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Digging deep into hot bosonic two-loop vacuum sum-integrals

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based on common work with York Schröder

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Finite-temperature quantum field theory, or Thermal quantum field theory

Basically, a set of methods to calculate expectation values of physical observables of the quantum field theory at finite temperature.

Important applications:

- cosmology (early universe, etc.)
- heavy-ion collisions (quark-gluon plasma, etc.)

Refer to the textbooks:

[Kapusta, 1989], [LeBellac, 2000], [Kapusta, Gale, 2006], [Laine, Vuorinen, 2017], ...

Sum-integrals in thermal quantum field theory

Definition:

$$\sum\!\!\!\!\!\!\int_P \equiv \mu^{2\varepsilon}\, T \sum_{P_0} \int \frac{\mathrm{d}^d \mathbf{p}}{(2\pi)^d}\,, \qquad \text{with} \quad P = (P_0, \mathbf{p}), \quad d = 3 - 2\varepsilon$$

$$P^2 = P_0^2 + \mathbf{p}^2 = \begin{cases} ([2n_p]\pi T)^2 + \mathbf{p}^2 & \text{for bosons} \quad \leftarrow \\ ([2n_p+1]\pi T)^2 + \mathbf{p}^2 & \text{for fermions} \end{cases}$$

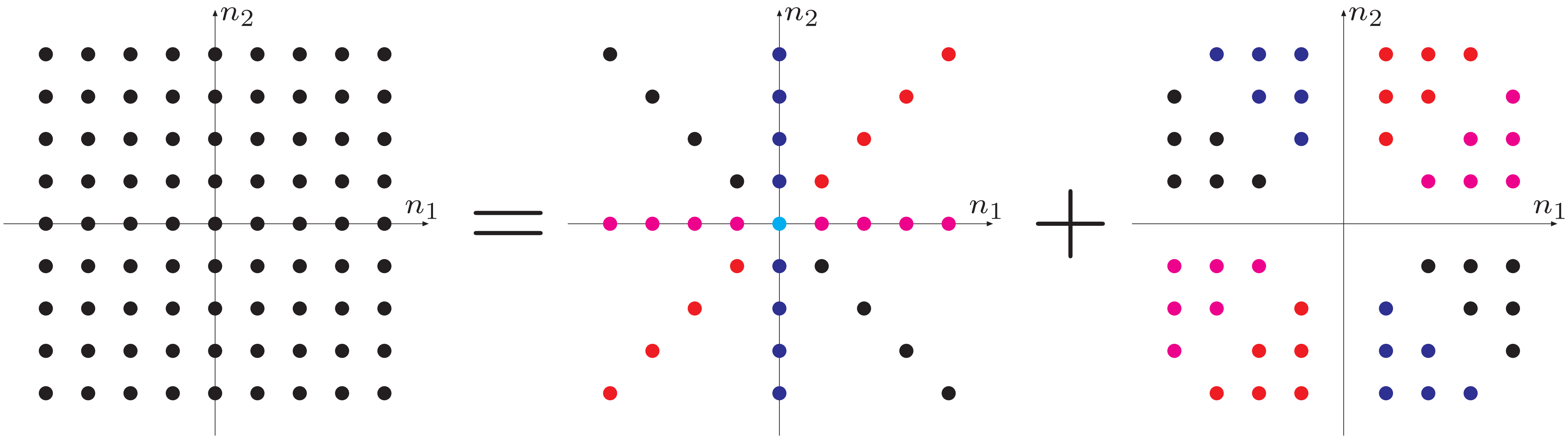
$$\sum_{P_0} = \sum_{n_p=-\infty}^{\infty} \qquad - \text{ Matsubara frequency sum}$$

Massless one-loop bosonic vacuum sum-integral:

$$I_\nu^\eta(d) = \sum\!\!\!\!\!\!\int_Q \frac{(Q_0)^\eta}{[Q^2]^\nu} = \frac{[1+(-1)^\eta]\ T}{(2\pi T)^{2\nu-\eta-d}}\, \frac{\Gamma\left(\nu - \frac{d}{2}\right)}{(4\pi)^{d/2}\,\Gamma(\nu)}\, \zeta(2\nu - \eta - d)$$

where $\Gamma(z)$ – Euler gamma function, $\qquad \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ – Riemann zeta function

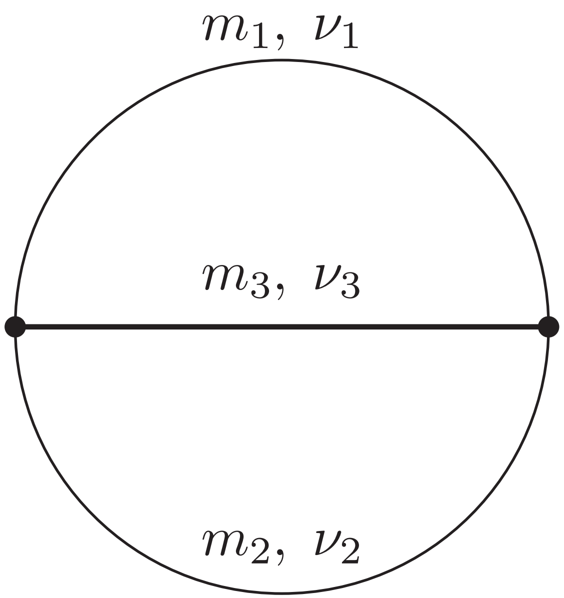
Two-loop massless bosonic vacuum sum-integrals



$$L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3}(d,T) = \sum\!\!\!\!\!\!\int_{P,Q} \frac{(P_0)^{\eta_1}\,(Q_0)^{\eta_2}\,(P_0-Q_0)^{\eta_3}}{[P^2]^{\nu_1}\,[Q^2]^{\nu_2}\,[(P-Q)^2]^{\nu_3}}$$

$$= \frac{T^2}{(2\pi T)^{2\Sigma\nu_i-\Sigma\eta_i-2d}} \sum_{n_1,n_2} n_1^{\eta_1}\,n_2^{\eta_2}\,(n_1-n_2)^{\eta_3} B^{\nu_1,\nu_2,\nu_3}_{n_1,n_2,n_1-n_2}(d)$$

Two-loop "cold" $(T = 0)$ massive vacuum integrals



$$B^{\nu_1,\nu_2,\nu_3}_{m_1,m_2,m_3} = \frac{1}{(2\pi)^{2d}} \int\int \frac{\mathrm{d}^d p \ \mathrm{d}^d q}{[m_1^2 + p^2]^{\nu_1} \ [m_2^2 + q^2]^{\nu_2} \ [m_3^2 + (p-q)^2]^{\nu_3}}$$

- In the bosonic case, $m_i = 2n_i\pi T$ play the role of the "masses"
- Massless hot sum-integral $\Leftrightarrow$ massive cold integral
- "Collinear" masses: one is equal to the sum of two others, e.g., $m_3 = m_1 + m_2$

Recurrence relation and boundary conditions

One-loop tadpoles:

$$T_m^{\nu} = \frac{g(\nu)}{m^{2\nu-2}}\, T_m^1, \qquad \text{with} \quad g(\nu) \equiv \frac{\Gamma(\nu - d/2)}{\Gamma(\nu)\Gamma(1-d/2)} \quad \text{and} \quad T_m^1 = \frac{\Gamma(1-d/2)}{(4\pi)^{d/2} m^{2-d}}$$

Recurrence relation (with $\Sigma\nu_i \equiv \nu_1 + \nu_2 + \nu_3$) [Tarasov, 1997]

$$B^{\nu_1\nu_2\nu_3} = \frac{-1}{2(d+3-2\Sigma\nu_i)m_1 m_2 m_3}\Big\{ \left[m_1(d+2-\Sigma\nu_i) + m_2\nu_3 - m_3\nu_2\right]\mathbf{1}^- + \left[m_1\nu_3 + m_2(d+2-\Sigma\nu_i) - m_3\nu_1\right]\mathbf{2}^- + \left[m_1\nu_2 + m_2\nu_1 - m_3(d+2-\Sigma\nu_i)\right]\mathbf{3}^- \Big\} B^{\nu_1\nu_2\nu_3}$$

Boundary conditions (products of tadpoles):

$$B^{\nu_1\nu_2 0} = \frac{g(\nu_1)}{m_1^{2\nu_1-2}}\frac{g(\nu_2)}{m_2^{2\nu_2-2}} B^{110}, \quad B^{\nu_1 0\nu_3} = \frac{g(\nu_1)}{m_1^{2\nu_1-2}}\frac{g(\nu_3)}{m_3^{2\nu_3-2}} B^{101}, \quad B^{0\nu_2\nu_3} = \frac{g(\nu_2)}{m_2^{2\nu_2-2}}\frac{g(\nu_3)}{m_3^{2\nu_3-2}} B^{011},$$

$$B^{\nu_1 00} = 0, \qquad B^{0\nu_2 0} = 0, \qquad B^{00\nu_3} = 0.$$

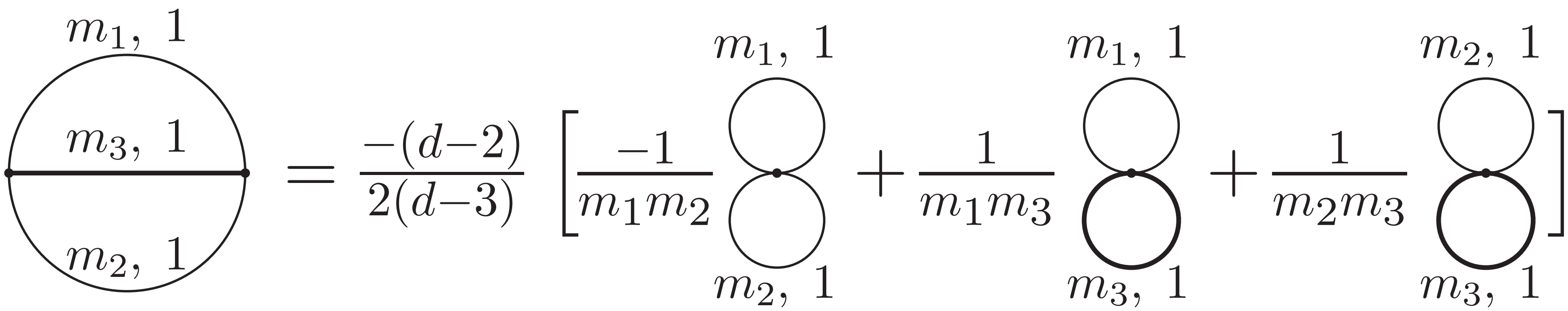
Recurrence relation for equal indices ν_i

For equal indices $\nu_1 = \nu_2 = \nu_3 \equiv \nu$ we have

$$B^{\nu\nu\nu} = \frac{-(d+2-4\nu)}{2(d+3-6\nu)} \left[\frac{\mathbf{1}^-}{m_2 m_3} + \frac{\mathbf{2}^-}{m_1 m_3} - \frac{\mathbf{3}^-}{m_1 m_2} \right] B^{\nu\nu\nu}$$

and, in particular, for $\nu = 1$ we get

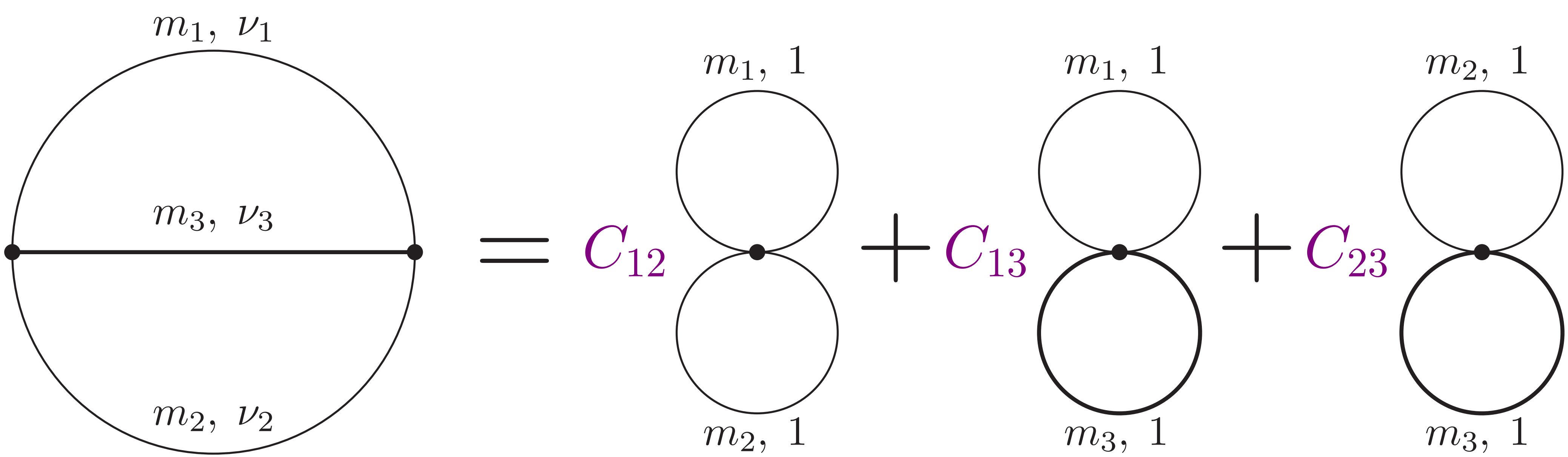
$$\begin{aligned} B^{111} &= \frac{-(d-2)}{2(d-3)} \left[\frac{\mathbf{1}^-}{m_2 m_3} + \frac{\mathbf{2}^-}{m_1 m_3} - \frac{\mathbf{3}^-}{m_1 m_2} \right] B^{111} \\ &= \frac{-(d-2)}{2(d-3)} \left[\frac{B^{011}}{m_2 m_3} + \frac{B^{101}}{m_1 m_3} - \frac{B^{110}}{m_1 m_2} \right] \end{aligned}$$



The master integrals

Therefore, any $B^{\nu_1\nu_2\nu_3}$ with integer ν_1, ν_2, ν_3 can be expressed in terms of three master integrals (remember that $m_3 = m_2 + m_2$)

$$B^{011} = \frac{(4\pi)^{-d}\Gamma^2(1-d/2)}{m_2^{2-d} m_3^{2-d}}, \quad B^{101} = \frac{(4\pi)^{-d}\Gamma^2(1-d/2)}{m_1^{2-d} m_3^{2-d}}, \quad B^{110} = \frac{(4\pi)^{-d}\Gamma^2(1-d/2)}{m_1^{2-d} m_2^{2-d}}.$$



Two-step reduction via Källen recursion

We can obtain the following recurrence relations: [Tarasov, 1997]

$$(d-2)(1-\nu_1)B^{\nu_1\nu_2\nu_3}(d) = \left[2m_2m_3 + \mathbf{1}^- - \mathbf{2}^- - \mathbf{3}^-\right] \mathbf{1}^-\mathbf{d}^- B^{\nu_1\nu_2\nu_3}(d) ,$$

$$(d-2)(d-\Sigma\nu_i)B^{\nu_1\nu_2\nu_3}(d) = \left[-2m_2m_3\mathbf{1}^- - 2m_1m_3\mathbf{2}^- + 2m_1m_2\mathbf{3}^-\right] \mathbf{d}^- B^{\nu_1\nu_2\nu_3}(d) ,$$

where the operator $\mathbf{d}^-$ shifts $d \to d-2$.

Combining these relations and their permutations we arrive at

$$(d-2)(d+3-\Sigma\nu_i)B^{\nu_1\nu_2\nu_3}(d) = \lambda(\mathbf{1}^-, \mathbf{2}^-, \mathbf{3}^-)\, \mathbf{d}^-\, B^{\nu_1\nu_2\nu_3}(d) ,$$

where λ is nothing but the well-known Källen "triangle" function,

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz ,$$

so that

$$\lambda(\mathbf{1}^-, \mathbf{2}^-, \mathbf{3}^-) = \mathbf{1}^-\mathbf{1}^- + \mathbf{2}^-\mathbf{2}^- + \mathbf{3}^-\mathbf{3}^- - 2\cdot\mathbf{1}^-\mathbf{2}^- - 2\cdot\mathbf{1}^-\mathbf{3}^- - 2\cdot\mathbf{2}^-\mathbf{3}^-$$

The coefficients g_{a_1,a_2,a_3}

Generating function for the coefficients g_{a_1,a_2,a_3} (note that $a_1+a_2+a_3$ is even):

$$\sum_{a_1,a_2,a_3\geq 0} g_{a_1,a_2,a_3}\, x^{a_1}y^{a_2}z^{a_3} = \frac{1}{1-\lambda(x,y,z)} = \frac{1}{1-x^2-y^2-z^2+2xy+2xz+2yz}$$

We can get a triple sum representation (with $A\equiv\frac{1}{2}(a_1+a_2+a_3)$),

$$g_{a_1,a_2,a_3} = \sum_{n_1=0}^{\lfloor a_1/2\rfloor}\sum_{n_2=0}^{\lfloor a_2/2\rfloor}\sum_{n_3=0}^{\lfloor a_3/2\rfloor} \frac{A!(-2)^{A-n_1-n_2-n_3}}{n_1!\, n_2!\, n_3!}$$
$$\times\frac{1}{(A-a_1+n_1-n_2-n_3)!(A-a_2-n_1+n_2-n_3)!(A-a_3-n_1-n_2+n_3)!}$$

or, with some effort, an explicit result

$$g_{a_1,a_2,a_3} = \frac{4^A\, A!}{a_1!\, a_2!\, a_3!\, \left(\frac{1}{2}\right)_{a_1-A}\left(\frac{1}{2}\right)_{a_2-A}\left(\frac{1}{2}\right)_{a_3-A}}$$

where $(\alpha)_j \equiv \dfrac{\Gamma(\alpha+j)}{\Gamma(\alpha)}$ is the Pochhammer symbol

Conjecture (proven by induction)

Explicit results for the lower cases suggest that in the general case (for non-negative integer values of ν_i) the solution looks like

$$
\begin{aligned}
B^{\nu_1\nu_2\nu_3} = & \frac{1}{(m_1 m_2)^{\Sigma\nu_i-2}} B^{110} \sum_{j=-(\nu_1-1)}^{\nu_2-1} c^{(\Sigma\nu_i)}_{\nu_1,\nu_2;j} \left(\frac{m_1}{m_2}\right)^j \\
& + \frac{1}{(-m_1 m_3)^{\Sigma\nu_i-2}} B^{101} \sum_{j=-(\nu_1-1)}^{\nu_3-1} c^{(\Sigma\nu_i)}_{\nu_1,\nu_3;j} \left(-\frac{m_1}{m_3}\right)^j \\
& + \frac{1}{(-m_2 m_3)^{\Sigma\nu_i-2}} B^{011} \sum_{j=-(\nu_2-1)}^{\nu_3-1} c^{(\Sigma\nu_i)}_{\nu_2,\nu_3;j} \left(-\frac{m_2}{m_3}\right)^j ,
\end{aligned}
$$

where the universal coefficients $c^{(\nu_a+\nu_b+\nu_c)}_{\nu_a,\nu_b;j}$ correspond to the given set $\{\nu_a, \nu_b, \nu_c\}$, and the indices ν_a and ν_b represent the non-zero positions of the arguments of the corresponding basis elements B^{110}, B^{101} and B^{011}. In particular,

$$
c^{(\nu_a+\nu_c)}_{\nu_a,0} = c^{(\nu_b+\nu_c)}_{0,\nu_b} = 0, \qquad c^{(\nu_a+\nu_b+\nu_c)}_{\nu_a,\nu_b;-j} = c^{(\nu_a+\nu_b+\nu_c)}_{\nu_b,\nu_a;j} .
$$

Solution for the coefficients $c^{(\nu_a+\nu_b+\nu_c)}_{\nu_a,\nu_b;j}$

Due to $c^{(\nu_a+\nu_b+\nu_c)}_{\nu_a,\nu_b;-j} = c^{(\nu_a+\nu_b+\nu_c)}_{\nu_b,\nu_a;j}$, it is enough to consider only the case $j \geq 0$. In this case we get an explicit solution:

$$(-1)^{\nu_a+\nu_b+\nu_c}\, c^{(\nu_a+\nu_b+\nu_c)}_{\nu_a,\nu_b;j}$$
$$= \sum_{k=0}^{\min(\nu_a-1,\nu_b-j-1)} \frac{(1-\frac{d}{2})_{n_j-1}(1-\frac{d}{2})_{n_j-j-1}}{(1-\frac{d}{2})_{n_j-j-k-1}(\frac{d+3}{2}-\nu_a-\nu_b-\nu_c)_{n_j-j-k-1}} \frac{G_{\nu_a-k-1,\nu_b-j-k-1,\nu_c-1}}{(-4)^{n_j-j-k-1}\, k!\, (j+k)!}$$

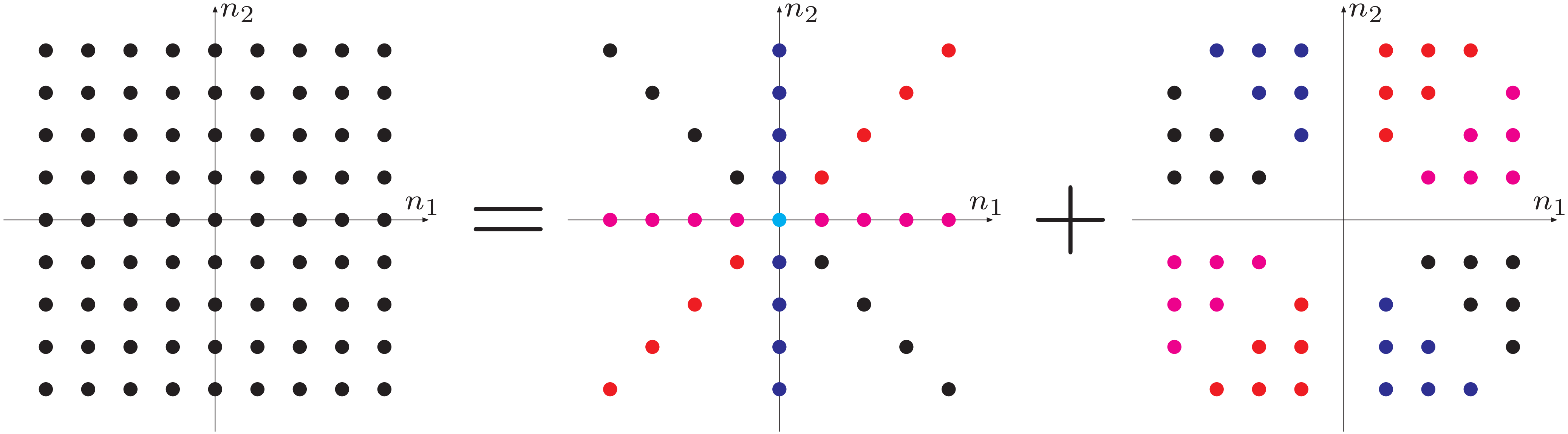
where

$$n_j \equiv \left\lceil \frac{\nu_a+\nu_b+\nu_c+j}{2} \right\rceil$$

and

$$G_{a_1,a_2,a_3} = \begin{cases} 2\, g_{a_1,a_2,a_3}, & a_1+a_2+a_3 \quad \text{even} \\ -g_{a_1-1,a_2,a_3} - g_{a_1,a_2-1,a_3} + g_{a_1,a_2,a_3-1}, & a_1+a_2+a_3 \quad \text{odd} \end{cases}$$

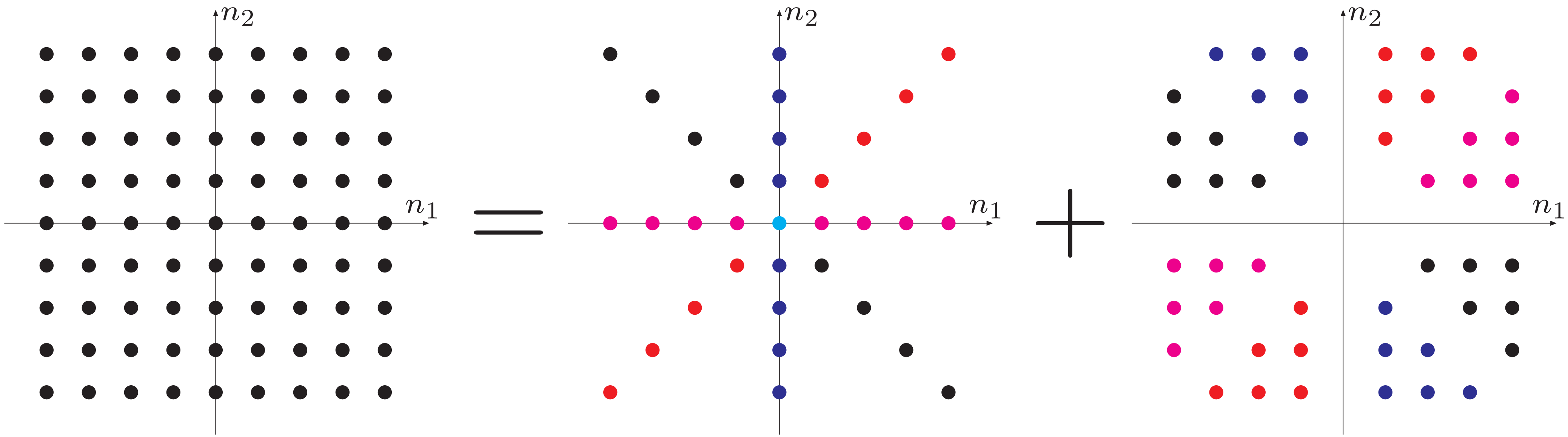
Back to the "hot" sum-integrals



$$L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3}(d,T) = \sum\!\!\!\!\!\!\!\int_{P,Q} \frac{(P_0)^{\eta_1}\,(Q_0)^{\eta_2}\,(P_0-Q_0)^{\eta_3}}{[P^2]^{\nu_1}\,[Q^2]^{\nu_2}\,[(P-Q)^2]^{\nu_3}}$$

$$= \frac{T^2}{(2\pi T)^{2\Sigma\nu_i-\Sigma\eta_i-2d}} \sum_{n_1,n_2} n_1^{\eta_1}\,n_2^{\eta_2}\,(n_1-n_2)^{\eta_3} B^{\nu_1,\nu_2,\nu_3}_{n_1,n_2,n_1-n_2}(d)$$

Back to the “hot” sum-integrals



Using the conjecture for B^{ν_1,ν_2,ν_3} we proved that the sums in $L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3}$ combine to

- evaluate to single and double zeta values only
- cancel all double zeta values $\zeta(i,j) = \sum_{n_2=1}^{\infty} \sum_{n_1=n_2+1}^{\infty} \frac{1}{n_1^i n_2^j}$
- cancel all remaining single $\zeta(i) = \sum_{n=1}^{\infty} \frac{1}{n^i}$
- leave us with products $\zeta(i)\zeta(j)$ containing only $\zeta(\text{even number} - d)$

The final result for two-loop bosonic sum-integrals

$$
\begin{aligned}
L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3} = {} & \frac{T^2\left[1+(-1)^{\Sigma\eta_i}\right]}{(2\pi T)^{2\Sigma\nu_i-\Sigma\eta_i-2d}} \frac{\Gamma^2\left(1-\frac{d}{2}\right)}{(4\pi)^d} \\
& \times \Bigg\{ \sum_{j=-(\nu_1-1)}^{\nu_2-1} (-1)^{\Sigma\nu_i}\, c^{(\Sigma\nu_i)}_{\nu_1,\nu_2;j} \sum_{k=0}^{\eta_3} \binom{\eta_3}{k} (-1)^{\eta_2} \left[1+(-1)^{\ell_1}\right] \\
& \qquad \times \zeta(\ell_1-d)\, \zeta(2\Sigma\nu_i-\Sigma\eta_i-\ell_1-d) \\
& \qquad + \text{ permutations} \Bigg\}
\end{aligned}
$$

with $\ell_1 = \Sigma\nu_i - \eta_1 - j - k$, etc.

Obviously, this also means that the two-loop bosonic sum-integral $L^{\eta_1,\eta_2,\eta_3}_{\nu_1,\nu_2,\nu_3}$ can be written in terms of the products of the one-loop bosonic sum-integrals I^{η}_{ν}.

Conclusions and outlook

- Massless two-loop bosonic vacuum sum-integrals completely dug out and understood (recursion-free solution, etc.)
  - Due to the "magic connection" [AD, Tausk, 1996], we also get (for free!) the recursion-free solution for the collinear configuration of the one-loop massless three-point function
- Fermionic case needs to be considered as well
- Massive particles (much more complicated, even at the one-loop level)
- Generalization to higher loops? (some results are available)

Something to celebrate?
Photo by Alex Kovner, December 2019