20 years of a holographic formula

based on work over the years with H. Dorn, R. Aros, F. Bugini and S. Acevedo

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Cosmology & Particles, UBB, Sep. 12-14, 2022

Aim of the talk

Report a fruitful interplay **AdS**/ **CFT** correspondence $\Leftrightarrow$ Conformal Geometry/Spectral Theory

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- **Holographic formula**: a nostalgic overview
- **Gravity as a tool**: one-loop partition functions and trace/Weyl/conformal anomalies

Plan

1. Maldacena's conjecture
2. Holographic Weyl anomaly
3. A subleading O(1) result
4. The holographic formula
5. Testing and tweaking
6. Outlook

Maldacena’s conjecture

Maldacena's AdS = CFT [Maldacena'97]

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$$\begin{cases} \text{the holographic principle} & \texttt{[G.'t Hooft / L.Susskind]} \\ \text{the string of the large-N gauge theory} & \texttt{[G.'t Hooft]} \end{cases}$$

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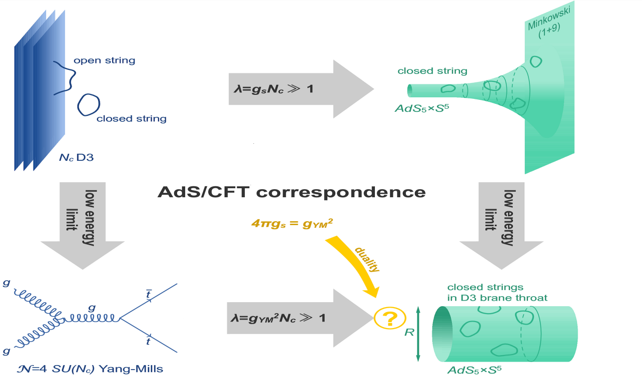

[Holography at work for nuclear and hadron physics, Y. Kim and D. Yi]

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String/M-theory: partition function

- $AdS_{d+1} \times X$
- prescribed asymptotics at the conformal infinity

CFT_d: **generating functional**

- at conformal boundary
- gauge invariant single-trace composite operators

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- Most of the initial tests: class. SUGRA/ leading large-N regimes
- weak/strong duality: comparison with perturbative gauge regime only for protected quantities

Holographic Weyl anomaly

Matching of trace anomaly

[Henningson+Skenderis'98]

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Bulk

- reconstruct a Poincare-Einstein metric from a given conformal infty
- $\frac{1}{G_5}(R - \Lambda)\, Vol_g$, but infinite volume

Boundary

- 1-loop effective potential (UV)
- QFT in curved spacetime: proper-time, heat-kernel, etc

$\star$ IR-UV connection [Susskind+Witten'98]

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$AdS_5 \times S^5$: $\quad \langle T_{ren\,\mu}{}^{\mu} \rangle = \frac{c}{8\pi^2}\left(Ric^2 - \frac{1}{3}R^2\right) \qquad c = \frac{N^2-1}{4}$

A subleading O(1) result

Beyond classical SUGRA: $l_p^4/L^4 \sim 1/N$

- ⋆ O(N): tree-level after inclusion of open or unoriented closed strings
- ⋆ O(1) taking $N^2 \to N^2 - 1$: loop in SUGRA, but needs whole KK-towers and SUSY [Bilal+Chu'99, Mansfield+Nolland+Ueno'02]

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Bulk

- scalar $-\frac{d^2}{4} < m^2 < -\frac{d^2}{4} + 1$, two AdS-inv. quantizations [Breitenlohner+Freedman'82]
- generalized boundary condition $\alpha = \widetilde{f}\beta$: the only two conformal inv. choices $\widetilde{f} = 0, \infty$

Boundary

- α/β-CFT: same hologram (but different asymptotics) [Klebanov+Witten'99]
- end points of RG-flow triggered by a relevant double-trace deformation $f\, O_\alpha^2$ of the α-CFT.

$c_\alpha - c_\beta$: **bulk** AdS_{d+1} [Gubser+Mitra'02]

- Background solution has $\phi = 0 \Rightarrow$ no effect on the classical SUGRA partition function
- But two AdS-invariant propagators $G_{\Delta_\pm} \Rightarrow$ quantum fluctuations of ϕ are sensitive to the boundary conditions ($\sim$ Casimir effect)

$$Z^{\pm}_{grav} = Z^{class}_{grav} \cdot \left[\det_{\pm}(-\Box + m^2)\right]^{-1/2}$$

- The ratio $Z^{+}_{grav}/Z^{-}_{grav}$ only contains the IR-divergence of the infinite AdS volume.

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AdS prediction: O(1) correction to the holographic anomaly, polynomial in ν

$c_\alpha - c_\beta$: boundary $\mathbb{S}^d$ [Gubser+Klebanov'02]

Can this correction be reproduced on the boundary? YES!!!

- exploit the RG-flow picture: $f\, O_\alpha^2$
- Hubbard-Stratonovich transf. (auxiliary field trick)

$$\left\langle e^{-\frac{f}{2}\int O_\alpha^2} \right\rangle \sim \int \mathcal{D}\sigma\, e^{\frac{1}{2f}\int \sigma^2} \left\langle e^{\int \sigma O_\alpha} \right\rangle$$

- large-N factorization

$$\left\langle e^{\int \sigma O_\alpha} \right\rangle \approx e^{\frac{1}{2}\int\int \sigma \langle O_\alpha O_\alpha \rangle \sigma}$$

- fluct. det. of the auxiliary field: $\Xi \sim \langle O_\alpha O_\alpha \rangle$ as $f \to \infty$

$$Z_\beta = Z_\alpha \cdot [\det(\Xi)]^{-1/2}$$

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CFT confirmation: O(1) correction to the trace anomaly (d=2,4,6,8)

The holographic formula

Shortcomings and plan for amended version [Dorn+DD'07]

1. Bulk: non-zero $V^+ - V^-$ for both even and odd d, but for odd d only via numerics [Hartman+Rastelli'06]
2. Confusion: for odd d, no anomaly in CFT vs. nonzero effective potential in the bulk
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$$\text{AdS/CFT} \Rightarrow \qquad \frac{\det_{+}(-\Box + m^2)}{\det_{-}(-\Box + m^2)} \qquad \stackrel{?}{=} \qquad \det \langle O_\alpha O_\alpha \rangle$$

Testing and tweaking

Unknown to our contemporary conformal geometers? GJMS ops.

Continuation: $\Delta_- \to d/2 - k$ $(k \in \mathbb{N}) \Rightarrow$ think of Ξ as inverse of *k*-th GJMS [Graham+Jenne+Mason+Sparling'91]

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d odd

- Analogous result for $d+1$ even, for a generalized notion of determinant of GJMS [Guillarmou'05]

d even

- "The delicate case of $d+1$ odd where things do not renormalize correctly", is still to be understood!

We anticipated that a proper treatment in $d = \textit{even}$ should unveil the Weyl anomaly.

Polyakov formulas for GJMS from AdS/CFT [DD'08]

Coefficient of the universal part (type A anomaly) of the Polyakov formulas, that agrees with the few ones known from heat kernel techniques.

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a			
	$k=1$	$k=2$	$k=3$
$n=2$	2 ✓	-	-
$n=4$	$-4/3$ ✓	$112/3$ ✓	-
$n=6$	$10/3$ ✓	$-64/3$	738

- A compact formula in terms of Plancherel measure [DD'08,Dowker'10]
- The very same numbers found in recent years: log-term of the entanglement entropy for a massless free scalar through an even-sphere [Casini+Huerta'10]

Polyakov formulas for GJMS from AdS/CFT [DD'08]

Maybe more important, the two chief roles of the Q-curvature are connected for the first time: Branson vs. Fefferman+Graham.

(i) regularized volume in the ambient construction [Graham+Zworski'01]

(ii) Polyakov f-la for GJMS [Branson'93] ($d = 2, 4, 6$ and conjecturally for all even)

Holographically induced Polyakov formulas:

$$-\log \frac{\det \widehat{P_{2k}}}{\det P_{2k}} = a \int_{\mathcal{M}} w(\widehat{\mathbf{Q}} + \mathbf{Q}) + \ldots$$

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- $\star$ From renormalized volume and its conformal variation under $\widehat{g} = e^{2w} g$ [Chang+Qing+Yang'05], or alternatively, induced action for the conformal mode [Carlip'05, Aros+Romo+Zamorano'06], one gets the same structure!

Quotients $X = \Gamma \setminus \mathrm{AdS}_{n+1}$ (e.g. thermal AdS and BTZ bh)

[DD'08,Aros+DD'09]

Poincaré patch

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2} \text{ identification } (z, \vec{x}) \sim e^{l}(z, \mathbb{A}\vec{x})$$

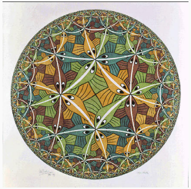
Gravity

- method of images
- in a nutshell, (Patterson-)Selberg zeta function Z_Γ [Patterson'89]

CFT

- thermal correlator $\langle\langle O_\lambda O_\lambda \rangle\rangle$ at $T \sim 1/l$, l
- read off from scattering in X [Perry+Williams'03]
- stationary Schrödinger in a Pöschl-Teller barrier

Quotients



$$\left\{ \frac{Z_\Gamma(n-\lambda)}{Z_\Gamma(\lambda)} \right\}^2 \cdot exp\left(\mathcal{A}_n \cdot \mathcal{V}\right)$$

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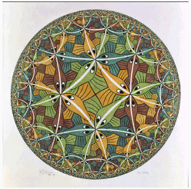

$$\boxed{\left\{\frac{Z_\Gamma(n-\lambda)}{Z_\Gamma(\lambda)}\right\}^2 \cdot \, exp\left(\mathcal{A}_n \cdot \mathcal{V}\right)}$$

In particular, for 'resonant' values of the scaling dimension $\lambda \to 2$ in two dimensions: the celebrated determinant of the Laplacian on the torus [Ray+Singer'73,Polchinski'86]

Higher spins

Bulk AdS_{odd} [Giombi+Klebanov+Pufu+Safdi+Tarnopolsky'13]

- Holographic derivation of the type-A trace anomaly coefficient of Fradkin-Tseytlin Conformal Higher Spins

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- Heat-kernel confirmation of the type-A trace anomaly coefficient

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Boundary Einstein [Tseytlin'13]

- Heat-kernel confirmation of the type-A trace anomaly coefficient
- Heat-kernel derivation of the type-B trace anomaly coefficient(s): remained holographically unaccounted for

Higher spins: building blocks

[Acevedo+Aros+Bugini+DD'17,Aros+Bugini+DD'19'21'22]

$$\boxed{\frac{\det_{-,TT}\left\{\hat{\Delta}_L^{(s)} + s(n+s-2) - \frac{n^2}{4} + k^2\right\}}{\det_{+,TT}\left\{\hat{\Delta}_L^{(s)} + s(n+s-2) - \frac{n^2}{4} + k^2\right\}} = \det_{TT} P_{2k}^{(s)} \cdot \ldots \cdot \det P_{2k}^{(0)}}$$

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$$\boxed{\frac{\det_{-,TT}\left\{\hat{\Delta}_L^{(s)}+s(n+s-2)-\frac{n^2}{4}+k^2\right\}}{\det_{+,TT}\left\{\hat{\Delta}_L^{(s)}+s(n+s-2)-\frac{n^2}{4}+k^2\right\}}=\det_{TT}P_{2k}^{(s)}\cdot\ldots\cdot\det P_{2k}^{(0)}}$$

Three key ingredients:

- Simple holographic recipe to read off type-B anomaly [Bugini+DD'16].
- Extrapolation of b_6 heat coefficient, recently computed by [Liu+McPeak'18]
- WKB-exactness of the heat-kernel for tranverse-traceless totally symmetric rank-s tensors.

Outlook

Summary

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Many Thanks For Your Attention

¡GRACIAS!