20 years of a holographic formula

based on work over the years with H. Dorn, R. Aros, F. Bugini and S. Acevedo

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Cosmology & Particles, UBB, Sep. 12-14, 2022

Aim of the talk

Report a fruitful interplay **AdS**/ **CFT** correspondence , Conformal Geometry/Spectral Theory

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- **Holographic formula**: a nostalgic overview
- **Gravity as a tool**: one-loop partition functions and trace/Weyl/conformal anomalies

Plan

1. Maldacena's conjecture
2. Holographic Weyl anomaly
3. A subleading O(1) result
4. The holographic formula
5. Testing and tweaking
6. Outlook

Maldacena's conjecture

Maldacena's AdS = CFT [Maldacena'97]

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Realization of two deeply-rooted ideas in physics:
(
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- the string of the large-N gauge theory [G.'t Hooft]

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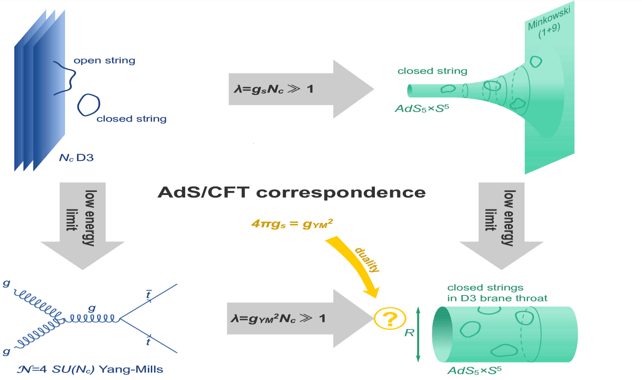

[Holography at work for nuclear and hadron physics, Y. Kim and D. Yi]

Calculational prescription [Gubser+Klebanov+Polyakov'98, Witten'98]

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String/M-theory: partition function

- $AdS_{d+1} \times X$
- prescribed asymptotics at the conformal infinity

CFT_d**: generating functional**

- at conformal boundary
- gauge invariant single-trace composite operators

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- Most of the initial tests: class. SUGRA/ leading large-N regimes
- weak/strong duality: comparison with perturbative gauge regime only for protected quantities

Holographic Weyl anomaly

Matching of trace anomaly

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Bulk

- reconstruct a Poincare-Einstein metric from a given conformal infty
- $\frac{1}{G_5}(R \quad) \, Vol_g$, but in nite volume

Boundary

- 1-loop e ective potential (UV)
- QFT in curved spacetime: proper-time, heat-kernel, etc

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$AdS_5 \times S^5$: $\langle T_{ren} \rangle = \frac{c}{8 \ \ 2} \ Ric^2 \quad \frac{1}{3} R^2 \qquad c = \frac{N^2 \quad 1}{4}$

A subleading O(1) result

Beyond classical SUGRA: $l_p^4=L^4 \quad 1=N$

? O(N): tree-level after inclusion of open or unoriented closed strings

? O(1) taking $N^2 \;!\; N^2 \quad 1$: loop in SUGRA, but needs whole KK-towers and SUSY [Bilal+Chu'99, Mansfield+Nolland+Ueno'02]

Beyond classical SUGRA: $l_p^4/L^4 \sim 1/N$

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Bulk

- scalar $\frac{d^2}{4} < m^2 < \frac{d^2}{4} + 1$, two AdS-inv. quantizations [Breitenlohner+Freedman'82]
- generalized boundary condition $= f$: the only two conformal inv. choices $f = 0; 1$

Boundary

- $=$ -CFT: same hologram (but di erent asymptotics) [Klebanov+Witten'99]
- end points of RG- ow triggered by a relevant double-trace deformation $f\,O^2$ of the -CFT.

C C : bulk AdS_{d+1} [Gubser+Mitra'02]

- Background solution has = 0) no e ect on the classical SUGRA partition function
- But two AdS-invariant propagators G) quantum uctuations of are sensitive to the boundary conditions (Casimir e ect)

$$Z_{grav} = Z_{grav}^{class} \quad \det (\quad + m^2) \quad ^{1=2}$$

- The ratio $Z^+_{grav}=Z_{grav}$ only contains the IR-divergence of the in nite AdS volume.

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AdS prediction: O(1) correction to the holographic anomaly, polynomial in

Can this correction be reproduced on the boundary? YES!!!

- exploit the RG- ow picture: $f\, O^2$
- Hubbard-Stratonovich transf. (auxiliary eld trick)

$$\langle e^{\frac{f}{2}\int O^2}\rangle \quad \int D \quad e^{\frac{1}{2f}\int \ ^2}\langle e^{\int \ O}\rangle$$

- large-N factorization

$$\langle e^{\int \ O}\rangle \quad e^{\frac{1}{2}\int\int \ \langle O\ O\ \rangle}$$

- uct. det. of the auxiliary eld: $\langle O\ O\ \rangle$ as $f\ !\ 1$

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CFT con rmation: O(1) correction to the trace anomaly (d=2,4,6,8)

The holographic formula

Shortcomings and plan for amended version [Dorn+DD'07]

1. Bulk: non-zero $V^+ \quad V$ for both even and odd d, but for odd d only via numerics [Hartman+Rastelli'06]
2. Confusion: for odd d, no anomaly in CFT vs. nonzero effective potential in the bulk
3. Boundary: overall coeff. of the anomaly? generic d?
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$$\mathrm{AdS/CFT}\ ) \qquad \frac{\det_+(\quad + m^2)}{\det\ (\quad + m^2)} \stackrel{?}{=} \det \langle O\ O\ \rangle$$

Testing and tweaking

Unknown to our contemporary conformal geometers? GJMS ops.

Continuation: Δ ! d=2 k (k 2 N)) think of Ξ as inverse of k-th GJMS [Graham+Jenne+Mason+Sparling'91]

$$P_{2k} = \Delta^k + LOT$$

Unknown to our contemporary conformal geometers? GJMS ops.

Continuation: $\Delta \to d/2 - k$ ($k \in \mathbb{N}$) $\Rightarrow$ think of Ξ as inverse of k-th GJMS [Graham+Jenne+Mason+Sparling'91]

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d odd

- Analogous result for $d+1$ even, for a generalized notion of determinant of GJMS [Guillarmou'05]

d even

- "The delicate case of $d+1$ odd where things do not renormalize correctly", is still to be understood!

We anticipated that a proper treatment in $d = even$ should unveil the Weyl anomaly.

Polyakov formulas for GJMS from AdS/CFT [DD'08]

Coefficient of the universal part (type A anomaly) of the Polyakov formulas, that agrees with the few ones known from heat kernel techniques.

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a			
	$k=1$	$k=2$	$k=3$
$n=2$	2 X	-	-
$n=4$	4=3 X	112=3 X	-
$n=6$	10=3 X	64=3	738

- A compact formula in terms of Plancherel measure [DD'08,Dowker'10]
- The very same numbers found in recent years: log-term of the entanglement entropy for a massless free scalar through an even-sphere [Casini+Huerta'10]

Polyakov formulas for GJMS from AdS/CFT [DD'08]

Maybe more important, the two chief roles of the Q-curvature are connected for the first time: Branson vs. Fefferman+Graham.

(i) regularized volume in the ambient construction [Graham+Zworski'01]

(ii) Polyakov f-la for GJMS [Branson'93] ($d = 2, 4, 6$ and conjecturally for all even)

Holographically induced Polyakov formulas:

$$\log \frac{\det \hat{P}_{2k}}{\det P_{2k}} = a \int_M w(\hat{Q} + Q) + \dots$$

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- ? From renormalized volume and its conformal variation under $\hat{g} = e^{2w} g$ [Chang+Qing+Yang'05], or alternatively, induced action for the conformal mode [Carlip'05, Aros+Romo+Zamorano'06], one gets the same structure!

Quotients $X = n\ \mathrm{AdS}_{n+1}$ (e.g. thermal AdS and BTZ bh)

[DD'08,Aros+DD'09]

Poincare patch

$$ds^2 = \frac{dz^2 + dx^2}{z^2} \text{ identi cation } (z; x) \quad e^{l}(z; Ax)$$

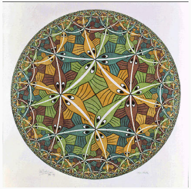
Gravity

- method of images
- in a nutshell, (Patterson-)Selberg zeta function Z [Patterson'89]

CFT

- thermal correlator $hhO \ O \ ii$ at $T \quad 1/l; l$
- read o from scattering in X [Perry+Williams'03]
- stationary Schrödinger in a Pöschl-Teller barrier

Quotients



$$\left(\frac{Z_\Gamma(n \quad\quad)}{Z_\Gamma(\quad)}\right)^2 \quad exp\,(A_n \quad V)$$

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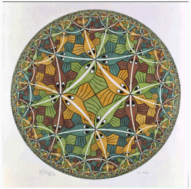

$$\left(\frac{Z_\Gamma(n \quad)}{Z_\Gamma(\ )} \right)^2 \quad exp\,(A_n \quad V)$$

In particular, for 'resonant' values of the scaling dimension $\ ! \ 2$ in two dimensions: the celebrated determinant of the Laplacian on the torus [Ray+Singer'73,Polchinski'86]

Higher spins

Bulk AdS_{odd} [Giombi+Klebanov+Pufu+Safdi+Tarnopolsky'13]

- Holographic derivation of the type-A trace anomaly coe cient of Fradkin-Tseytlin Conformal Higher Spins

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Boundary Einstein [Tseytlin'13]

- Heat-kernel confirmation of the type-A trace anomaly coefficient

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Boundary Einstein [Tseytlin'13]

- Heat-kernel con rmation of the type-A trace anomaly coe cient
- Heat-kernel derivation of the type-B trace anomaly coe cient(s): remained holographically unaccounted for

Higher spins: building blocks

[Acevedo+Aros+Bugini+DD'17,Aros+Bugini+DD'19'21'22]

$$\frac{\det_{-;TT}\left\{\hat\Delta^{(s)}_L + s(n+s-2) - \frac{n^2}{4} + k^2\right\}}{\det_{+;TT}\left\{\hat\Delta^{(s)}_L + s(n+s-2) - \frac{n^2}{4} + k^2\right\}} = \det_{TT} P^{(s)}_{2k} \cdots \det P^{(0)}_{2k}$$

Higher spins: building blocks

[Acevedo+Aros+Bugini+DD'17,Aros+Bugini+DD'19'21'22]

$$\frac{\det_{-;TT}\left\{\Delta_L^{(s)} + s(n+s-2) - \left(\frac{n^2}{4}+k^2\right)\right\}}{\det_{+;TT}\left\{\Delta_L^{(s)} + s(n+s-2) - \left(\frac{n^2}{4}+k^2\right)\right\}} = \det_{TT} P_{2k}^{(s)} \cdots \det P_{2k}^{(0)}$$

Three key ingredients:

- Simple holographic recipe to read off type-B anomaly [Bugini+DD'16].
- Extrapolation of b_6 heat coefficient, recently computed by [Liu+McPeak'18]
- WKB-exactness of the heat-kernel for tranverse-traceless totally symmetric rank-s tensors.

Outlook

Summary

- "It is very likely that the holographic formula is right"

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MANY THANKS FOR YOUR ATTENTION

¡GRACIAS!