Energy of anti-de Sitter black holes in odd-dimensional Quadratic Curvature Gravity

Based on: Phys.Rev.D **106, 044048,**

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September 9, 2022

Outline

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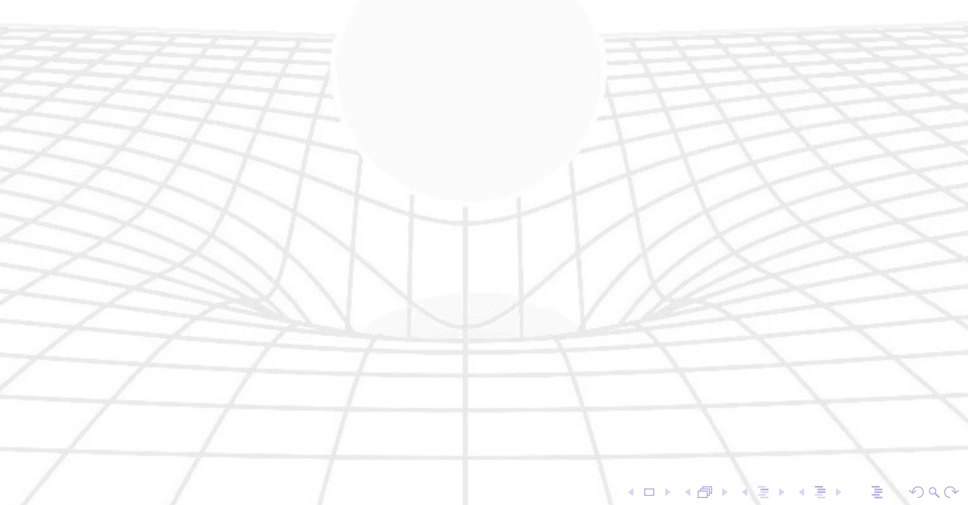
AdS/CFT Correspondence

- The AdS/CFT correspondence: duality between the short distance regime of conformal quantum field theory (CFT) and the long distance regime of classical gravitational theory (AdS)



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- However, $I_{\mathrm{gr}}(AdS)|_{\mathrm{on-shell}} \to \infty \implies$ Holographic Renormalization (Infrared Renormalization)

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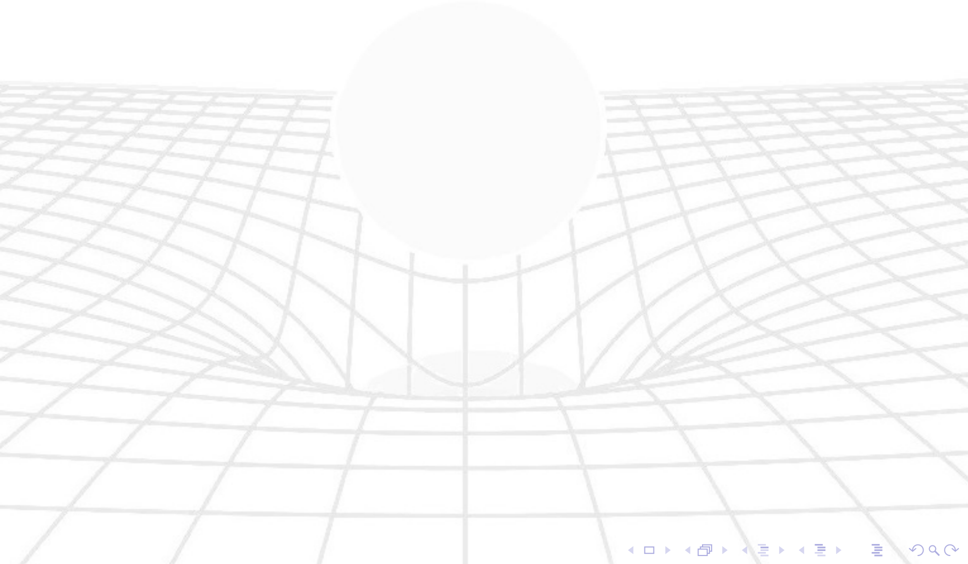

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with $\Lambda = -\frac{d(d-1)}{2\ell^2}$ and K is the trace of extrinsic curvature $K_{ij} = -\frac{1}{2}n^{\mu}\partial_{\mu}h_{ij}$, and
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- Holographic information, in FG coordinates

$$\boxed{T^{ij}[g_{(0)}] = \lim_{z\to 0}\left(\frac{1}{z^{d-2}}\,T^{ij}_{\rm ren}[h]\right) \equiv \langle T_{ij}\rangle_{CFT}}$$ Holographic stress-tensor

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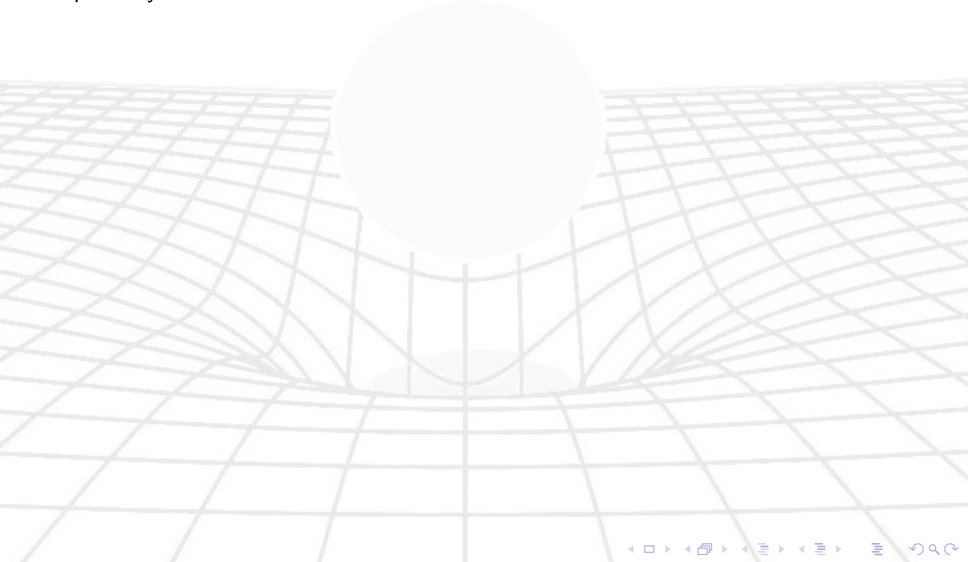
Kounterterms in AdS Gravity

- No general expression for L_{ct} in any dimension $\Longrightarrow$ need to Explore a different pathway to renormalization in AdS



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$$\boxed{\tilde{I}_{\rm ren} = I_{\rm bulk} + c_d \int_{\partial M} d^d x \sqrt{-h}\, B_d(f(h), K)}$$

$\Longrightarrow$ Well possed variational principle for Dirichlet boundary conditions only for $g_{(0)ij}$

$$\delta \tilde{I}_{\rm ren} = \delta \tilde{I}_{\rm ren}(\delta h, \delta K) = \frac{1}{2}\int_{\partial M} d^d x \sqrt{-g_{(0)}}\tau^{ij}\delta g_{(0)ij}\,,$$ Action holographically finite !!!

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- Another observation: in $d+1=5$

$$T_{\rm ren}{}^{i}{}_{j} = -\left(\mathcal{R}^{i}_{j} - K^{i}_{j}K + K^{i}_{k}K^{k}_{j} + \frac{3}{\ell^2}\delta^{i}_{j}\right) + \Delta^{i}_{j}\,,$$

which can be truncated as $T_{\rm ren}{}^{i}{}_{j} = \pi^{i}_{j} + \mathcal{O}(K^2) \implies B_d$ non-linear in K.

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- In odd dimensions $d+1=2n+1$

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   - AdS Gravity
   - Kounterterms in AdS Gravity
2. Quadratic Curvature Gravity
   - Quadratic Curvature Gravity Action
   - Renormalized Action
   - Energy for Static Black Holes
3. Discussions

Quadratic Curvature Gravity

- The Quadratic Curvature Gravity (QCG) action is

$$I_{\mathrm{QCG}} = \int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left[\frac{1}{\kappa}\left(R - 2\Lambda_0\right) + \alpha R_{\mu\nu}R^{\mu\nu} + \beta R^2 + \gamma\, GB\right]$$

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Einstein's Tensor: $G_{\mu\nu} = R_{\mu\nu}-\frac{1}{2}\,g_{\mu\nu}\,R+\Lambda_0\,g_{\mu\nu}$

Lanczos tensor: $H_{\mu\nu} = -\frac{1}{8}g_{\mu\lambda}\,\delta^{\lambda\mu_1\cdots\mu_4}_{\nu\nu_1\cdots\nu_4}R^{\nu_1\nu_2}_{\mu_1\mu_2}R^{\nu_3\nu_4}_{\mu_3\mu_4}\,.$

Quadratic terms:
$$P_{\mu\nu} = 2\beta R\left(R_{\mu\nu}-\frac{1}{4}\,g_{\mu\nu}R\right)+(\alpha+2\beta)\left(g_{\mu\nu}\Box-\nabla_\mu\nabla_\nu\right)R$$
$$+\alpha\Box G_{\mu\nu}+2\alpha\left(R_{\mu\sigma\nu\lambda}-\frac{1}{4}\,g_{\mu\nu}R_{\sigma\lambda}\right)R^{\sigma\lambda}$$

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Einstein branch: Λ^{+}_{eff}, non-Einstein branch: Λ^{-}_{eff}

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- Asymptoticaly anti-de Sitter (AAdS) spacetimes: $\Lambda_{\text{eff}} = -\frac{d(d-1)}{2\ell^2_{\text{eff}}}$,

$$\frac{1}{\ell_0^2} = \frac{1}{\ell^2_{\text{eff}}} - \frac{\kappa d\,(d-3)}{\ell^4_{\text{eff}}}\left(\frac{\alpha+(d+1)\beta}{d-1} + \gamma\,\frac{d-2}{d}\right)$$

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where, $B_d = B_d\left(K, \mathcal{R}, \ell_{\mathrm{eff}}\right)$.

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- For AAdS spaces, the boundary term vanished $\Longrightarrow$ specific coupling c_d. In odd dimensions $d+1 = 2n+1$

$$c_{2n} = \frac{(-1)^n \ell_{\rm eff}^{2n-2}}{2^{2n-2}\kappa n(n-1)!^2}\left[1 - \frac{4n\kappa}{\ell_{\rm eff}^2}\left(\alpha + (2n+1)\,\beta + \gamma\,\frac{(2n-1)\,(2n-2)}{2n}\right)\right]$$

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  where, $B_d = B_d\left(K, \mathcal{R}, \ell_{\mathrm{eff}}\right)$.

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$$\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\,\nabla_\alpha \Theta^\alpha = \int_{\partial\mathcal{M}} d^d x\,\sqrt{-h}\,n_\mu \Theta^\mu(\delta g, \delta\Gamma) = \int_{\partial\mathcal{M}} d^d x\, N\sqrt{-h}\,\Theta^z$$

- The variation of renormalized action

$$\delta I_{\mathrm{QCG}}^{\mathrm{ren}} = \int_{\partial\mathcal{M}} d^d x\,\left(\sqrt{-h}N\Theta^z + c_d\delta(\sqrt{-h}B_d)\right)$$

- For AAdS spaces, the boundary term vanished $\Longrightarrow$ specific coupling c_d. In odd dimensions $d+1 = 2n+1$

$$c_{2n} = \frac{(-1)^n \ell_{\mathrm{eff}}^{2n-2}}{2^{2n-2}\kappa n(n-1)!^2}\left[1 - \frac{4n\kappa}{\ell_{\mathrm{eff}}^2}\left(\alpha + (2n+1)\,\beta + \gamma\,\frac{(2n-1)\,(2n-2)}{2n}\right)\right]$$

- Renormalized action $\Longrightarrow$ Finite conserved charge.

Renormalized conserved charge

- Invariant action under $x'^{\mu} = x^{\mu} + \xi^{\mu}$

$$\Longrightarrow \text{Noether current:} \qquad J^{\mu} = \Theta^{\mu} + \mathcal{L}\xi^{\mu}$$

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$$Q^{\mu\nu} \rightarrow \; Q^{\mu\nu} + c_d \xi^{[\mu} B_d^{\nu]} = Q^{\mu\nu}_{\text{ren}} \,,$$

where $B_d = n_{\mu} B_d^{\mu}$.

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where $B_d = n_{\mu} B_d^{\mu}$.
- The Kounterterm charge $n_{\nu} Q^{i\nu}_{\text{ren}} = q^i_j \xi^j$,

$$Q[\xi] = \int_{\Sigma} d^{d-1}x \sqrt{\sigma}\, u_i\, q^i_j\, \xi^j ,$$

where $u_i u^i = -1$ and $\xi = \xi^i \partial_i$ are asymptotics isometries.

Renormalized conserved charge in odd dimensions

- In the odd dimensional case

$$Q[\xi] = \int_\Sigma d^{d-1}x\,\sqrt{\sigma}\,u_i\left(\,q^i_j + q^i_{(0)j}\right)\xi^j = q[\xi] \;+\; \underbrace{q_{(0)}[\xi]}_{\sim\text{AdS vacuum energy}}$$

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- For the QCG case

$$q_i^j = \frac{1}{2^{n-2}}\delta^{jj_1\cdots j_{2n-1}}_{ki_1\cdots i_{2n-1}} K_i^k \delta^{i_1}_{j_1}\Bigg[nc_{2n}\int_0^1 du\left(R^{i_2i_3}_{j_2j_3} + \frac{u^2}{\ell^2_{\text{eff}}}\delta^{i_2i_3}_{j_2j_3}\right)\cdots\left(R^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}} + \frac{u^2}{\ell^2_{\text{eff}}}\delta^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}\right)$$

$$-\frac{1}{(2n-1)!}\left(\frac{1}{\kappa}\delta^{i_2i_3}_{j_2j_3} + \alpha R^z_z\delta^{i_2i_3}_{j_2j_3} + 2\beta R\delta^{i_2i_3}_{j_2j_3} + 2(2n-1)(2n-2)\gamma R^{\ i_2i_3}_{\ j_2j_3}\right)\delta^{i_4i_5}_{j_4j_5}\cdots\delta^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}\Bigg]$$

$$-2N\left[\alpha\left(\nabla^z R^j_i - \nabla^j R^z_i + \nabla^z R^z_z\delta^j_i + \nabla^k R^z_k\delta^j_i\right) + 2\beta\,\nabla^z R\,\delta^j_i\right] - 2\alpha\,K^j_k R^k_i$$

$$-N\left[\alpha\left(\nabla^k R^z_k\delta^j_i - \nabla^j R^z_i\right) + 4\gamma\left(\nabla^j R^z_i - \delta^j_i\nabla^k R^z_k + \nabla^k R^{jz}_{ik}\right)\right],$$

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$$Q[\xi] = \int_{\Sigma} d^{d-1}x \sqrt{\sigma} u_i \left(q^i_j + q^i_{(0)j} \right) \xi^j = q[\xi] + \underbrace{q_{(0)}[\xi]}_{\sim \text{AdS vacuum energy}}$$

- For the QCG case

$$q^j_i = \frac{1}{2^{n-2}} \delta^{jj_1 \cdots j_{2n-1}}_{ki_1 \cdots i_{2n-1}} K^k_i \delta^{i_1}_{j_1} \left[nc_{2n} \int_0^1 du \left(R^{i_2 i_3}_{j_2 j_3} + \frac{u^2}{\ell^2_{\text{eff}}} \delta^{i_2 i_3}_{j_2 j_3} \right) \cdots \left(R^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}} + \frac{u^2}{\ell^2_{\text{eff}}} \delta^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}} \right) \right.$$
$$\left. - \frac{1}{(2n-1)!} \left(\frac{1}{\kappa} \delta^{i_2 i_3}_{j_2 j_3} + \alpha R^z_z \delta^{i_2 i_3}_{j_2 j_3} + 2\beta R \delta^{i_2 i_3}_{j_2 j_3} + 2(2n-1)(2n-2)\gamma R^{\ i_2 i_3}_{\ j_2 j_3} \right) \delta^{i_4 i_5}_{j_4 j_5} \cdots \delta^{i_{2n-2} i_{2n-1}}_{j_{2n-2} j_{2n-1}} \right]$$
$$-2N \left[\alpha \left(\nabla^z R^j_i - \nabla^j R^z_i + \nabla^z R^z_z \delta^j_i + \nabla^k R^z_k \delta^j_i \right) + 2\beta \nabla^z R \delta^j_i \right] - 2\alpha K^j_k R^k_i$$
$$- N \left[\alpha \left(\nabla^k R^z_k \delta^j_i - \nabla^j R^z_i \right) + 4\gamma \left(\nabla^j R^z_i - \delta^j_i \nabla^k R^z_k + \nabla^k R^{jz}_{ik} \right) \right],$$

-

$$q^i_{(0)j} = nc_{2n} \delta^{ii_1 \cdots i_{2n-1}}_{kj_1 \cdots j_{2n-1}} \int_0^1 du\, u \left(K^k_j \delta^{j_1}_{i_1} + K^k_{i_1} \delta^{j_1}_j \right) \left(\frac{1}{2} \mathcal{R}^{j_2 j_3}_{i_2 i_3} - u^2 K^{j_2}_{i_2} K^{j_3}_{i_3} + \frac{u^2}{\ell^2_{\text{eff}}} \delta^{j_2}_{i_2} \delta^{j_3}_{i_3} \right) \times$$
$$\cdots \times \left(\frac{1}{2} \mathcal{R}^{j_{2n-2} j_{2n-1}}_{i_{2n-2} i_{2n-1}} - u^2 K^{j_{2n-2}}_{i_{2n-2}} K^{j_{2n-1}}_{i_{2n-1}} + \frac{u^2}{\ell^2_{\text{eff}}} \delta^{j_{2n-2}}_{i_{2n-2}} \delta^{j_{2n-1}}_{i_{2n-1}} \right).$$

Outline

1. Motivation
   - AdS/CFT Correspondence
   - AdS Gravity
   - Kounterterms in AdS Gravity
2. Quadratic Curvature Gravity
   - Quadratic Curvature Gravity Action
   - Renormalized Action
   - Energy for Static Black Holes
3. Discussions

Topological Black Holes

- Static ansatz, in coordinates $x^{\mu} = (t, r, \varphi^{n})$,

$$ds^2 = -f^2(r)dt^2 + f^{-2}(r)dr^2 + r^2\gamma_{mn}(\varphi)\, d\varphi^m d\varphi^n\,, \qquad \varphi^m \in \Sigma^{d-1},$$

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- Here, $\gamma_{mn}(\varphi)$ is the metric of a $(d-1)$-dimensional Riemann space Σ^{d-1} with constant curvature k,

$$\mathcal{R}^{mn}_{pq}(\gamma) = k\,\delta^{mn}_{pq}\,, \quad \text{where} \;\; k \in \{-1, 0, 1\} \qquad \Longrightarrow \text{Topological BH (TBH)}$$

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- In AAdS in QCG, one can assume an asymptotic behavior of the form

$$f^2(r) \approx k - \left(\frac{r_0}{r}\right)^{d-2} + \frac{r^2}{\ell^2_{\text{eff}}} + \cdots\,,$$

where the constant r_0^{d-2} is related with the mass of the TBH.

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- Putting $\kappa = 2\mathrm{Vol}(S^{d-1})G$. The energy/mass corresponds to $q[\partial_t] = E$,

$$E = \left[1 + \frac{4\kappa\Lambda_{\text{eff}}}{(2n-1)}\left(\alpha + \beta(2n+1) + \gamma\frac{(2n-2)(2n-3)}{2n}\right)\right]\frac{\mathrm{Vol}(\Sigma^{2n-1})}{\mathrm{Vol}(S^{2n-1})}\frac{(2n-1)r_0^{2n-2}}{4G}$$

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- And the vacuum energy $q_{(0)}[\partial_t] = E_{\mathrm{vac}}$,

$$E_{\mathrm{vac}} = k^n\,(2n-1)!\,c_{2n}\,\mathrm{Vol}(\Sigma^{2n-1})$$

$$= (-k)^n\frac{2(2n-1)!!^2}{(2n)!}\frac{\mathrm{Vol}(\Sigma^{2n-1})}{\kappa}\ell_{\mathrm{eff}}^{2n-2}\left[1 - \frac{4n\kappa}{\ell_{\mathrm{eff}}^2}\left(\alpha + (2n+1)\beta + \gamma\frac{(2n-1)(2n-2)}{2n}\right)\right]$$

Outline

Discussions

- Kounterterms renormalization scheme allows us to define a finite action and, in consecuence, a finite conserved charges for QCG in AAdS, for any dimension

$$I_{\mathrm{QCG}}^{\mathrm{ren}} = I_{\mathrm{QCG}} + c_d \int_{\partial\mathcal{M}} d^d x \sqrt{-h}\, B_d(\mathcal{R}, K)\,,$$

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- In general, q^i_j contains higher order curvatures terms. However, if we expand to first order

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$$Q_{\mathrm{Lovelock}}[\xi] = -\frac{\ell_{\mathrm{eff}}\,\Delta'}{8\pi G(d-2)} \int_{\Sigma} d^{d-1} x \sqrt{\sigma}\, u_i\, \mathcal{E}^i_j\, \xi^j\,,$$

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where $\Delta' = 0$ give us the degenerate vacuum condition.

- Unlike Lovelock, in QCG case, $\Xi_d = 0$ is not the degenerate vacuum condition, then we can have a well-defined charge, even in the degenerate case

$$\Xi_d - \Delta' = -4\kappa d\, \frac{\alpha + (d+1)\beta}{(d-1)\ell^2_{\rm eff}}\,.$$

Acknowledgment

This work is supported by the ANID National M.Sc. Scholarship N° 22210771, and ANID-SCIA-ANILLO ACT210100 *Holography and its applications to High Energy Physics, Quantum Gravity and Condensed Matter Systems*.

Muchas Gracias a Todos y Todas !!!