Non-relativistic expansions of gravity theories

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Patrick Concha Aguilera

Facultad de Ingeniería, Universidad Católica de la Santísima Concepción

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Outline

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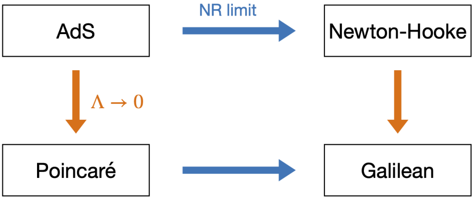
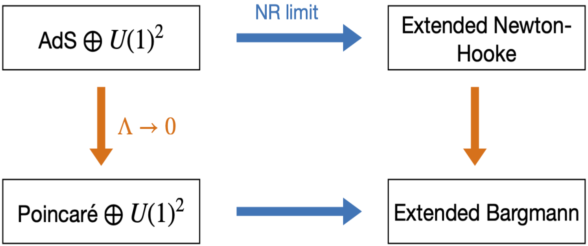
Introduction

- There has been a renewed interest in non-relativistic (NR) theories due to their utilities to approach strongly coupled condensed matter systems as well as NR effective field theories.
- A NR theory can be obtained by a suitable **limiting process** from a relativistic theory.
- In particular, through this talk, the NR limit corresponds to the limit in which $c \to \infty$.



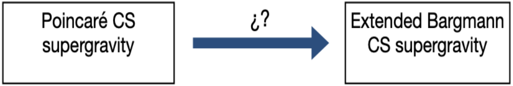
Introduction

- In the limit $c \to \infty$ there might appear infinities in the contraction of the original Lagrangian.
- In $D = 2 + 1$, the Chern-Simons (CS) formalism allows us to construct NR gravity actions whose underlying symmetry can be obtained as a NR limit of a relativistic algebra. The infinities and degeneracy can be avoided by considering additional $\mathfrak{u}(1)$ generators.



Introduction

- The formulation of a NR gravity theory in presence of **supersymmetry**, **higher-spin** or in **higher spacetime dimensions** remains as a challenging task.
- A limiting process to obtain a NR supergravity, a NR higher-spin gravity or a four-dimensional NR gravity is not trivial.



- One way to circumvent the difficulty to establish a well-defined NR limit in the presence of supersymmetry or higher-spin is through the Lie algebra expansion methods. [J. de Azcarraga, D. Gútiez, J. M. Izquierdo (2019)], [P. Concha, M. Ipinza, L. Ravera, E. Rodríguez (2020)], [P. Concha, C. Henríquez-Baez, E. Rodríguez (2022)],

Outline

Semigroup expansion method

The Semigroup expansion (S-expansion) procedure consists in obtaining a new Lie algebra $\mathfrak{G}$, by combining the elements of a semigroup S with the structure constants of a Lie algebra $\mathfrak{g}$.

$$\mathfrak{G} = S \times \mathfrak{g}$$

Advantages of the S-expansion procedure

- It not only provides us with the commutation relations of the expanded algebra, but also allows us to compute the non-vanishing components of the invariant tensor of the expanded algebra in terms of the original ones.
- It can reproduce the Maurer-Cartan forms power series expansion for a particular choice of the semigroup S.

[F. Izaurieta, E. Rodríguez, P. Salgado (2006)]

Non-relativistic expansion

A NR algebra can be obtained by expanding its relativistic counterpart with a particular semigroup. To this end, we require to consider a $\mathbb{Z}_2$-graded subspace decomposition

$$[V_0, V_0] \subset V_0 \qquad [V_0, V_1] \subset V_1 \qquad [V_1, V_1] \subset V_0$$

Then, the NR version of the Lie original algebra $\mathfrak{g}$ is obtained by considering a resonant $S_E^{(1)}$-expansion of the relativistic algebra $\mathfrak{g}$.

$S_E^{(1)}$ semigroup

$$\begin{array}{c|ccc}
\lambda_2 & \lambda_2 & \lambda_2 & \lambda_2 \\
\lambda_1 & \lambda_1 & \lambda_2 & \lambda_2 \\
\lambda_0 & \lambda_0 & \lambda_1 & \lambda_2 \\
\hline
 & \lambda_0 & \lambda_1 & \lambda_2
\end{array}$$

Resonant decomposition

$$S_0 \cdot S_0 \subset S_0$$
$$S_0 \cdot S_1 \subset S_1$$
$$S_1 \cdot S_1 \subset S_0$$

where $S_0 = \{\lambda_0, \lambda_2\}$ and $S_1 = \{\lambda_1, \lambda_2\}$. [J. Gomis, A. Kleinschmidt, J. Palmkvist, P. Salgado-Rebolledo (2020)] [P. Concha, C. Henríquez-Baez, E. Rodríguez (2022)]

Non-relativistic expansion of the $\mathfrak{so}(3,2)$ algebra

Let us consider the $\mathfrak{so}(3,2)$ algebra:

$$\begin{aligned}\left[\hat{J}_{AB},\hat{J}_{CD}\right] &= \eta_{[A[C}\,\hat{J}_{D]B]}\\ \left[\hat{J}_{AB},\hat{P}_{C}\right] &= \eta_{C[B}\,\hat{P}_{A]}\\ \left[\hat{P}_{A},\hat{P}_{B}\right] &= \frac{1}{\ell^2}\hat{J}_{AB}\end{aligned}$$

Before applying a NR expansion we first decompose the relativistic A index in terms of space and time components $A=(0,a)$ with $a=1,2,3$.

$$\begin{aligned}\left[\hat{J}_{ab},\hat{J}_{cd}\right] &= \delta_{[a[c}\,\hat{J}_{d]b]} & \left[\hat{J}_{ab},\hat{J}_{c}\right] &= \delta_{c[b}\,\hat{J}_{a]}\\ \left[\hat{J}_{a},\hat{J}_{b}\right] &= \hat{J}_{ab} & \left[\hat{J}_{ab},\hat{P}_{c}\right] &= \delta_{c[b}\,\hat{P}_{a]}\\ \left[\hat{J}_{a},\hat{P}\right] &= \hat{P}_{a} & \left[\hat{J}_{a},\hat{P}_{b}\right] &= \delta_{ab}\hat{P}\\ \left[\hat{P}_{a},\hat{P}_{b}\right] &= \frac{1}{\ell^2}\hat{J}_{ab} & \left[\hat{P},\hat{P}_{a}\right] &= \frac{1}{\ell^2}\hat{J}_{a}\end{aligned}$$

where we have relabelled the AdS generators as,

$$\hat{J}_a=\hat{J}_{0a} \qquad \hat{J}_{ab}=\hat{J}_{ab} \qquad \hat{P}=\hat{P}_0 \qquad \hat{P}_a=\hat{P}_a$$

Non-relativistic expansion of the $\mathfrak{so}\,(3,2)$ algebra

One can see that the subspace decomposition $V_0 = \{\hat{J}_{ab}, \hat{P}\}$ and $V_1 = \{\hat{J}_a, \hat{P}_a\}$ satisfies

$$[V_0, V_0] \subset V_0 \qquad [V_0, V_1] \subset V_1 \qquad [V_1, V_1] \subset V_0$$

On the other hand, let us consider the subset decomposition $S_E^{(1)} = S_0 \cup S_1$, where $S_0 = \{\lambda_0, \lambda_2\}$ and $S_1 = \{\lambda_1, \lambda_2\}$ is said to be resonant. One finds a NR algebra after applying a resonant expansion,

$$\mathfrak{G}_R = S_0 \times V_0 \oplus S_1 \times V_1 \tag{1}$$

and considering the 0_S-reduction conduction $0_S T_A = 0$. The expanded generators are related to the relativistic $\mathfrak{so}\,(3,2)$ ones through the semigroup elements as

Resonant expansion

λ_2		
λ_1		G_a, P_a
λ_0	J_{ab}, H	
	$\hat{J}_{ab}\,, \hat{P}$	$\hat{J}_a\,, \hat{P}_a$

Non-relativistic expansion of the $\mathfrak{so}(3,2)$ algebra

How to obtain the explicit commutation relation of the expanded generators?

Let us consider for instance $[H, P_a]$:

$$
\begin{aligned}
[H, P_a] &= \left[\lambda_0 \hat{P}, \lambda_1 \hat{P}_a\right] \\
&= \lambda_0 \lambda_1 \left[\hat{P}, \hat{P}_a\right] \\
&= \lambda_1 \frac{1}{\ell^2} \hat{J}_a \\
&= \frac{1}{\ell^2} G_a
\end{aligned}
$$

Thus the expanded generators satisfy the Newton-Hooke algebra:

Newton-Hooke algebra

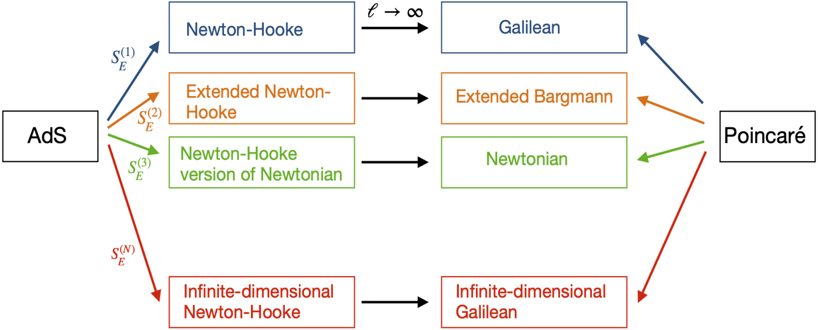
$$
[J_{ab}, J_{cd}] = \delta_{[a[c} \, J_{\ d]b]} \qquad [J_{ab}, P_c] = \delta_{c[b} \, P_{\ a]} \qquad [J_{ab}, G_c] = \delta_{c[b} \, G_{\ a]}
$$

$$
[G_a, H] = P_a \qquad [H, P_a] = \frac{1}{\ell^2} G_a
$$

Vanishing cosmological constant limit $\ell \to \infty \implies$ Galilean algebra

Non-relativistic expansions

What happen if we consider a bigger semigroup?



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Relativistic MacDowell-Mansouri gravity

A four-dimensional action for gravity can be written considering the MacDowell-Mansouri formalism. Such unified geometric approach is based on the relativistic $\mathfrak{so}(3,2)$ algebra. The gauge connection one-form $A = A^a_\mu T_a \otimes dx^\mu$ for the $\mathfrak{so}(3,2)$ algebra is given by

$$A = \frac{1}{2} W^{AB} \hat{J}_{AB} + E^A \hat{P}_A$$

where W^{AB} is the spin-connection one-form and E^A is the vierbein one-form. The corresponding curvature two-form $F = dA + \frac{1}{2}[A, A]$ reads

$$F = \frac{1}{2} \mathcal{R}^{AB} \hat{J}_{AB} + T^A \hat{P}_A$$

with $\mathcal{R}^{AB}$ and T^A being the respective AdS curvature and torsion,

$$\begin{aligned} \mathcal{R}^{AB} &= dW^{AB} + W^A{}_C W^{CB} + \frac{1}{\ell^2} E^A E^B \\ T^A &= dE^A + W^A{}_B E^B \end{aligned}$$

Relativistic MacDowell-Mansouri gravity

The MacDowell-Mansouri gravity action reads

$$I_{MM} = 2\int_{\mathcal{M}_4} \langle FF \rangle$$

where $\langle \cdots \rangle$ denotes the bilineal invariant trace for the $\mathfrak{so}\,(3,1)$ algebra.

$$\langle \hat{J}_{AB} \hat{J}_{CD} \rangle = \sigma \epsilon_{ABCD}$$

it is possible to construct a gravity action:

$$I_{MM} = \frac{\sigma}{2} \int_{\mathcal{M}_4} \epsilon_{ABCD} \mathcal{R}^{AB} \mathcal{R}^{CD}$$

which can be written considering the explicit components of the curvature two-form as

$$I_{MM} = \frac{\sigma}{2} \int_{\mathcal{M}_4} \epsilon_{ABCD} \left(R^{AB} R^{CD} + \frac{2}{\ell^2} R^{AB} E^C E^D + \frac{1}{\ell^4} E^A E^B E^C E^D \right)$$

where $R^{AB} = dW^{AB} + W^A_{\ C} W^{CB}$ is the usual Lorentz curvature two-form.

Newtonian gravity à la MacDowell-Mansouri

The minimal algebraic structure allowing us to construct a non-relativistic gravity action with cosmological constant considering the MacDowell-Mansouri approach is

Newton-Hooke version of the Newtonian algebra

$$
\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{[a[c}\, J_{\ d]b]} & [J_{ab}, P_c] &= \delta_{c[b}\, P_{\ a]} & [G_a, G_b] &= S_{ab} \\
[J_{ab}, S_{cd}] &= \delta_{[a[c}\, S_{\ d]b]} & [S_{ab}, G_c] &= \delta_{c[b}\, B_{\ a]} & [G_a, P_b] &= \delta_{ab} M \\
[J_{ab}, G_c] &= \delta_{c[b}\, G_{\ a]} & [G_a, H] &= P_a & [P_a, P_b] &= \frac{1}{\ell^2} S_{ab} \\
[J_{ab}, B_c] &= \delta_{c[b}\, B_{\ a]} & [G_a, M] &= T_a & [H, P_a] &= \frac{1}{\ell^2} G_a \\
[J_{ab}, T_c] &= \delta_{c[b}\, T_{\ a]} & [B_a, H] &= T_a & [H, T_a] &= \frac{1}{\ell^2} B_a \\
[S_{ab}, P_c] &= \delta_{c[b}\, T_{\ a]} & [M, P_a] &= \frac{1}{\ell^2} B_a & &
\end{aligned}
$$

$\ell \to \infty \implies$ Newtonian algebra

[P. Concha, E. Rodríguez, G. Rubio (work in progress)]

Newtonian gravity à la MacDowell-Mansouri

The gauge connection one-form A for the Newton-Hooke-Newtonian algebra is given by

$$A = \frac{1}{2}\omega^{ab}J_{ab} + \tau H + \omega^{a}G_{a} + e^{a}P_{a} + \frac{1}{2}s^{ab}S_{ab} + mM + b^{a}B_{a} + t^{a}T_{a}$$

where ω^{ab}, ω^{a}, τ and e^{a} are the time and spatial components of the spin-connection and vierbein, respectively. The corresponding curvature two-form $F = F^{A}T_{A}$ reads

$$\begin{aligned} F =& \frac{1}{2}R^{ab}\left(\omega\right)J_{ab} + R\left(\tau\right)H + R^{a}\left(\omega\right)G_{a} + R^{a}\left(e\right)P_{a} \\ &+ \frac{1}{2}R^{ab}\left(s\right)S_{ab} + R\left(m\right)M + R^{a}\left(b\right)B_{a} + R^{a}\left(t\right) \end{aligned}$$

where

$$\begin{aligned} R^{a}\left(\omega\right) &= d\omega^{a} + \omega^{a}{}_{c}\omega^{c} + \frac{1}{\ell^{2}}\tau e^{a} & R^{ab}\left(\omega\right) &= d\omega^{ab} + \omega^{a}{}_{c}\omega^{cb} \\ R^{ab}\left(s\right) &= ds^{ab} + 2\omega^{a}{}_{c}s^{cb} + \omega^{a}\omega^{b} + \frac{1}{\ell^{2}}e^{a}e^{b} & R\left(\tau\right) &= d\tau \\ R^{a}\left(b\right) &= db^{a} + \omega^{a}{}_{c}b^{c} + s^{a}{}_{c}\omega^{c} + \frac{1}{\ell^{2}}\tau t^{a} + \frac{1}{\ell^{2}}me^{a} & R^{a}\left(e\right) &= de^{a} + \omega^{a}{}_{c}e^{c} + \omega^{a}\tau \\ R^{a}\left(t\right) &= dt^{a} + \omega^{a}{}_{c}t^{c} + s^{a}{}_{c}e^{c} + \omega^{a}m + b^{a}\tau & R\left(m\right) &= dm + \omega^{a}e_{a} \end{aligned}$$

Newtonian gravity à la MacDowell-Mansouri

The Newton-Hooke-Newtonian algebra admits the following non-vanishing components of the invariant tensor

$$\langle J_{ab} G_c \rangle = \alpha \epsilon_{abc}$$
$$\langle J_{ab} B_c \rangle = \beta \epsilon_{abc}$$
$$\langle S_{ab} G_c \rangle = \beta \epsilon_{abc}$$

where α and β are arbitrary constants related to the $\mathfrak{so}(3,1)$ constant through the semigroup elements:

$$\alpha = \lambda_1 \sigma \qquad\qquad \beta = \lambda_3 \sigma$$

Let us note that the invariant tensor breaks the symmetry to an extended Nappi-Witten subalgebra spanned by $\{J_{ab}, G_a, S_{ab}, B_a\}$. A four-dimensional non-relativistic gravity action is obtained considering the invariant tensor and the curvature two-forms of the Newton-Hooke-Newtonian algebra in the general expression of the MacDowell-Mansouri action,

$$I_{MM}^{NR} = 2 \int_{\mathcal{M}_4} \alpha\, \epsilon_{abc}\; R^{ab}(\omega)\, R^{c}(\omega) + \beta\, \epsilon_{abc}\; \left[R^{ab}(\omega)\, R^{c}(b) + R^{ab}(s)\, R^{c}(\omega) \right]$$

Newtonian gravity à la MacDowell-Mansouri

$$
\begin{aligned}
I_{MM}^{NR} = 2\int_{\mathcal{M}_4} & \frac{\alpha}{\ell^2}\epsilon_{abc}\ R^{ab}(\omega)\,\tau e^{c} \\
& + \frac{\beta}{\ell^2}\epsilon_{abc}\left[\mathcal{R}^{a}(\omega)\,e^{b}e^{c} + \frac{1}{\ell^2}e^{a}e^{b}\tau e^{c} + \mathcal{R}^{ab}(s)\,\tau e^{c} + R^{ab}(\omega)\,\tau t^{c} + R^{ab}(\omega)\,me^{c}\right]
\end{aligned}
$$

where we have omitted the boundary terms and considered the redefinition

$$
\begin{aligned}
\mathcal{R}^{a}(\omega) &= d\omega^{a} + \omega^{a}{}_{c}\omega^{c} \\
\mathcal{R}^{ab}(s) &= ds^{ab} + 2\omega^{a}{}_{c}s^{cb} + \omega^{a}\omega^{b}
\end{aligned}
$$

Considering the variation of the action under an arbitrary spatial spin-connection ω^{a} reproduces the ω^{a} field equation:

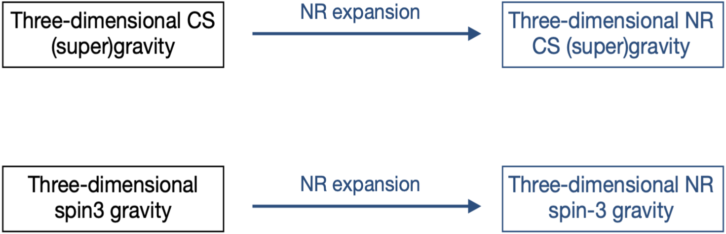
$$
\delta_{\omega} I_{MM}^{NR} = \epsilon_{abc}\ R^{b}(e)\,e^{c} = 0
$$

Then, demanding invariance of the action for arbitrary $\delta\omega^{a}$ yields $R^{a}(e) = 0$ allowing to express the spatial and time spin-connection in terms of the spatial and time vierbein.

Outline

Comments and further developments

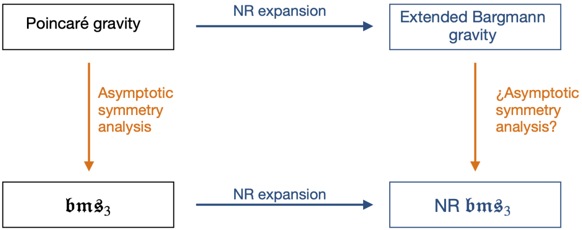
The expansion procedure considering S_E as the relevant semigroup is a powerful tool which, under certain conditions, allows us to obtain the corresponding NR counterpart of a (super)gravity theory.



These results along with those presented here could be used to approach several issues and open questions.

Comments and further developments

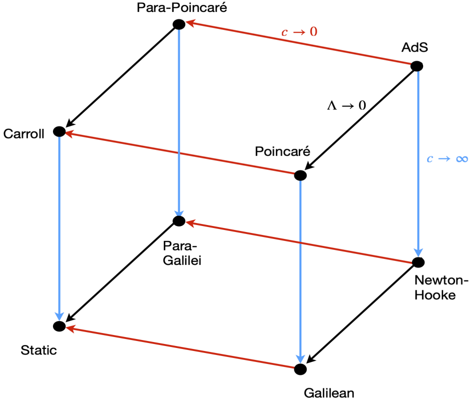
- It would be interesting to explore the physical implications of larger NR Newtonian symmetries and analyze its relations to **post-Newtonian** theory.
- It would be worth it to study the NR expansion of known **asymptotic symmetries**. Our procedure could be useful to elucidate the corresponding NR versions of the asymptotic symmetries of three-dimensional AdS Chern-Simons gravity and its Poincaré limit.



- Another aspect that deserve to be explored is the study of three-dimensional NR gravity coupled to spin higher than 3/2 by analyzing the NR expansion of hypersymmetric extension of gravity. A NR **hypergravity** is unknown and would probably requires the presence of spin-4 generators.

Comments and further developments

An ultra-relativisic or Carrollian expansion can be obtained by considering a diverse decomposition of the original (super)algebra. It would be interesting to explore how a semigroup expansion can be seen as a physical limit. Our objective is to elucidate the relation between the semigroup expansion method and the cube summarizing the sequential limits starting from the AdS algebra.



Thank you!