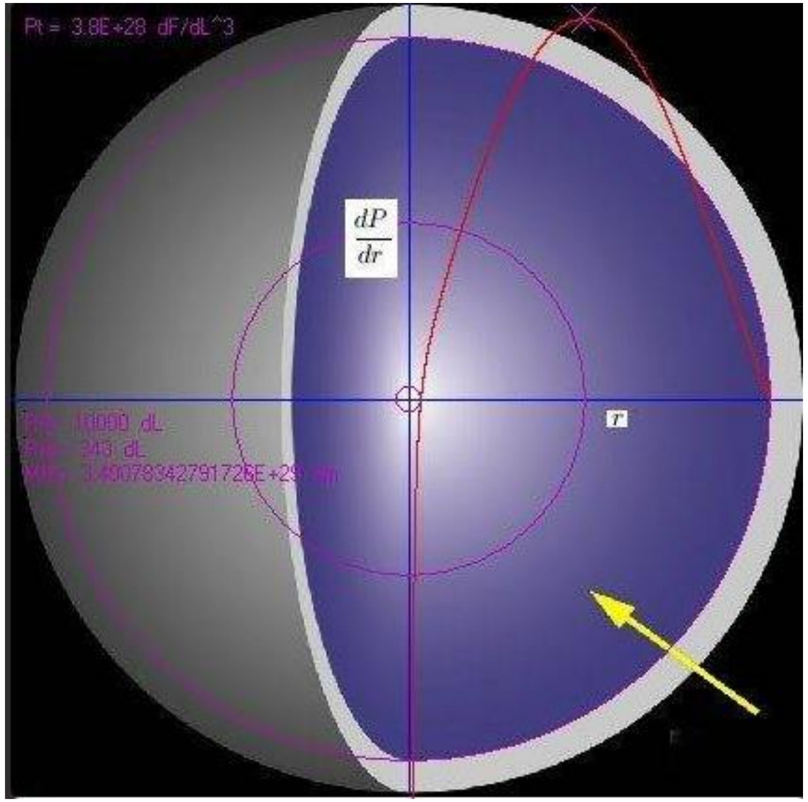
Source: NASA



Source: Kent Yagi (U. Virginia)

TOV EQUATIONS

- Describes gravitational pressure of a spherically symmetric compact object in equilibrium



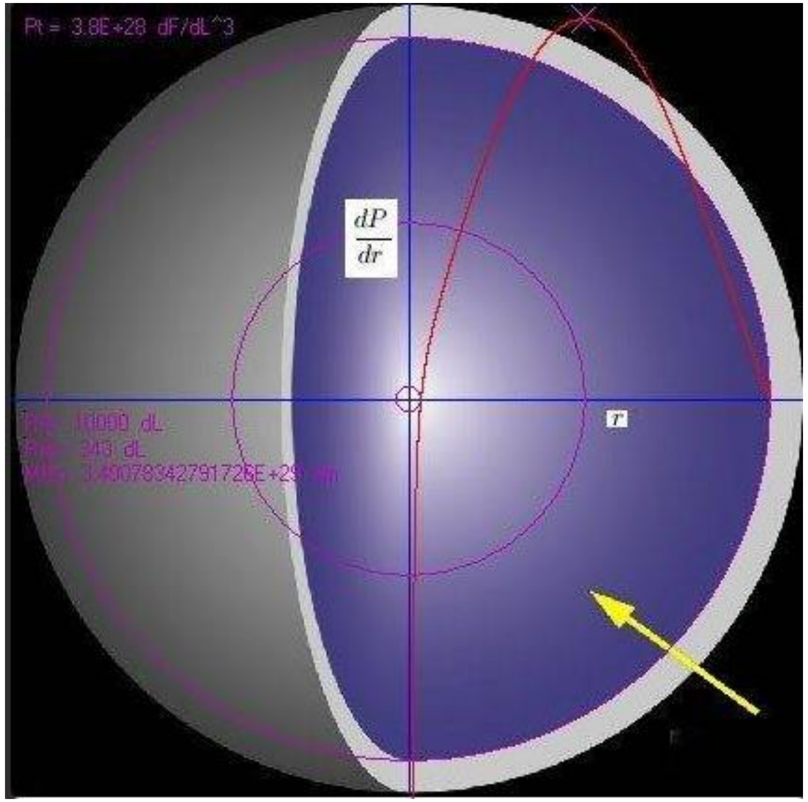
Source: Physics Forums

TOV EQUATIONS

- Describes gravitational pressure of a spherically symmetric compact object in equilibrium
- General form of NS metric:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $$e^{-\lambda(r)} = 1 - \frac{2m(r)}{r}$$



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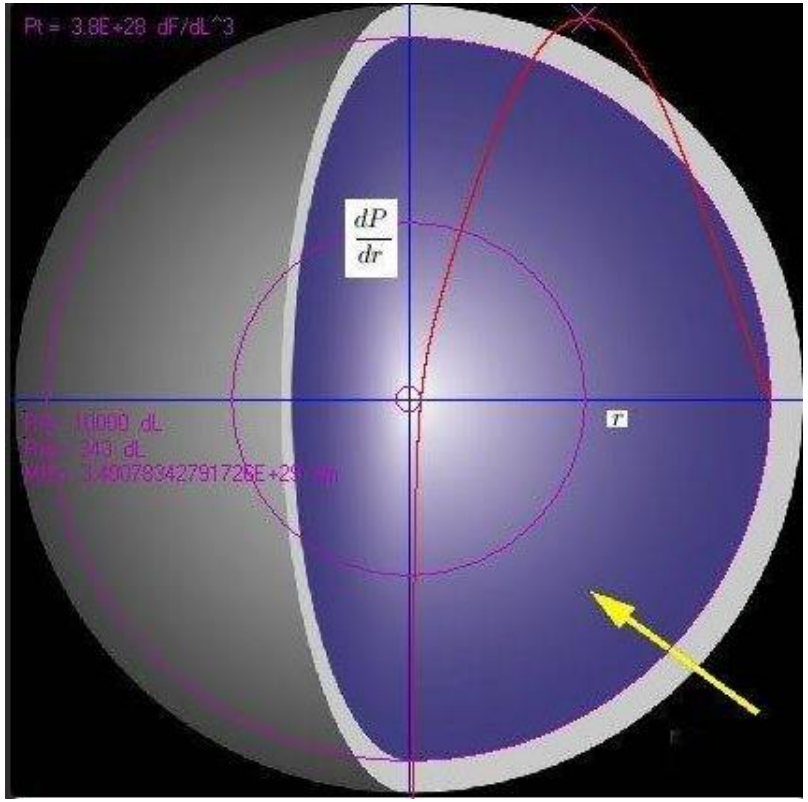
- **Will be modified if corrections to GR**

  → Can be used to probe new physics (via GWs)



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DCS + EDGB GRAVITY

- New: Dynamical Chern-Simons gravity (dCS) and Einstein-dilaton-Gauss-Bonnet gravity (EdGB)

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$$S = \frac{1}{16\pi}\int d^4x\sqrt{|g|}\left\{R + \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}e^{2\phi}(\partial\varphi)^2 + \frac{\alpha'}{8}\Big(\underbrace{e^{-\phi}\mathcal{X}_4}_{\text{EdGB}} - \underbrace{\varphi R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}}_{\text{dCS}}\Big) + \mathcal{O}(\alpha'^2)\right\}$$

$$+ S_{\rm mat}\left[\psi, e^{\phi}g_{\mu\nu}\right].$$

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- Can we derive observational constraints?

Transform action from Jordan to Einstein frame → Field and TOV equations → Solve TOV numerically (current progress) → Observe modifications to mass-radius relation, universal relations

FIELD EQUATIONS

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$$\nabla^2\phi = e^{2\phi}(\partial\varphi)^2 - \frac{\alpha'}{8}e^{-\phi}\mathcal{X}_4 - 8\pi T,$$

$$\nabla_\mu(e^{2\phi}\nabla^\mu\varphi) = -\frac{\alpha'}{8}R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma},$$

$$G_{\mu\nu} + \frac{\alpha'}{8}\left(D^{(\phi)}_{\mu\nu} + 2C_{\mu\nu}\right) = 8\pi\left(T^{\mathrm{mat}}_{\mu\nu} + T^{(\phi)}_{\mu\nu} + T^{(\varphi)}_{\mu\nu}\right),$$

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where

$$T^{(\varphi)}_{\mu\nu} = e^{2\phi}\nabla_\mu\varphi\nabla_\nu\varphi - \frac{1}{2}g_{\mu\nu}e^{2\phi}\nabla_a\varphi\nabla^a\varphi,$$

$$T^{\mathrm{mat}}_{\mu\nu} = e^{2\phi}\Big[(\rho + P)u_\mu u_\nu + g_{\mu\nu}P\Big],$$

$$T \equiv g^{\mu\nu}T^{\mathrm{mat}}_{\mu\nu}.$$

SOLVING TOV EQUATIONS: PROCEDURE

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Impose initial conditions for mass, pressure and scalar field central values

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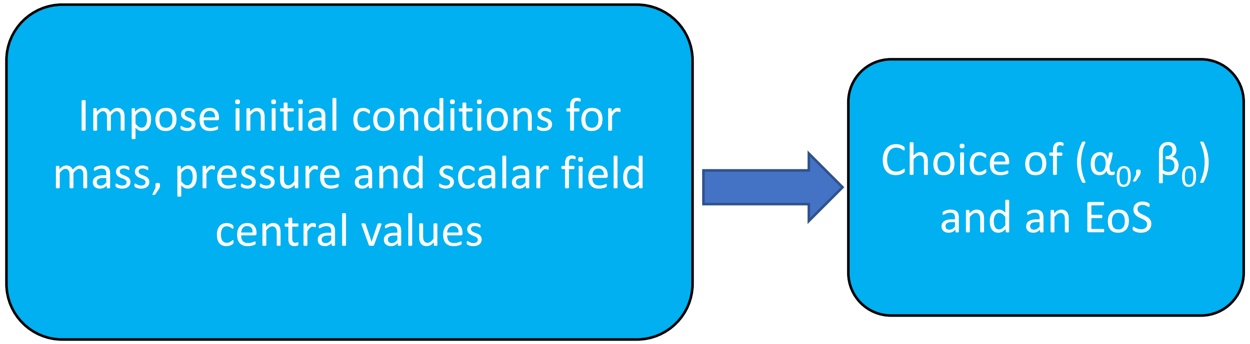


SOLVING TOV EQUATIONS: PROCEDURE

Impose initial conditions for mass, pressure and scalar field central values

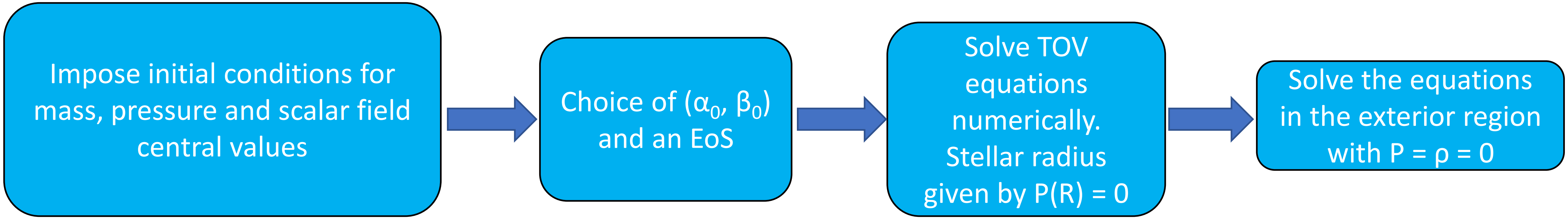
→

Choice of (α_0, β_0) and an EoS

→

Solve TOV equations numerically. Stellar radius given by $P(R) = 0$

SOLVING TOV EQUATIONS: PROCEDURE



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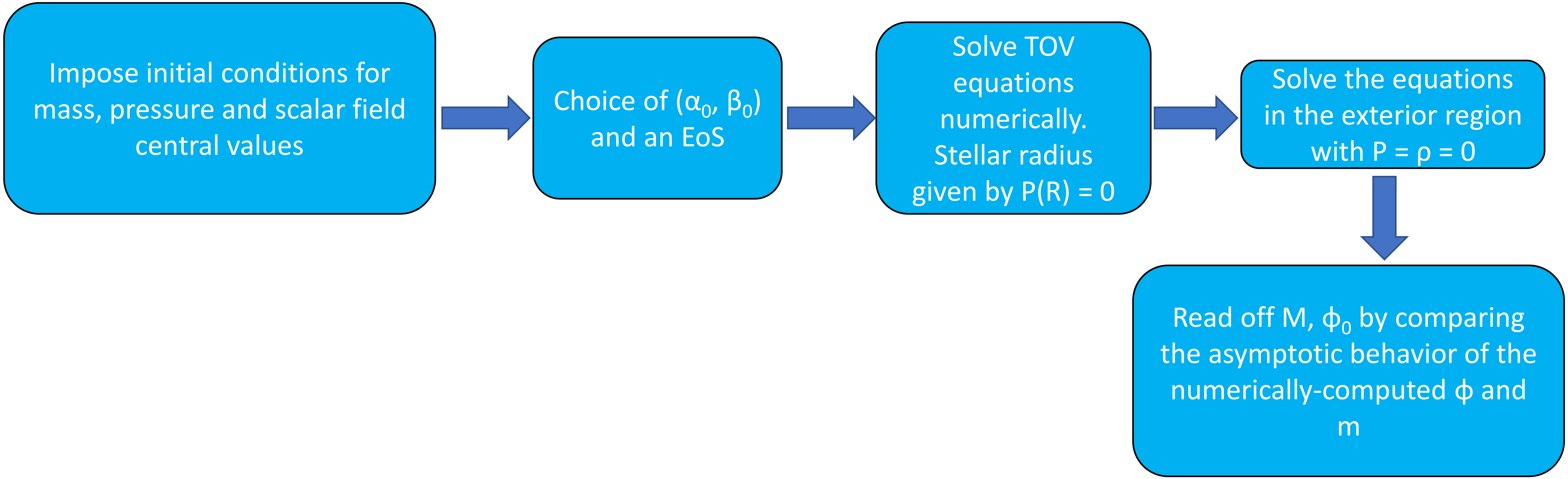

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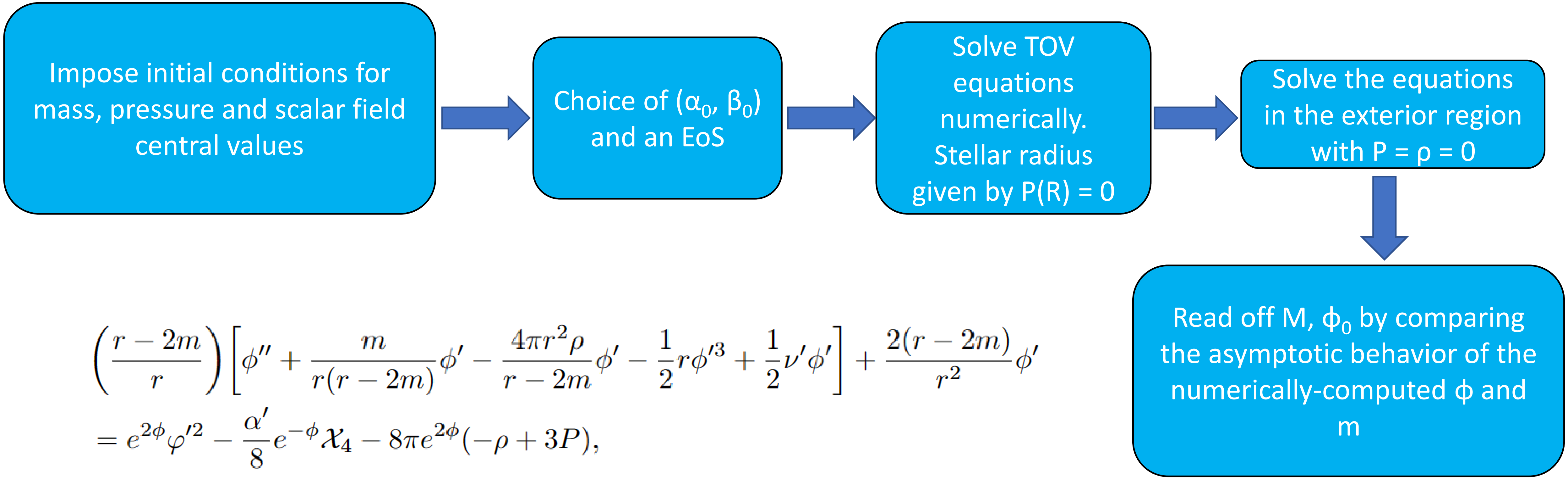

$$\left(\frac{r-2m}{r}\right)\left[\phi'' + \frac{m}{r(r-2m)}\phi' - \frac{4\pi r^2 \rho}{r-2m}\phi' - \frac{1}{2}r\phi'^3 + \frac{1}{2}\nu'\phi'\right] + \frac{2(r-2m)}{r^2}\phi'$$
$$= e^{2\phi}\varphi'^2 - \frac{\alpha'}{8}e^{-\phi}\mathcal{X}_4 - 8\pi e^{2\phi}(-\rho + 3P),$$

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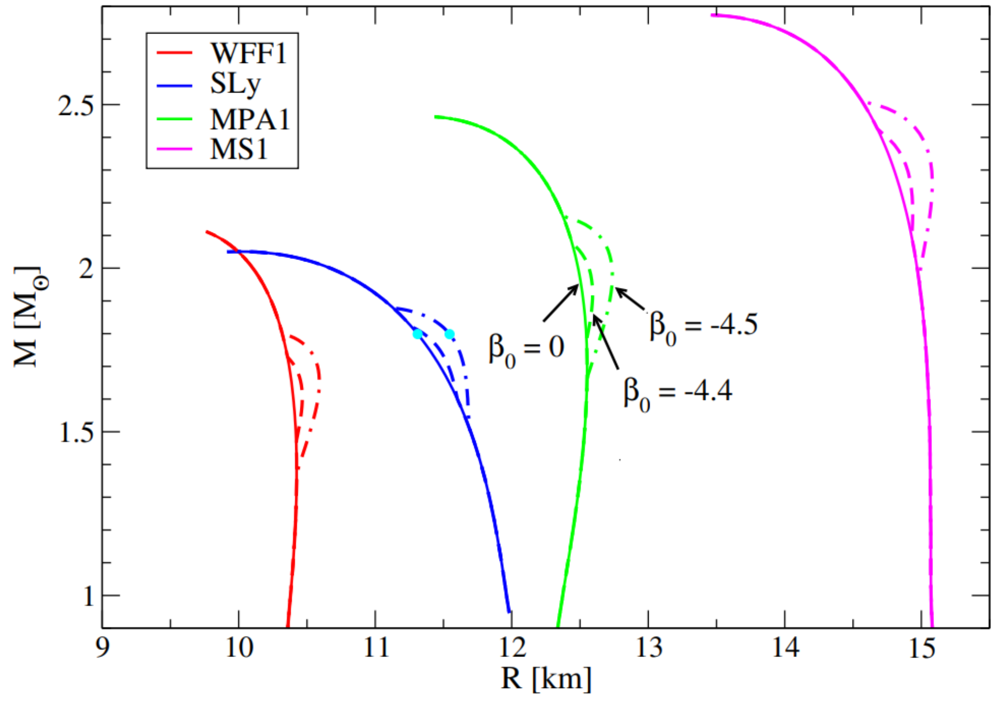

Yagi and Stepniczka, 2021 (Phys. Rev. D 104, 044017)

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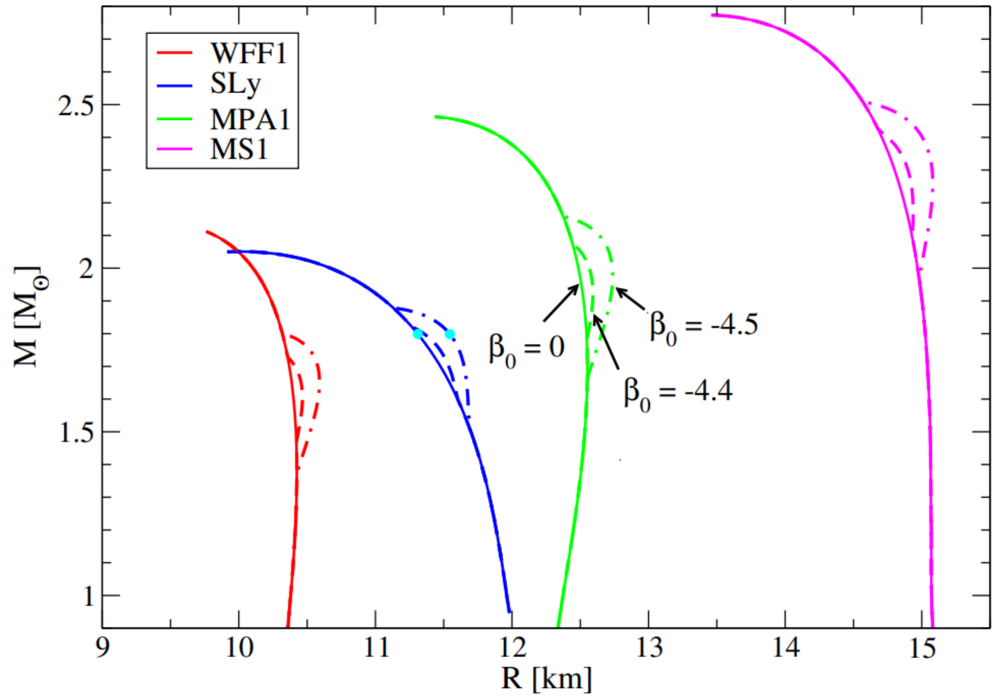

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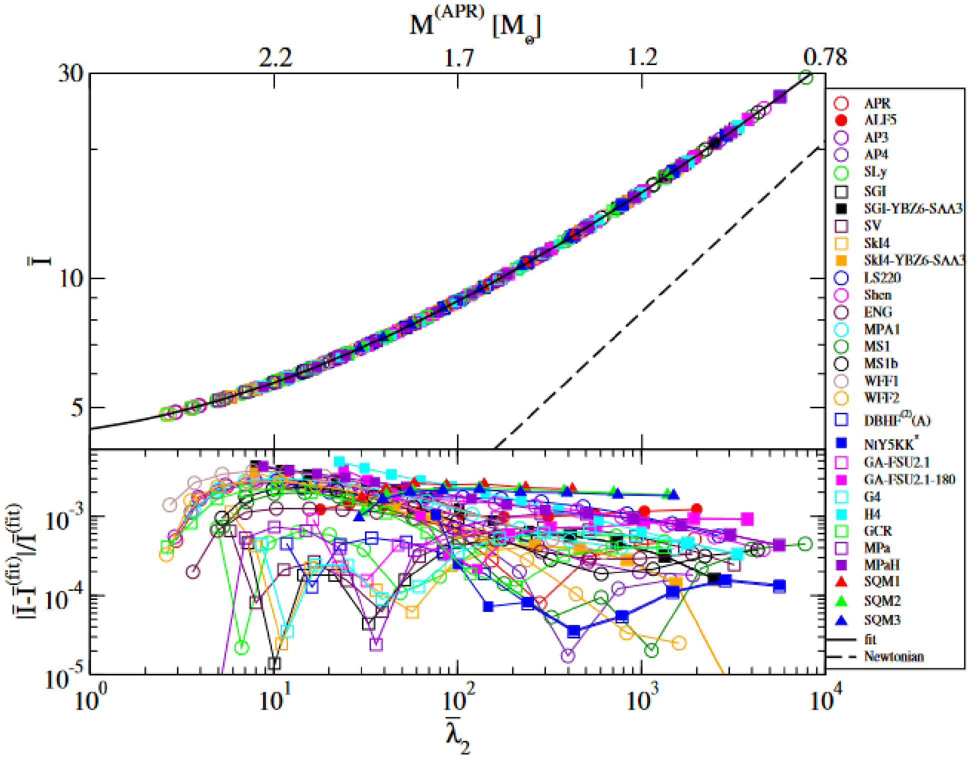

Yagi and Yunes, 2017
(Physics Reports, Volume 681)

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SUMMARY

- **Scalar-tensor theories** are modified theories of gravity
- Observational constraints placed on scalar-tensor theories from weak gravity tests, but not strong gravity
- Considering two ST theories (Chern-Simons and Gauss-Bonnet), which are well-motivated from string theory, to probe strong gravity using binary NS systems
- GW deviations from GR would be an interesting hint that GR needs modifications in strong gravity regime

BACKUP SLIDES

SCALAR-TENSOR THEORIES

- Introduce scalar fields into action

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- Introduce scalar fields into action
- A benchmark theory (Damour and Esposito-Varèse, 1996 PRD):

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi} \left(R - \underbrace{2g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi}_{\text{Scalar field coupling to metric}}\right) + S_{\text{mat}}\left[\psi, \underbrace{A^2(\varphi)}_{\text{“Coupling function”}} g_{\mu\nu}\right]$$

Scalar field coupling to metric

“Coupling function”

FIELD EQUATIONS OF ST THEORIES

$$R_{\mu\nu} = 2\partial_\mu\varphi\partial_\nu\varphi + 8\pi\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right),$$

$$\Box\varphi = -4\pi\alpha(\varphi)T\,,$$

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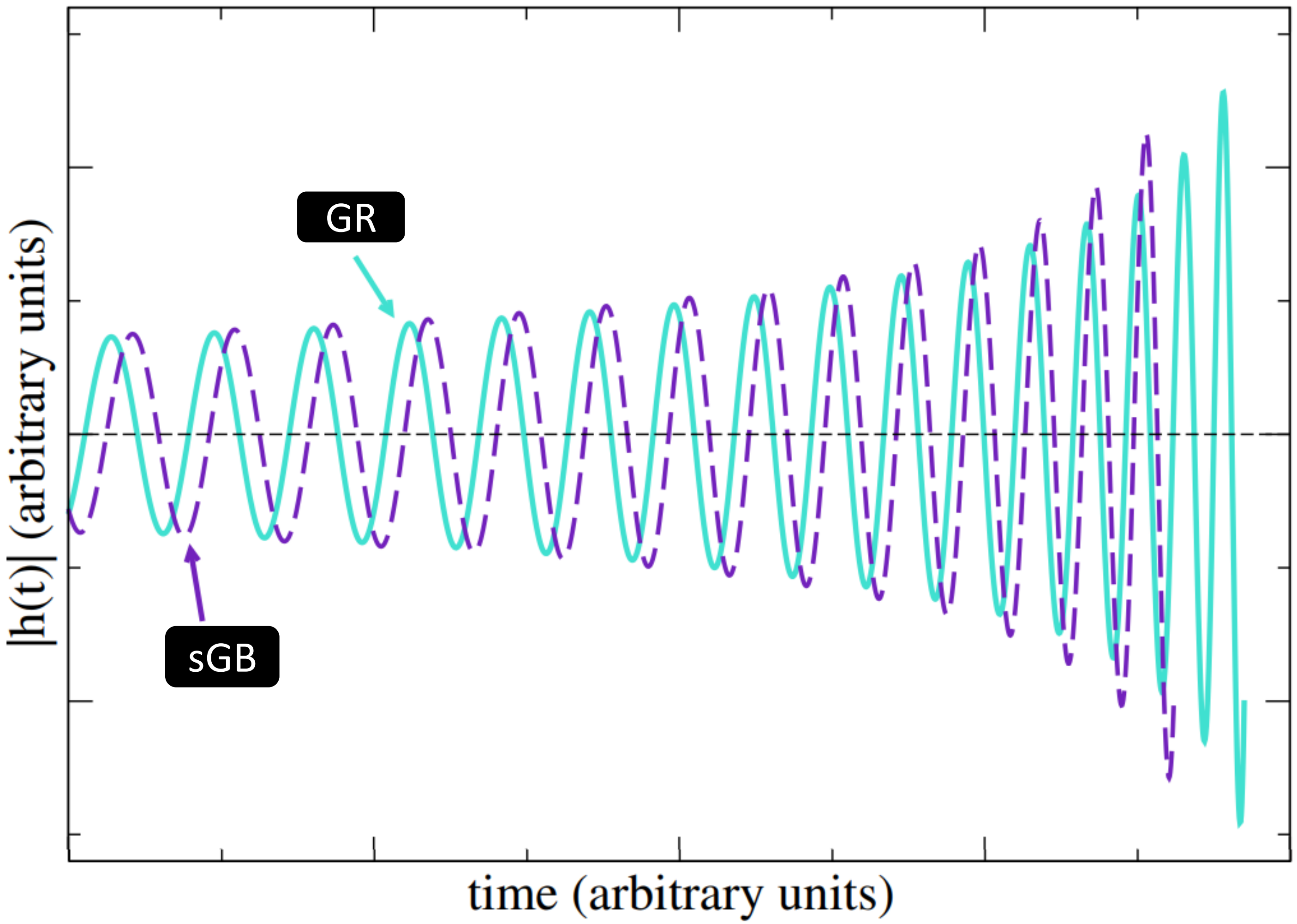
Damour and Esposito-Varèse (1996, PRD):

$$A(\varphi) = \exp\left(\frac{\beta_0}{2}\varphi^2\right) \qquad \alpha_0 \equiv \alpha(\varphi_0) = \beta_0\varphi_0$$

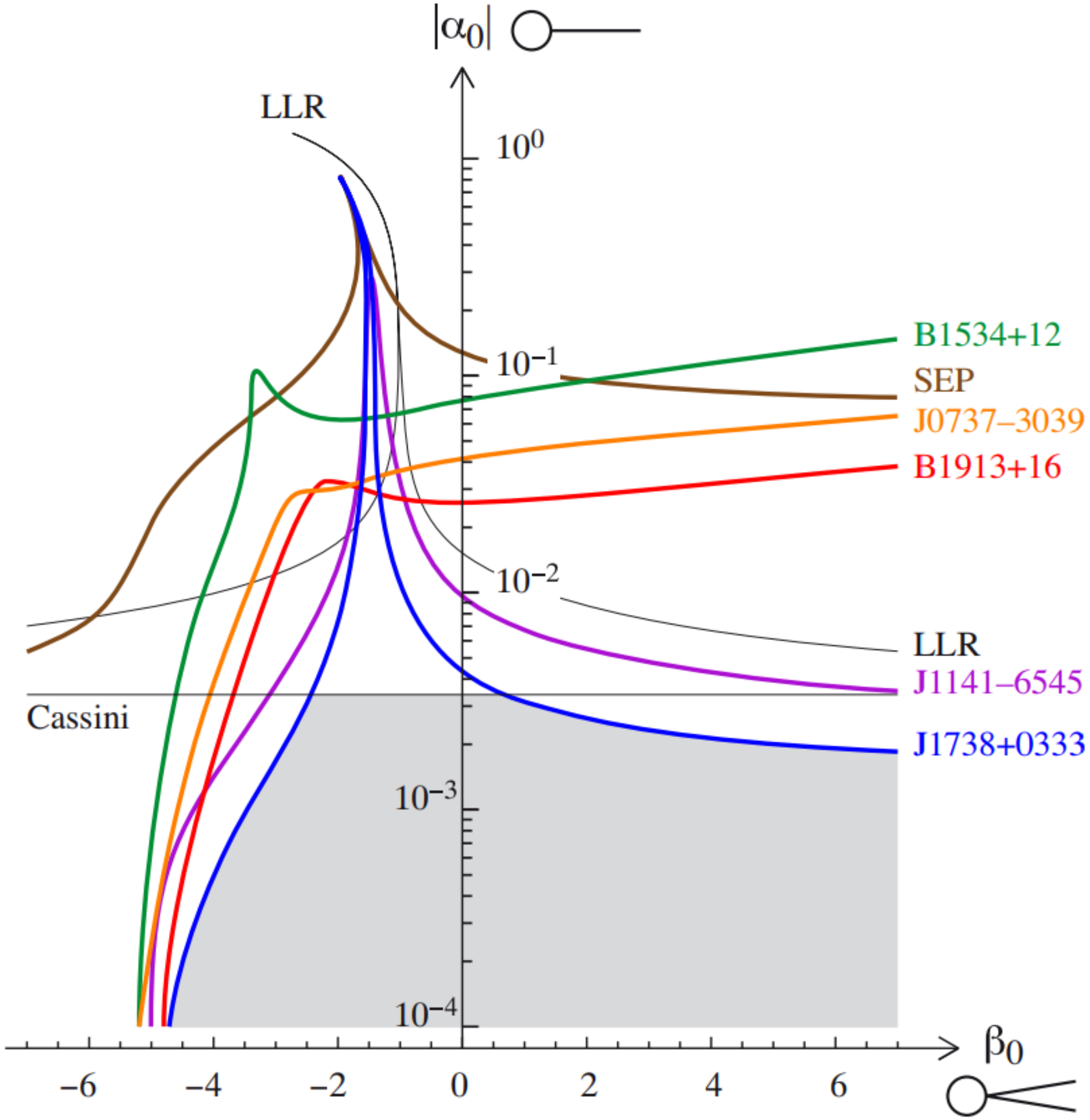
SPONTANEOUS SCALARIZATION

- When β is sufficiently negative, NSs can have nontrivial scalar field distribution profile
  - Parametrized by (α_0, β_0)
- NSs undergoing tachyonic instability



Carson, 2020 (ArXiv:2010.04745)

OBSERVABLE CONSTRAINTS ON ST THEORIES



Freire et al. 2012,
ArXiv:1205.1450

TOV EQUATIONS

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$$m' = 4\pi r^2 \left[e^{2\phi}\rho + \frac{1}{2}\left(\phi'^2 + e^{2\phi}\varphi'^2\right)\right] - \frac{\alpha'}{16}\left(8m\nu' - 4m' + \frac{4m}{r} - 2r\nu' - 2r^2\nu'^2 + 4m'\nu' r - 4r^2\nu''\right.$$
$$+ 4mr\nu' - 8m + 4mr^2\nu'^2 + 8mr^2\nu'' + 8m'r - r^3\nu'^2 + 2r^2m'\nu' - 2r^3\nu'' + \frac{16m^2}{r} + 4m^2\nu'$$
$$\left. - 4m^2r\nu'^2 - 8m^2r\nu'' - 16m'm - 4rmm'\nu'\right)\phi'',$$

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$$\left.- 4m^2r\nu'^2 - 8m^2r\nu'' - 16m'm - 4rmm'\nu'\right)\phi'',$$

$$2\phi' e^{2\phi}\varphi' + e^{2\phi}\left(\frac{r-2m}{r}\right)\left[\varphi'' - \frac{1}{4}\left(\frac{r^2}{r-2m}\right)\varphi'\phi'^2 - \frac{1}{2}\left(\frac{r^2}{r-2m}\right)e^{2\phi}\varphi'^3 + \frac{\alpha' m'}{4r\sqrt{-g}(r-2m)}\phi'\varphi' e^{-\phi}\right.$$
$$- \frac{3m\alpha'}{4r^2\sqrt{-g}(r-2m)}\phi'\varphi' e^{-\phi} + \frac{m\alpha'}{4r\sqrt{-g}(r-2m)}\phi''\varphi' e^{-\phi} - \frac{m\alpha'}{4r\sqrt{-g}(r-2m)}\phi'^2\varphi'$$
$$\left.- \frac{4\pi r^2 e^{2\phi}\rho}{r-2m}\varphi' + \frac{m}{r(r-2m)}\varphi'\right] + \frac{1}{2}e^{2\phi}\left(\frac{r-2m}{r}\right)\nu'\varphi' + e^{2\phi}\left(\frac{r-2m}{r^2}\right)\varphi' + e^{2\phi}\left(\frac{r-2m}{r^2}\right)\varphi' = 0.$$

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$$- 4m^2r\nu'^2 - 8m^2r\nu'' - 16m'm - 4rmm'\nu'\Bigg)\phi'',$$

$$2\phi' e^{2\phi}\varphi' + e^{2\phi}\left(\frac{r-2m}{r}\right)\Bigg[\varphi'' - \frac{1}{4}\left(\frac{r^2}{r-2m}\right)\varphi'\phi'^2 - \frac{1}{2}\left(\frac{r^2}{r-2m}\right)e^{2\phi}\varphi'^3 + \frac{\alpha' m'}{4r\sqrt{-g}(r-2m)}\phi'\varphi' e^{-\phi}$$
$$- \frac{3m\alpha'}{4r^2\sqrt{-g}(r-2m)}\phi'\varphi' e^{-\phi} + \frac{m\alpha'}{4r\sqrt{-g}(r-2m)}\phi''\varphi' e^{-\phi} - \frac{m\alpha'}{4r\sqrt{-g}(r-2m)}\phi'^2\varphi'$$
$$- \frac{4\pi r^2 e^{2\phi}\rho}{r-2m}\varphi' + \frac{m}{r(r-2m)}\varphi'\Bigg] + \frac{1}{2}e^{2\phi}\left(\frac{r-2m}{r}\right)\nu'\varphi' + e^{2\phi}\left(\frac{r-2m}{r^2}\right)\varphi' + e^{2\phi}\left(\frac{r-2m}{r^2}\right)\varphi' = 0.$$

$$(\rho + P)\nu' + 4P\phi' + 2P' = (-\rho + 3P)\phi'.$$

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$$= e^{2\phi}\varphi'^2 - \frac{\alpha'}{8}e^{-\phi}\mathcal{X}_4 - 8\pi e^{2\phi}(-\rho+3P),$$

$$\nu' = \frac{2m}{r(r-2m)} + 8\pi r\left\{e^{2\phi}\left(\frac{r}{r-2m}\right)P + \frac{r-4m}{2(r-2m)}\phi'^2 + e^{2\phi}\frac{r-4m}{2(r-2m)}\varphi'^2\right\}$$
$$-\frac{\alpha'}{8r^2}\left(\frac{4mr^3\nu' - 12m^2\nu' - 8mr^2\nu'' + 16m^2r\nu'' + 24m'm - 4rm\nu' - 2r^2m\nu'^2 + 4m^2r\nu'^2 + 4m'mr\nu' - 24m^2}{r-2m}\right)\phi''.$$

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$$=e^{2\phi}\varphi'^2-\frac{\alpha'}{8}e^{-\phi}\mathcal{X}_4-8\pi e^{2\phi}(-\rho+3P),$$

$$\begin{aligned}\mathcal{X}_4 = &\frac{6m^2}{r^2}\nu'^4-\frac{6m^2}{r^3}\nu'^3+\frac{2m'm}{r^2}\nu'^3-\frac{m}{r}\nu'^4+\frac{20m^2}{r^2}\nu''\nu'^2-\frac{4m}{r}\nu''\nu'^2+\frac{5m^2}{r^4}\nu'^2-\frac{2m'm}{r^3}\nu'^2\\
&+\frac{m}{r^2}\nu'^3-\frac{4m^2}{r^3}\nu''\nu'+\frac{2m}{r^2}\nu''\nu'+\frac{m'^2}{r^3}\nu'^2-\frac{m'}{r}\nu'^3+\frac{4m'm}{r^2}\nu''\nu'-\frac{2m'}{r}\nu''\nu'+\frac{1}{4}\nu'^4+\nu''\nu'^2\\
&-\frac{4m}{r}\nu''^2+\frac{4m^2}{r^2}\nu''^2+\nu''^2-\frac{4m^3}{r^3}\nu'^4+\frac{8m^3}{r^3}\nu'^3-\frac{8m'm^2}{r^3}\nu'^3-\frac{16m^3}{r^3}\nu''\nu'^2-\frac{4m^3}{r^5}\nu'^2\\
&+\frac{8m'm^2}{r^4}\nu'^2+\frac{16m^3}{r^4}\nu''\nu'-\frac{16m^2}{r^3}\nu''\nu'-\frac{4m'm}{r^4}\nu'^2+\frac{4m'm}{r^2}\nu'^3-\frac{16m'm^2}{r^3}\nu''\nu'\\
&+\frac{8m'm}{r^2}\nu''\nu'-\frac{m}{r}\nu'^4-\frac{4m}{r}\nu''\nu'^2+\frac{16m^2}{r^2}\nu''^2-\frac{16m^3}{r^3}\nu''^2-\frac{4m}{r}\nu''^2+\frac{4m^4}{r^4}\nu'^4-\frac{8m^4}{r^5}\nu'^3\\
&+\frac{8m'm^3}{r^4}\nu'^3-\frac{4m^3}{r^3}\nu'^4+\frac{16m^4}{r^4}\nu''\nu'^2-\frac{16m^3}{r^3}\nu''\nu'^2+\frac{4m^4}{r^6}\nu'^2-\frac{8m'm^3}{r^5}\nu'^2+\frac{4m^3}{r^4}\nu'^3\\
&-\frac{16m^4}{r^5}\nu''\nu'+\frac{32m^3}{r^4}\nu''\nu'+\frac{4m'^2m^2}{r^5}\nu'^2-\frac{4m'm^2}{r^3}\nu'^3+\frac{16m'm^3}{r^4}\nu''\nu'-\frac{8m'm^2}{r^3}\nu''\nu'\\
&+\frac{4m^2}{r^2}\nu''\nu'^2-\frac{16m^3}{r^3}\nu''^2+\frac{16m^4}{r^4}\nu''^2+\frac{8m^2}{r^2}\nu''^2-144m^2\nu'^2-96m^2r\nu'^3+192mr\nu'^2\\
&+112mr^2\nu'^3-96m'mr\nu'^2+224mr^2\nu'\nu''-192m^2r\nu'\nu''-16m^2r^2\nu'^4+16mr^3\nu'^4\\
&-32m'mr^2\nu'^3+64mr^3\nu''\nu'^2-32m^2r^2\nu''\nu'^2-64r^2\nu'^2-32r^3\nu'^3+64m'r^2\nu'^2-64r^3\nu''\nu'\\
&+16m'r^3\nu'^3-16r^4\nu''\nu'^2-16m'^2r^2\nu'^2+32m'r^3\nu'\nu''-64m'mr^2\nu'\nu''-16r^4\nu''^2+16mr^3\nu''^2\\
&-32m^2r^2\nu'^2\nu''+32mr^3\nu''^2-64m^2r^2\nu''^2-\frac{8m'm}{r^4}\nu'+\frac{8m^2}{r^5}\nu'+\frac{6m}{r^3}\nu'^2+\frac{8m'm}{r^3}\nu''-\frac{8m^2}{r^4}\nu''\\
&-\frac{4m}{r^2}\nu'\nu''+\frac{4m'^2}{r^4}-\frac{8m'm}{r^5}-\frac{2m'}{r^3}\nu'+\frac{4m^2}{r^6}+\frac{4m}{r^4}\nu'+\frac{1}{r^2}\nu'^2.\end{aligned}$$

$$\nu'=\frac{2m}{r(r-2m)}+8\pi r\left\{e^{2\phi}\left(\frac{r}{r-2m}\right)P+\frac{r-4m}{2(r-2m)}\phi'^2+e^{2\phi}\frac{r-4m}{2(r-2m)}\varphi'^2\right\}$$
$$-\frac{\alpha'}{8r^2}\left(\frac{4mr^3\nu'-12m^2\nu'-8mr^2\nu''+16m^2r\nu''+24m'm-4rm\nu'-2r^2m\nu'^2+4m^2r\nu'^2+4m'mr\nu'-24m^2}{r-2m}\right)\phi''.$$