Quantum computing (2+1)-dimensional QED

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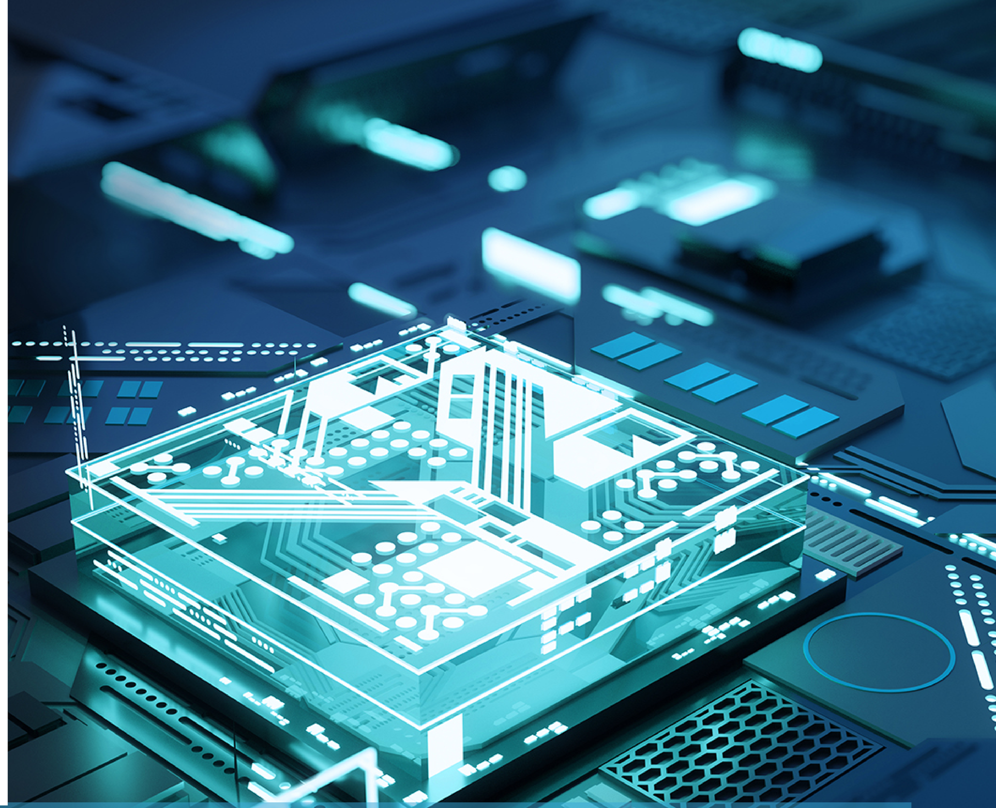


Abstract

We propose to compute the running coupling for asymptotically free (2+1)-dimensional QED in the small and intermediate coupling regime using quantum computing techniques. To this end, we provide a Hamiltonian formulation of QED on a 2-dimensional spatial lattice. Using a variational quantum approach we compute the energy gap and the plaquette expectation value which can be related to the running coupling [1, 2]. We discuss different methods for an efficient encoding of the system on a quantum circuit and for the classical optimization. The overarching goal of the project is to match physical quantities such as the **energy gap** or the **static force** with Markov Chain Monte Carlo (MCMC) calculations in the regime where both approaches can be applied. This would allow to obtain a physical scale from the MCMC simulations and to follow the running of the coupling deep into the perturbative regime using quantum computations. The techniques and algorithms used here for asymptotically free QED as a prototype model can eventually also be used for future studies of QCD in (3+1)-dimensions on quantum computers.

Running coupling

- Compute short distance quantities from Quantum Computing results, e.g. renormalized coupling $g_{ren}(\mu)$ at scale μ.
- Use static force at short distances (perturbative) to set the renormalization scale.
- Compute the expectation value of the plaquette operator $\langle\Box\rangle$ → define a *boosted coupling* (converges more rapidly than bare coupling): $g^2_{\Box} = \frac{g^2}{\langle\Box\rangle}$.
- Use mass gap at intermediate coupling to match to MCMC simulations (will provide the physical value of the lattice spacing).

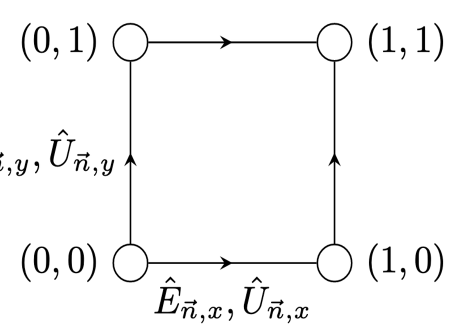
Hamiltonian

- Using the Kogut-Susskind formulation [3], fermions and antifermions are represented by single component field operators $\hat{\phi}_{\vec{n}}$, with $\vec{n} = (n_x, n_y)$.
- Hamiltonian:

$$H_{tot} = H_E + H_B + H_m + H_{kin}$$

$$\hat{H}_E = \frac{g^2}{2}\sum_{\vec{n}}\left(\hat{E}^2_{\vec{n},x} + \hat{E}^2_{\vec{n},y}\right), \qquad \hat{H}_B = -\frac{1}{2g^2}\sum_{\vec{n}}\left(\hat{P}_{\vec{n}} + \hat{P}^{\dagger}_{\vec{n}}\right),$$

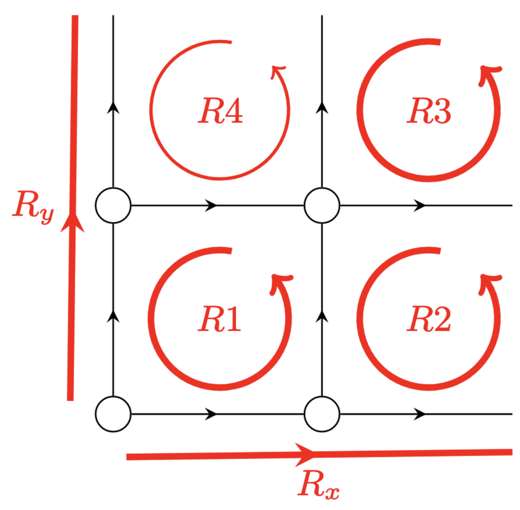
$$\hat{H}_m = m\sum_{\vec{n}}(-1)^{n_x+n_y}\hat{\phi}^{\dagger}_{\vec{n}}\hat{\phi}_{\vec{n}}, \qquad \hat{H}_{kin} = \Omega\sum_{\vec{n}}\sum_{\mu=x,y}(\hat{\phi}^{\dagger}_{\vec{n}}\hat{U}_{\vec{n},\mu}\hat{\phi}_{\vec{n}+\mu} + H.c.).$$



Where $\hat{P}_{\vec{n}} = \hat{U}_{\vec{n},x}\hat{U}_{\vec{n}+x,y}\hat{U}^{\dagger}_{\vec{n}+y,x}\hat{U}^{\dagger}_{\vec{n},y}$ is the plaquette operator.

- The Hamiltonian is gauge invariant, i.e. it commutes with the Gauss' law operators

$$G_{\vec{n}} = \left[\sum_{\mu=x,y}\left(\hat{E}_{\vec{n},\mu} - \hat{E}_{\vec{n}-\mu,\mu}\right) - \hat{q}_{\vec{n}} - \hat{Q}_{\vec{n}}\right]$$



- We consider a periodic boundary condition system with four fermionic sites [4]: operators (*rotators* and *strings*) to simplify the expressions.

Encodings

One-hot

- Maps the N fermionic states into an equal number of qubits and gauge physical states onto $2l+1$ qubits [5]

$$|-l+j\rangle_{\text{phy}} \mapsto |\overbrace{0\ldots0}^{j}1\overbrace{0\ldots0}^{2l-j}\rangle.$$

- Truncated electric field and link operators

$$\hat{E} = \sum_{i=-l}^{l} i\,|i\rangle_{\text{phy}}\langle i|_{\text{phy}}\,, \hat{U} = \sum_{i=-l+1}^{l} |i-1\rangle_{\text{phy}}\langle i|_{\text{phy}}\,.$$

- Example (l=1):

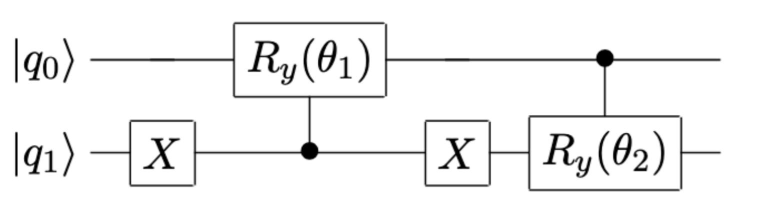
$$|-1\rangle_{\text{phy}} \mapsto |100\rangle\,,\ |0\rangle_{\text{phy}} \mapsto |010\rangle\,,\ |1\rangle_{\text{phy}} \mapsto |001\rangle\,.$$

- NOT resource efficient: needs $(2l+1)^N$ qubits for N gauge variables.

Gray Encoding

- Minimum number of qubits required per gauge variable is $q_{\min} = \lceil \log_2(2l+1)\rceil$.
- Example (l=1): there are three physical states $|i\rangle_{\text{phy}}$ for $i \in \{-1, 0, 1\}$, which can be encoded using only 2 qubits using a Gray encoding pattern

$$|-1\rangle_{\text{phy}} \to |00\rangle \quad |0\rangle_{\text{phy}} \to |01\rangle \quad |1\rangle_{\text{phy}} \to |11\rangle$$



- More resource efficient than one-hot encoding.

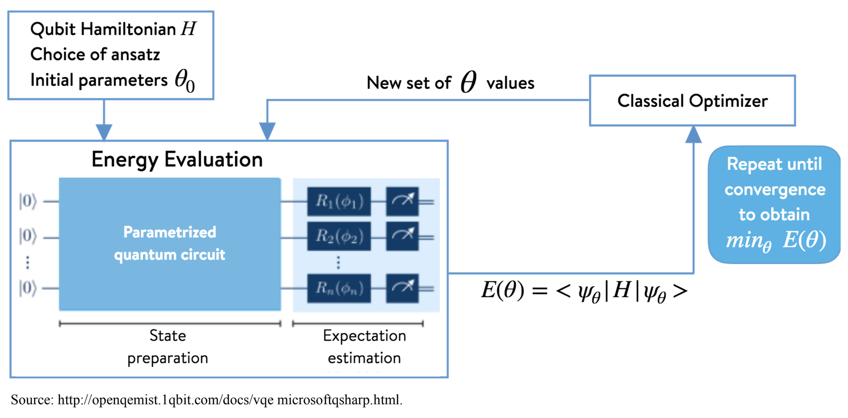
Methods

Variational Quantum Algorithms

- In order to find the eigenvalue of a given operator H, the **Variational Quantum Eigensolver** (VQE) algorithm [6] finds the eigenvector $|\psi\rangle$ which corresponds to the lowest eigenvalue and that minimizes

$$E(\vec{\theta}) := \left\langle\psi(\vec{\theta})\middle|H\middle|\psi(\vec{\theta})\right\rangle.$$

- Done by varying the $\vec{\theta}$ parameters through the combination of a classical and a quantum part.



- The **Variational Quantum Deflation** (VQD) method [7] extends VQE to compute excited states by optimizing the cost function

$$C(\theta_k) = \langle\psi(\theta_k)|H|\psi(\theta_k)\rangle + \sum_{i=0}^{k-1}\beta_i\,|\,\langle\psi(\theta_k)|\psi(\theta_i)\rangle\,|^2,$$

where β is a real-valued coefficient.

- Since our goal is to compute the energy gap between the ground state E_0 and first excited state E_1, we follow three main steps:

1) Perform the VQE and obtain optimal parameters and an approximate ground state $|\psi(\theta^*_0)\rangle$;

2) For E_1 define a Hamiltonian:

$$H_1 = H + \beta\,|\psi(\theta^*_0)\rangle\,\langle\psi(\theta^*_0)|\,,$$

β is arbitrary (must be larger than the energy gap);

3) Perform the VQE with the Hamiltonian H_1 to find an approximation of the first excited state.

Ansatz and penalty term

- Instead of constrain reachable states to the physical ones in the ansatz, we define a **penalty term** in H that suppresses unphysical contributions on the final states [8]:

$$\Delta H_{\text{suppr.}} = \lambda{\sum_s}'\,\Pi_s,$$

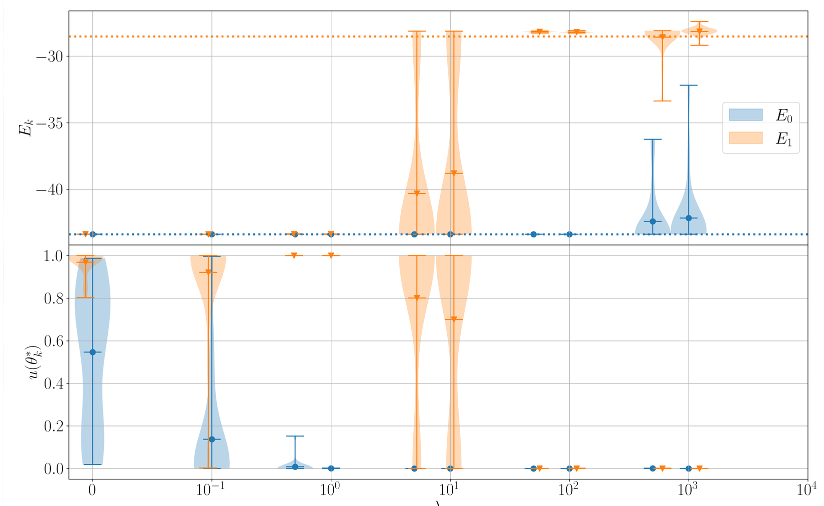
λ is the *suppression coefficient*, while Π_s are the corresponding single-state projectors (i.e. $\frac{I \pm Z}{2}$).

- Assess how much the optimal state reached is *unphysical* by computing the expectation value

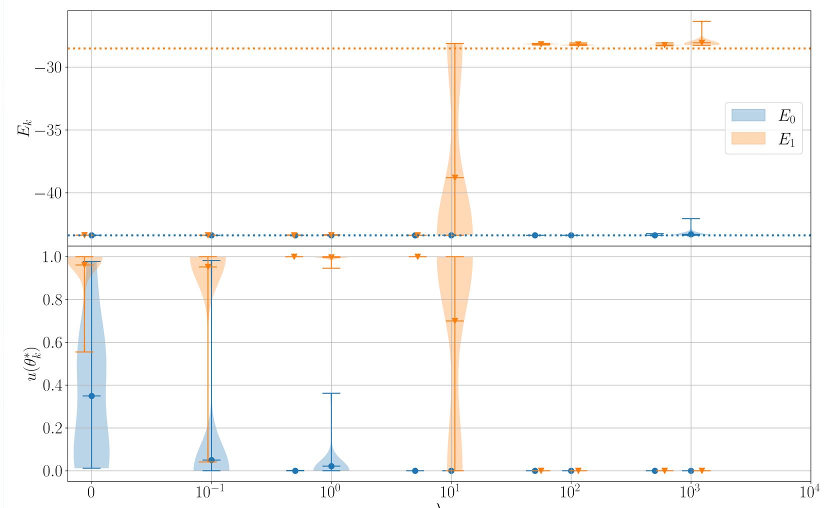
$$u(\theta^*) \equiv \langle\psi(\theta^*)|\Pi_{\text{unphys.}}|\psi(\theta^*)\rangle$$

of the projector into $\mathcal{H}_{\text{unphys.}}$

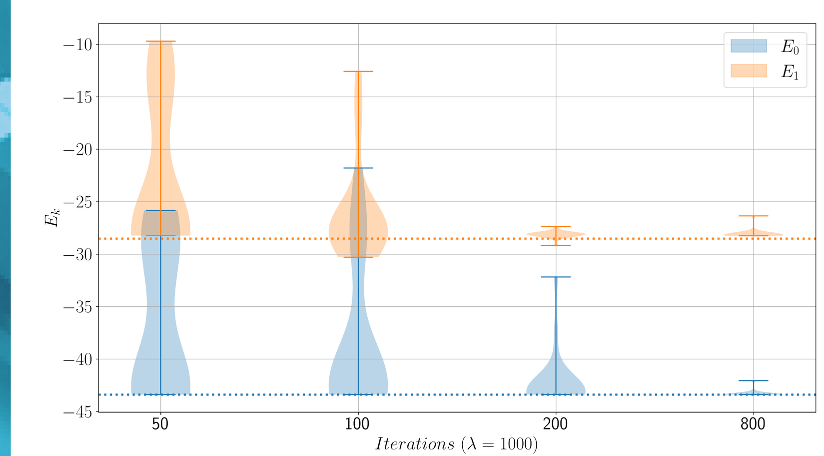
- We consider pure gauge case with 200 iterations:



- With 800 iterations the optimizer can converge even if λ is large:



- With λ = 1000 and different number of iterations:



- **Fermionic system**: bisection method to choose the best λ. Compute percentage of unphysical state and tune λ: threshold of 99% of physical component.

Results

Ground state Energy

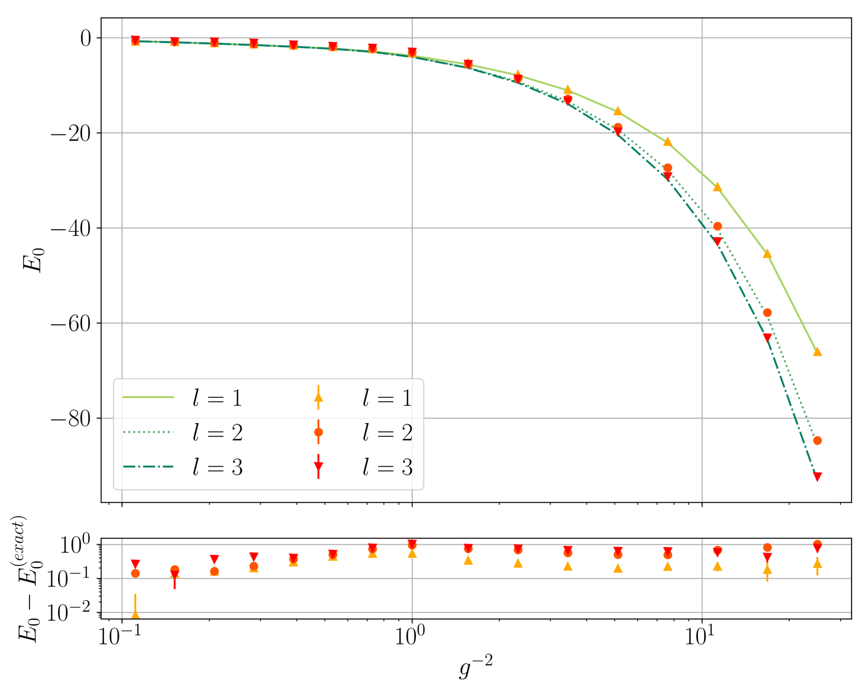


Fig. A: Best results for VQD ground state energy as a function of the coupling in the electric basis (dots) at some values of truncation level l and exact diagonalization (lines). *Bottom panel*: discrepancies with the exact values.

Energy gap

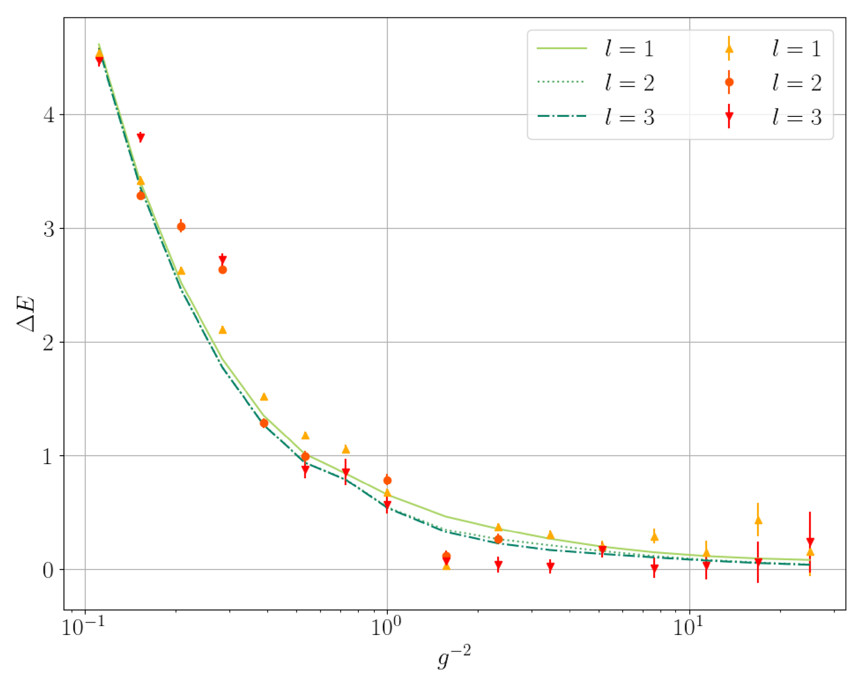


Fig. B: Best results for spectral gap as a function of the coupling g in the electric basis. (Notation as in Fig. A)

Plaquette operator

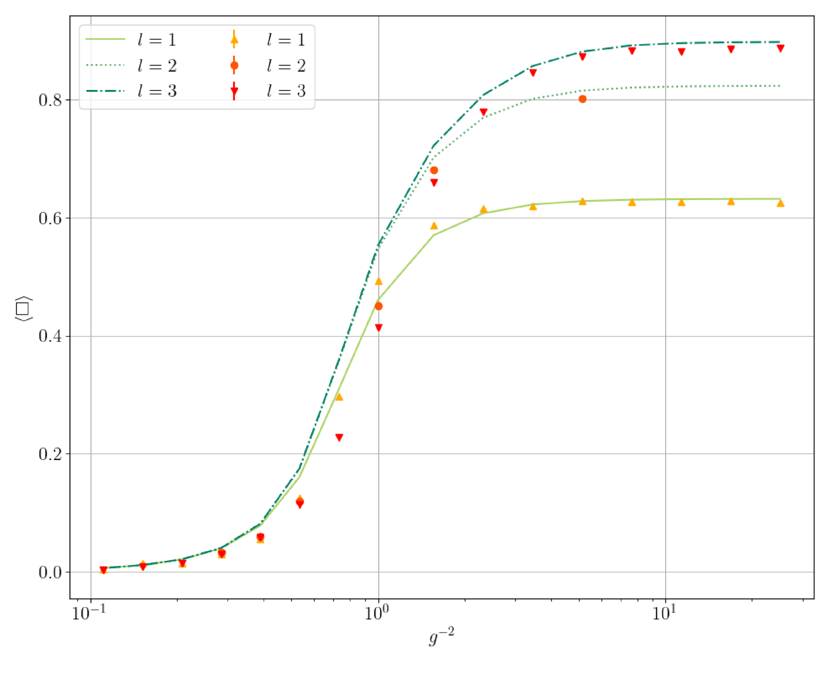


Fig. C: Plaquette measurements on the ground state (Fig. A) in the electric basis. (Notation as in Fig. A)

Conclusions

- Developed a resource efficient encoding for (2+1)-dimensional QED → can eventually be brought on a quantum computer.
- Demonstrated that suppression terms can be used to avoid unphysical states.
- Accurate results for E_0 in broad range of couplings → can obtain static force.
- Accurate results for energy gap in intermediate range of the coupling → can make contact to MC simulations.
- Accurate results for $\langle\Box\rangle$ in broad range of coupling → can assess the renormalized coupling.
- Setup of (2+1)-dimensional QED developed here is a basis for extensions, e.g. adding topological terms, chemical potential or real time simulations.

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