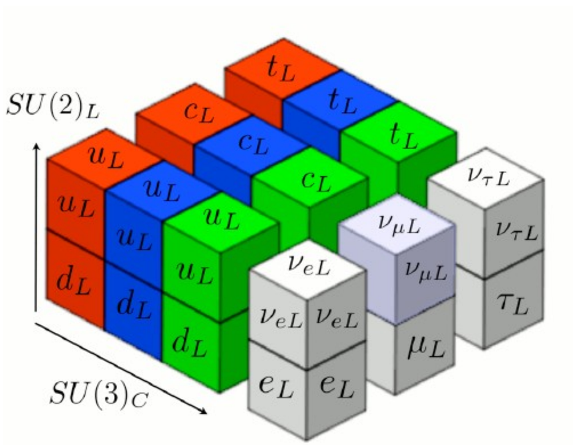
International Conference on Neutrinos and Dark Matter (NuDM-2022)

Modular symmetries and the flavor problem

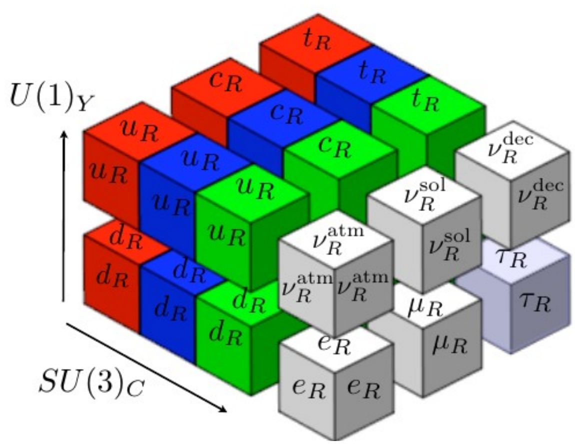
Davide Meloni
Dipartimento di Matematica e Fisica, Roma Tre

The Standard Model of Particle Physics

Left-handed

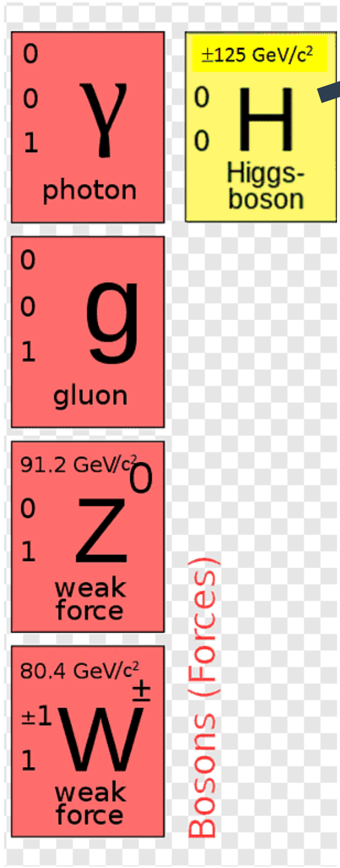


Right-handed



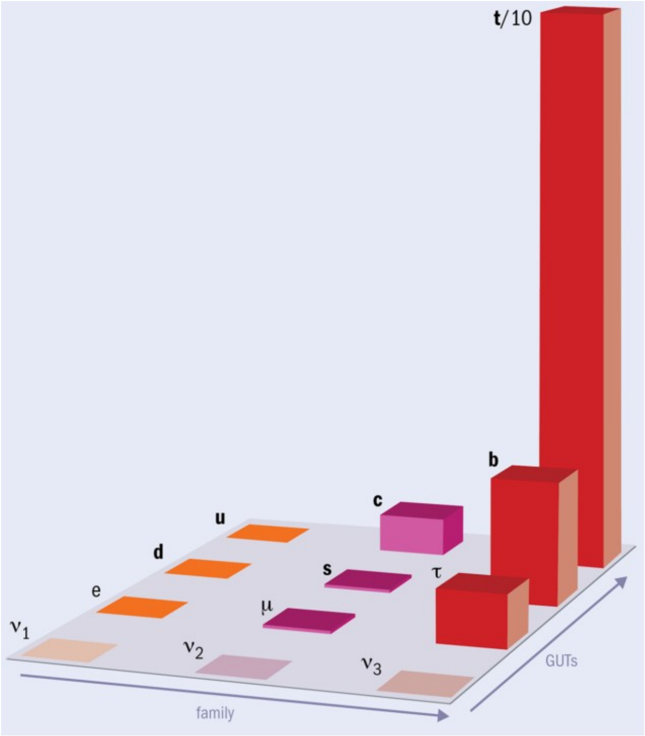
Scalar sector

Gauge boson sector



The Flavor Problem

Mass hierarchies



$$m_d \ll m_s \ll m_b \,, \quad \frac{m_d}{m_s} = 5.02 \times 10^{-2} \,,$$

$$m_u \ll m_c \ll m_t \,, \quad \frac{m_u}{m_c} = 1.7 \times 10^{-3} \,,$$

$$\frac{m_s}{m_b} = 2.22 \times 10^{-2} \,, \quad m_b = 4.18 \;\; \text{GeV};$$

$$\frac{m_c}{m_t} = 7.3 \times 10^{-3} \,, \quad m_t = 172.9 \;\; \text{GeV};$$

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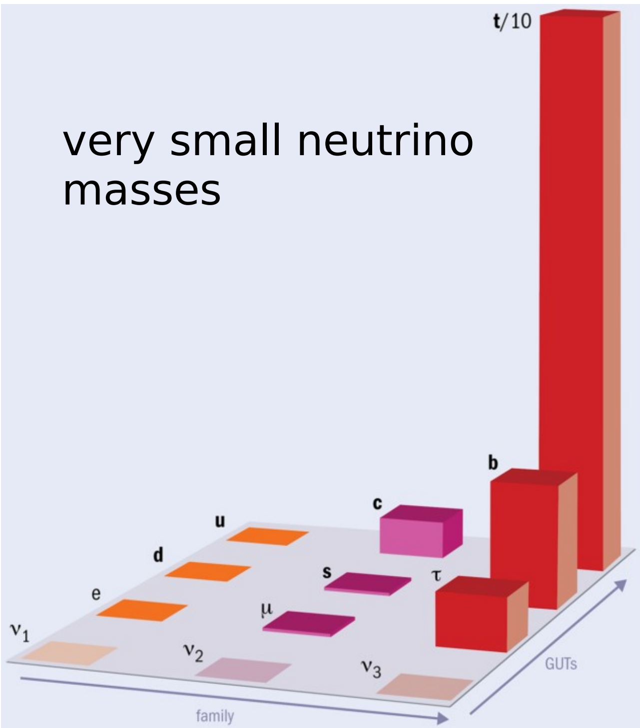

very small neutrino masses

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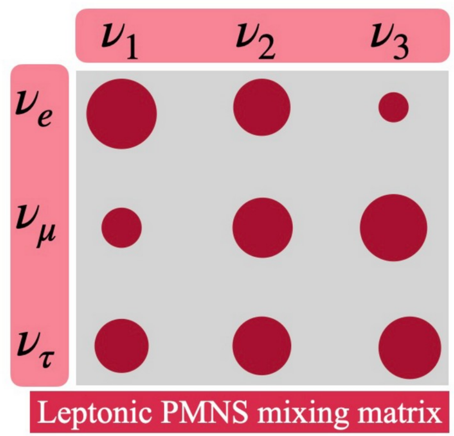
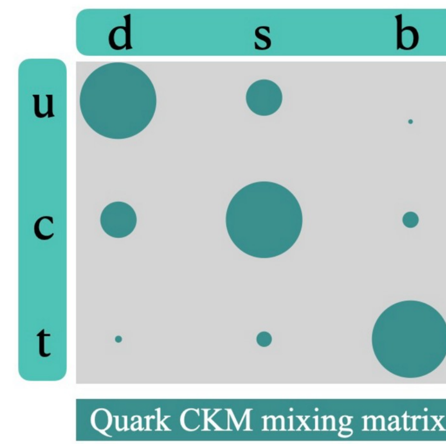
Fermion mixing





all mixing are large but the 13 element

almost a diagonal matrix

Suggested solutions

* Smallness of neutrino masses:

See-saw

$$\mathcal{M} = \begin{bmatrix} m_M^L & m_D \\ m_D & m_M^R \end{bmatrix}$$

$$m_{light} \sim \frac{m_D^2}{M_M^R}$$

No clue on mixing !

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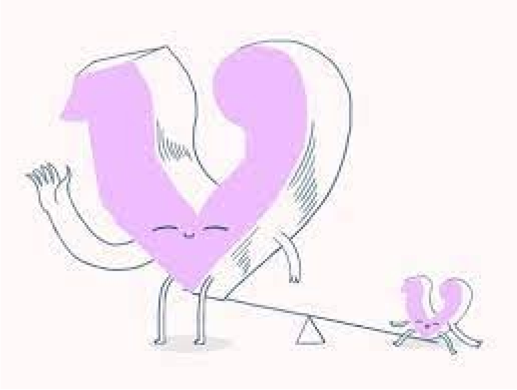

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* Hierarchical
Pattern

Froggatt-Nielsen
mechanism

$$L \sim \overline{\Psi}_L H \Psi_R \left(\frac{\theta}{\Lambda}\right)^n \rightarrow e^{(-q_L + q_H + q_R + n * q_\theta)}$$

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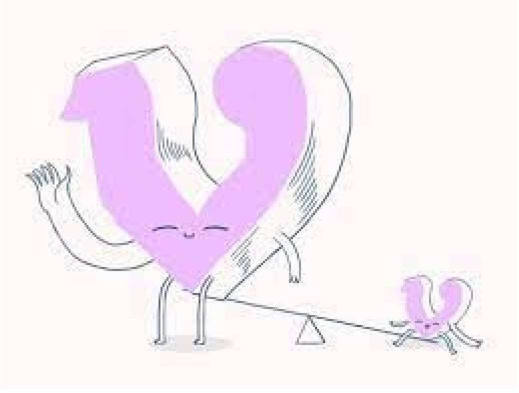

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Works better for small mixing

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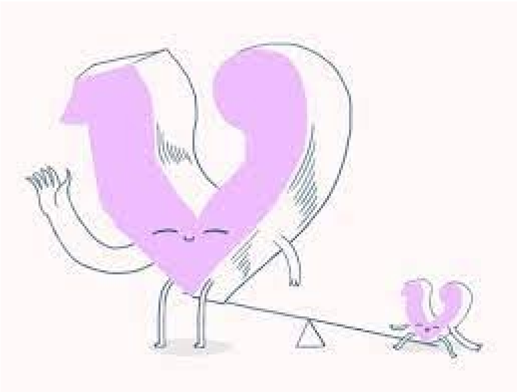

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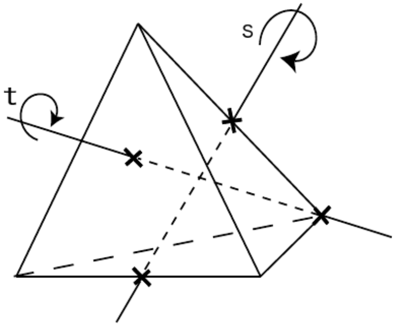
* mixing angles

elegant explanation:
non-Abelian
discrete flavour symmetries



Complicated scalar sector

Modular Symmetry

We start from

Feruglio, 1706.08749

$$\Gamma(N)=\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,Z),\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}(Mod\,N)\right\}$$

the group of 2x2 matrices with integer entries
modulo N and determinant equals to one modulo N

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Γ(N), N>=2 are infinite normal subgroups of Γ

the group Γ(N) acts on the complex variable τ (Im τ >0)

$$\gamma\,\tau=\frac{a\,\tau+b}{c\,\tau+d}$$

Modular Symmetry

Important observation for $N=1$: a transformation characterized by parameters {a, b, c, d} is identical to the one defined by {-a, -b, -c, -d}

Γ**(1)** is isomorphic to **PSL(2, Z) = SL(2, Z)/{±1} =** Γ

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In addition:

$$\bar{\Gamma}(2) = \Gamma(2)/\{1, -1\}$$

since 1 and -1 **cannot** be distinguished

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Finite Modular Group:

$$\boxed{\Gamma_N = \frac{\bar{\Gamma}}{\bar{\Gamma}(N)}}$$

Modular Symmetry

Generators of Γ_N : elements S and T satisfying

$$S^2 = 1, \quad (ST)^3 = 1, \quad T^N = 1$$

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

corresponding to:

$$\tau \xrightarrow{S} -\frac{1}{\tau} \qquad \tau \xrightarrow{T} \tau + 1$$

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<u>relevant for model building:</u>

for N ≤ 5, the finite modular groups Γ_N are isomorphic to non-Abelian discrete groups

$$\boxed{\Gamma_2 \simeq S_3 \qquad \Gamma_3 \simeq A_4 \qquad \Gamma_4 \simeq S_4 \qquad \Gamma_5 \simeq A_5}$$

Then the question is: why Modular Symmetry ?

Modular Forms

Modular Forms:

holomorphic functions of the complex variable τ with well-defined transformation properties under the group **Γ(N)**

$$f(\gamma\tau)=(c\,\tau+d)^{2k}f(\tau), \qquad \gamma=\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\Gamma(N)$$

2k = weigth, N = level

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2k = weigth, N = level

K < 0:
no modular forms

K = 0:
constant functions

K > 0:
linear space of finite dimension

N	$d_{2k}(\Gamma(N))$
2	$k+1$
3	$2k+1$
4	$4k+1$
5	$10k+1$
6	$12k$
7	$28k-2$

R. C. Gunning, Lectures on Modular Forms, Princeton, New Jersey USA, Princeton University Press 1962

Model Building

<u>Key points:</u>

1. Modular forms of weight 2k and level N ≥ 2 are invariant, up to the factor $(c\tau + d)^{2k}$ under $\boldsymbol{\Gamma(N)}$ but they transform under Γ_N !

$$f_i(\gamma\tau) = (c\tau + d)^{2k} \rho(\gamma)_{ij} f_j(\tau)$$

representative element of Γ_N

unitary representation of Γ_N

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2. in addition, one assumes that the fields of the theory χ_i transforms non-trivially under Γ_N

$$\chi(x)_i \to (c\,\tau+d)^{-k_i}\rho(\gamma)_{ij}\chi(x)_j$$

not modular forms !
No restrictions on ki

Model Building

<u>Building blocks:</u>

1. Modular forms and fields: $L_{eff} \in \ f(\tau) \times \phi^{(1)} \ldots \phi^{(n)}$

Model Building

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1. Modular forms and fields: $L_{eff} \in f(\tau) \times \phi^{(1)} \dots \phi^{(n)}$

2. Invariance under modular transformation requires:

$$2k = \Sigma_i k_i$$

$$\rho_f \otimes \rho_{\chi_1} \otimes \dots \otimes \rho_{\chi_n} \supset I$$

To start playing the game:

Can someone give me the Modular Forms?

Model Building

Long list from **S.T. Petcov, Bethe Forum, University of Bonn, 04/05/2022**

For ($\Gamma_3 \simeq A_4$), the generating (basis) modular forms of weight 2 were shown to form a 3 of A_4 (expressed in terms of log derivatives of Dedekind η-function η'/η of 4 different arguments).

F. Feruglio, arXiv:1706.08749

For ($\Gamma_2 \simeq S_3$), the two basis modular forms of weight 2 were shown to form a 2 of S_3 (expressed in terms of η'/η of 3 different arguments).

T. Kobayashi, K. Tanaka, T.H. Tatsuishi, arXiv:1803.10391

For ($\Gamma_4 \simeq S_4$), the 5 basis modular forms of weight 2 were shown to form a 2 and a $3'$ of S_4 (expressed in terms of η'/η of 6 different arguments).

J. Penedo, STP, arXiv:1806.11040

For ($\Gamma_5 \simeq A_5$), the 11 basis modular forms of weight 2 were shown to form a 3, a $3'$ and a 5 of A_5 (expressed in terms of Jacobi theta function $\theta_3(z(\tau), t(\tau))$ for 12 different sets of $z(\tau), t(\tau)$).

P.P. Novichkov et al., arXiv:1812.02158; G.-J. Ding et al., arXiv:1903.12588

Multiplets of higher weight modular forms have been also constructed from tensor products of the lowest weight 2 multiplets:
i) for $N = 4$ (i.e., S_4), multiplets of weight 4 (weight $k \leq 10$) were derived in arXiv:1806.11040 (arXiv:1811.04933);
ii) for $N = 3$ (i.e., A_4) multiplets of weight $k \leq 6$ were found in arXiv:1706.08749;
iii) for $N = 5$ (i.e., A_5), multiplets of weight $k \leq 10$ were derived in arXiv:1812.02158.

Model Building

Constructing the Modular Forms

Crucial observation:

if $$g(\tau) \rightarrow e^{i\alpha}(c\tau+d)^k g(\tau)$$

then $$\frac{d}{d\tau}\log[g(\tau)] \rightarrow (c\tau+d)^2\frac{d}{d\tau}\log[g(\tau)]+\underbrace{kc(c\tau+d)}$$

this term prevents of having a modular form of weight **2 k = 2**

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The inhomogeneous term can be removed if we combine several $f_i(\tau)$ with weights k_i

$$\frac{d}{d\tau}\Sigma_i\log[g_i(\tau)] \rightarrow (c\tau+d)^2\frac{d}{d\tau}\Sigma_i\log[g_i(\tau)]+(\Sigma_i k_i)\,c\,(c\tau+d)$$

with $\Sigma_i k_i=0$

A case study: $\Gamma_2 \sim S_3$

Let us find the functions f(τ) !

The group S_3 contains 1 + 1’ + 2

N	$d_{2k}(\Gamma(N))$
2	$k+1$
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two independent modular forms can fit into a doublet of S_3

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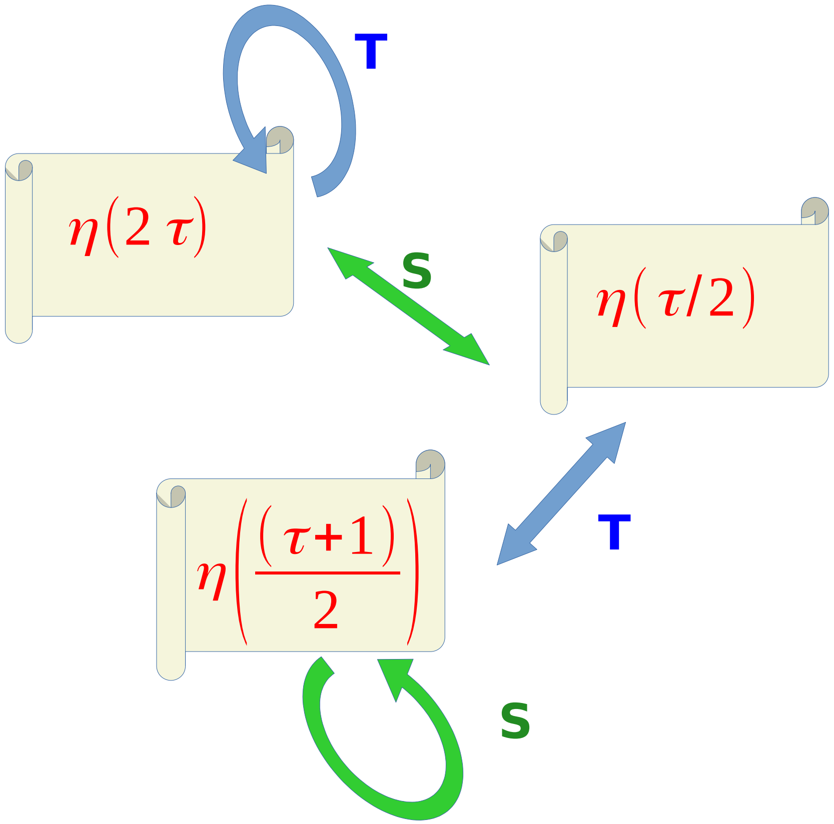
<u>Dedekind eta functions</u> $\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1-q^n)$ $\qquad q \equiv e^{i2\pi\tau}$

S: $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$, T: $\eta(\tau+1) = e^{i\pi/12}\,\eta(\tau)$

η^{24} is a modular form of weight 12

A case study: $\Gamma_2 \sim S_3$

<u>Constructing the Modular Forms</u>



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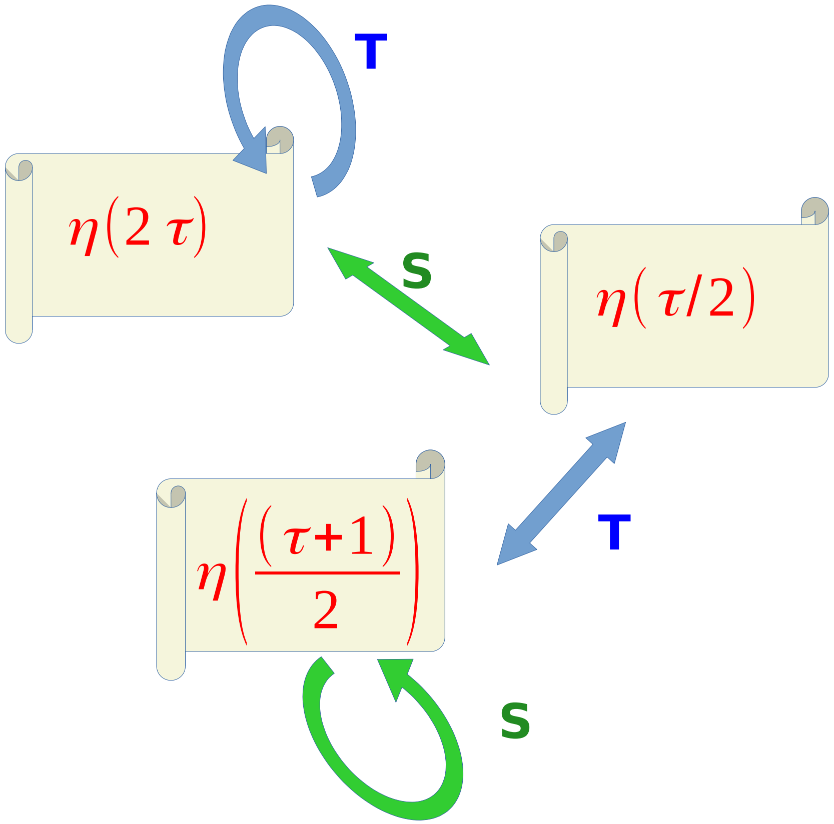

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candidate modular form

$$Y(\alpha,\beta,\gamma)=\frac{d}{d\tau}\left[\alpha\ \log\eta(\tau/2)+\beta\ \log\eta((\tau+1)/2)+\gamma\ \log\eta(2\tau)\right]$$

$$\alpha+\beta+\gamma=0$$

A case study: $\Gamma_2 \sim S_3$

Constructing the Modular Forms

Equations to be satisfied:

$$\left(\begin{array}{c} Y_1(-1/\tau) \\ Y_2(-1/\tau) \end{array}\right) = \tau^2 \rho(S) \left(\begin{array}{c} Y_1(\tau) \\ Y_2(\tau) \end{array}\right), \qquad \left(\begin{array}{c} Y_1(\tau+1) \\ Y_2(\tau+1) \end{array}\right) = \rho(T) \left(\begin{array}{c} Y_1(\tau) \\ Y_2(\tau) \end{array}\right)$$

$\Downarrow$

representation of generators

$$\rho(S) = \frac{1}{2}\left(\begin{array}{cc} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{array}\right), \qquad \rho(T) = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$$

$$(\rho(S))^2 = \mathbb{I}, \qquad (\rho(S)\rho(T))^3 = \mathbb{I}, \qquad (\rho(T))^2 = \mathbb{I}$$

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<u>Constructing the Modular Forms</u>

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$$(\rho(S))^2 = \mathbb{I}, \qquad (\rho(S)\rho(T))^3 = \mathbb{I}, \qquad (\rho(T))^2 = \mathbb{I}$$

$Y_1(\alpha, \beta, \gamma) \sim Y(1, 1, -2)$ $\qquad$ $Y_2(\alpha, \beta, \gamma) \sim Y(1, -1, 0)$

$$\begin{aligned} Y_1(\tau) &= \frac{i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} + \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)} - \frac{8\eta'(2\tau)}{\eta(2\tau)}\right) \\ Y_2(\tau) &= \frac{\sqrt{3}i}{4\pi}\left(\frac{\eta'(\tau/2)}{\eta(\tau/2)} - \frac{\eta'((\tau+1)/2)}{\eta((\tau+1)/2)}\right), \end{aligned}$$

doublet of S3: Y

A case study: $\Gamma_2 \sim S_3$

<u>How to predict the Neutrino mass matrix</u> (from the Weinberg operator, wrong path...)

For a satisfactory model, we ask:

1. small number of operators → ***predictability***

2. no new scalar fields beside Higgs(es) → ***symmetry breaking dictated by the vev of*** τ

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	S_3	SU(2)	k_i
$L_{e\mu}=(e,\mu)$	2	2	-1
L_τ	1	2	-1
H_u	1	2	0

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using one power of Y (modular form of lowest weight)

	S_3	SU(2)	k_i
$L_{e\mu}$ =(e,μ)	2	2	-1
L_τ	1	2	-1
H_u	1	2	0

$$\boxed{L = h_u^2 \left[a \left(\left(L_{e\mu} L_{e\mu} \right)_2, Y \right)_1 + b\, L_\tau \left(L_{e\mu} Y \right)_1 \right]}$$

$$\Downarrow$$

$$m_\nu = \begin{pmatrix} a Y_2 & a Y_1 & b Y_1/2 \\ a Y_1 & -a Y_2 & b Y_2/2 \\ b Y_1/2 & b Y_2/2 & 0 \end{pmatrix}$$

A case study: $\Gamma_2 \sim S_3$

How to predict the Neutrino mass matrix (from the Weinberg operator, wrong path...)

Mass matrix against the experimental data

$$m_\nu = \begin{pmatrix} aY_2 & aY_1 & bY_1/2 \\ aY_1 & -aY_2 & bY_2/2 \\ bY_1/2 & bY_2/2 & 0 \end{pmatrix}$$

$\sin^2\theta_{12}/10^{-1}$	$2.97^{+0.17}_{-0.16}$
$\sin^2\theta_{13}/10^{-2}$	$2.15^{+0.07}_{-0.07}$
$\sin^2\theta_{23}/10^{-1}$	$4.25^{+0.21}_{-0.15}$
δ_{CP}/π	$1.38^{+0.23}_{-0.20}$
r	$2.92^{+0.10}_{-0.11} \times 10^{-2}$

5 observables, 2 complex parameters: a/b and τ → very difficult task!

large χ^2 of O(100) mainly driven by θ_{13}

Conclusions

Modular symmetries offer an alternative way for model building

Yukawa couplins dictated by modular forms

unified description of quarks and leptons

symmetry breaking by the vev of tau only

A lot to do:

mass hierarchy

more than one modulus

more pheno: leptogenesis, LFV...

Backup slides

Kahler potential

Under Γ:

$$\begin{cases} \tau \to \dfrac{a\tau+b}{c\tau+d} \\ \varphi^{(I)} \to (c\tau+d)^{-k_I}\rho^{(I)}(\gamma)\varphi^{(I)} \end{cases}$$

Tte invariance of the action requires the invariance of the superpotential w(Φ) and the invariance of the Kahler potential up to a Kahler transformation:

$$\begin{cases} w(\Phi) \to w(\Phi) \\ K(\Phi,\bar{\Phi}) \to K(\Phi,\bar{\Phi}) + f(\Phi) + f(\overline{\Phi}) \end{cases}$$

Kahler potential:

$$\sum_I (-i\tau + i\bar{\tau})^{-k_I} |\varphi^{(I)}|^2$$

modular invariant kinetic terms

$$\frac{h}{\langle -i\tau + i\bar{\tau}\rangle^2}\partial_\mu\bar{\tau}\partial^\mu\tau + \sum_I \frac{\partial_\mu\overline{\varphi}^{(I)}\partial^\mu\varphi^{(I)}}{\langle -i\tau + i\bar{\tau}\rangle^{k_I}}$$

Some definitions

a <u>normal subgroup</u> (also known as an invariant subgroup or self-conjugate subgroup) is a *subgroup* which is invariant under conjugation by members of the group of which it is a part:

a subgroup N of the group G is normal in G if and only if $(g\, n\, g^{-1}) \in N$ for all $g \in G$ and $n \in N$

Γ(N), N>=2 are infinite normal subgroups of Γ, called *principal congruence subgroups*

the group Γ(N) acts on the complex variable τ (Im τ >0)

$$\gamma\,\tau = \frac{a\,\tau + b}{c\,\tau + d}$$

And it can be shown that the upper half-plane is mapped to itself under this action.
The complex variable is henceforth restricted to have positive imaginary part

Some definitions

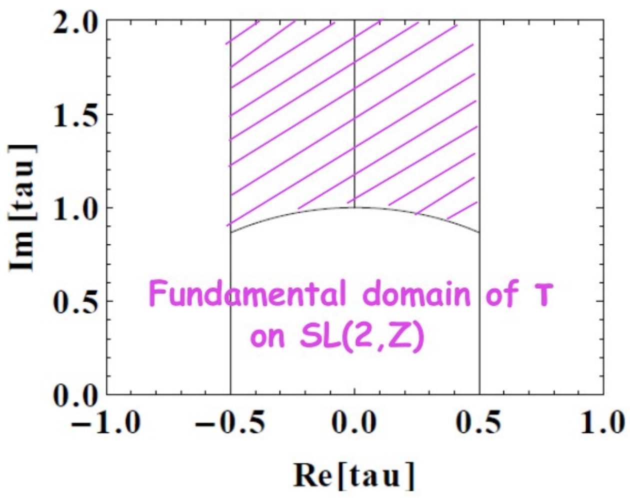
Modular Functions and Modular Forms
J. S. Milne

DEFINITION 0.2. A holomorphic function $f(z)$ on $\mathbb{H}$ is a *modular form of level N and weight* 2k if

(a) $f(\alpha z) = (cz+d)^{2k} \cdot f(z)$, all $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N)$;

(b) $f(z)$ is "holomorphic at the cusps".

Fundamental domain of τ on SL(2,Z): connected open subset such that no two points of D are equivalent under SL(2,Z)



THEOREM 2.12. *Let* $D = \{z \in \mathbb{H} \mid |z| > 1,\ |\Re(z)| < 1/2\}$.

(a) *D is a fundamental domain for $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$; moreover, two elements z and z' of $\bar{D}$ are equivalent under $\Gamma(1)$ if and only if*

(i) $\Re(z) = \pm 1/2$ *and* $z' = z \pm 1$, *(then* $z' = Tz$ *or* $z = Tz'$*), or*

(ii) $|z| = 1$ *and* $z' = -1/z = Sz$.

A case study: $\Gamma_2 \sim S_3$

Constructing the Modular Forms

Under **T**: $Y(\alpha,\beta,\gamma) \rightarrow Y(\gamma,\beta,\alpha)$

Under **S**: $Y(\alpha,\beta,\gamma) \rightarrow \tau^2 \; Y(\gamma,\alpha,\beta)$

representation of generators

$$\rho(S) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}, \qquad \rho(T) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$(\rho(S))^2 = \mathbb{I}, \qquad (\rho(S)\rho(T))^3 = \mathbb{I}, \qquad (\rho(T))^2 = \mathbb{I}$$

A case study: $\Gamma_2 \sim S_3$

q-expansion of the Modular Forms

$$\begin{aligned} Y_1(\tau) &= \frac{1}{8} + 3q + 3q^2 + 12q^3 + 3q^4 \cdots , \\ Y_2(\tau) &= \sqrt{3} q^{1/2} (1 + 4q + 6q^2 + 8q^3 \cdots). \end{aligned}$$

$$\downarrow$$

$Y_1(\tau) \gg Y_2(\tau)$ for Im(τ) >>1

Weinberg operators for $\Gamma_2 \sim S_3$ (1)

Neutrino mass matrices from the Weinberg operator

	S_3	SU(2)	k_i
$L_{e\mu}$ =(e,μ)	2	2	$k_{e\mu}$
L_τ	1	2	k_τ
H_u	1	2	0

Case a) $(L_{e\mu}^2)_1 \otimes (Y^2)_1, (Y^3)_1, \ldots, (Y^n)_1$ $\Rightarrow$ $-2k_{e\mu}+2n=0, \quad n=2\ldots$

Case b) $(L_{e\mu}^2)_2 \otimes Y, (Y^2)_2, (Y^3)_2, \ldots, (Y^n)_2$ $\Rightarrow$ $-2k_{e\mu}+2n=0, \quad n=1\ldots$

Case c) $(L_{e\mu} L_\tau)_2 \otimes Y, (Y^2)_2, (Y^3)_2, \ldots, (Y^n)_2$ $\Rightarrow$ $-k_{e\mu}-k_{e\tau}+2n=0, \quad n=1\ldots$

Case d) $(L_\tau)^2 \otimes (Y^2)_1, (Y^3)_1, \ldots, (Y^n)_1$ $\Rightarrow$ $-2k_{e\tau}+2n=0, \quad n=2\ldots$

Weinberg operators for $\Gamma_2 \sim S_3$ (2)

Neutrino mass matrices from the Weinberg operator

(n=1)

Case b) $(L^2_{e\mu})_2 \otimes Y, (Y^2)_2, (Y^3)_2, \ldots, (Y^n)_2$ $\Rightarrow$ $-2k_{e\mu}+2n=0, \quad n=1\ldots$

Case c) $(L_{e\mu}L_{\tau})_2 \otimes Y, (Y^2)_2, (Y^3)_2, \ldots, (Y^n)_2$ $\Rightarrow$ $-k_{e\mu}-k_{e\tau}+2n=0, \quad n=1\ldots$

Solutions: $[k_{e\mu}=1 \quad k_{e\tau}=0]$ $[k_{e\mu}=0 \quad k_{e\tau}=2]$ $[k_{e\mu}=1 \quad k_{e\tau}=1]$

$$m_\nu = \begin{pmatrix} bY_2 & bY_1 & cY_1/2 \\ bY_1 & -bY_2 & cY_2/2 \\ cY_1/2 & cY_2/2 & 0 \end{pmatrix}$$

Weinberg operators for $\Gamma_2 \sim S_3$ (3)

Neutrino mass matrices from the Weinberg operator

(n=2)

Case a) ➡ $-2k_{e\mu}+4=0$

Case b) ➡ $-2k_{e\mu}+4=0$

Case c) ➡ $-k_{e\mu}-k_{e\tau}+4=0$

Case d) ➡ $-2k_{e\tau}+4=0$

Solutions: $[k_{e\mu}=2 \quad k_{e\tau}=2]$ $[k_{e\mu}=2 \quad k_{e\tau}\neq 2]$ $[k_{e\mu}\neq 2 \quad k_{e\tau}=2]$

$$m_\nu=\begin{pmatrix} (a+b)y_1^2+(a-b)y_2^2 & 2b\,y_1 y_2 & c\,y_1 y_2 \\ * & (a-b)y_1^2+(a+b)y_2^2 & 1/2\,c\,(y_1^2-y_2^2) \\ * & * & d\,(y_1^2+y_2^2) \end{pmatrix}$$

A case study: $\Gamma_2 \sim S_3$

Dedekind eta functions

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \qquad q \equiv e^{i2\pi\tau}$$

Under **T**:

$$\begin{cases} \eta(2\tau) \rightarrow e^{i\pi/6}\,\eta(2\tau) \\ \eta(\tau/2) \rightarrow \eta((\tau+1)/2) \\ \eta((\tau+1)/2) \rightarrow e^{i\pi/12}\,\eta(\tau/2) \end{cases}$$

Under **S**:

$$\begin{cases} \eta(2\tau) \rightarrow \sqrt{-i\,\tau/2}\;\eta(\tau/2) \\ \eta(\tau/2) \rightarrow \sqrt{-2i\,\tau}\;\eta(2\tau) \\ \eta\left(\frac{(\tau+1)}{2}\right) \rightarrow e^{-i\pi/12}\sqrt{-i\,\tau(\sqrt{3}-i)}\;\eta\left(\frac{(\tau+1)}{2}\right) \end{cases}$$

Mod

```
Id[a_, b_] := {{Mod[a, b], 0}, {0, Mod[a, b]}}
```

Id[-1, 2] $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Id[-1, 3] $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$