Multiplicity distributions and entropy of the produced particles in one dimensional models and QCD

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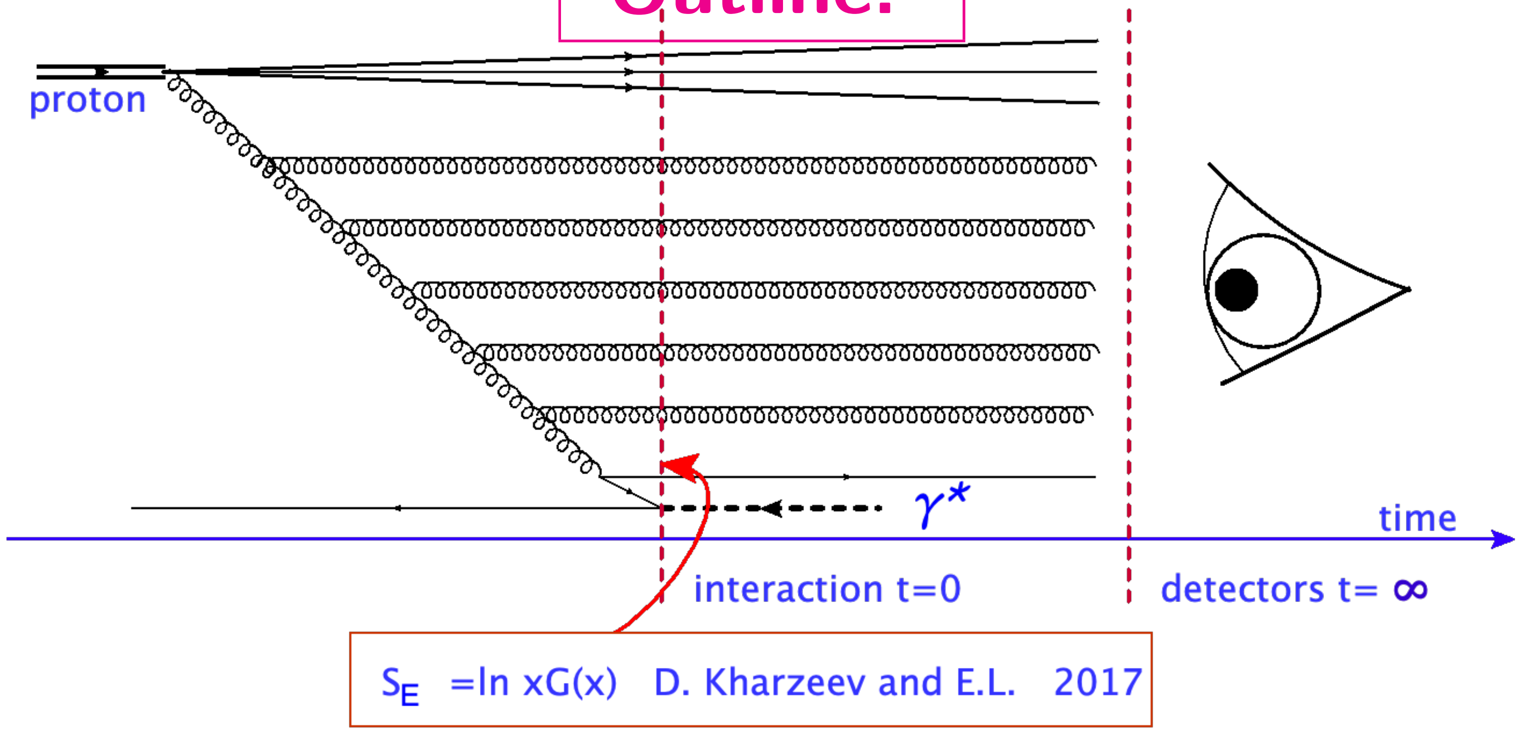
Outline:



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   E.L& M. Lublinsky,(2023) (in preparation)

2. **What is S_E at $t \to \infty$ in QCD for DIS?**

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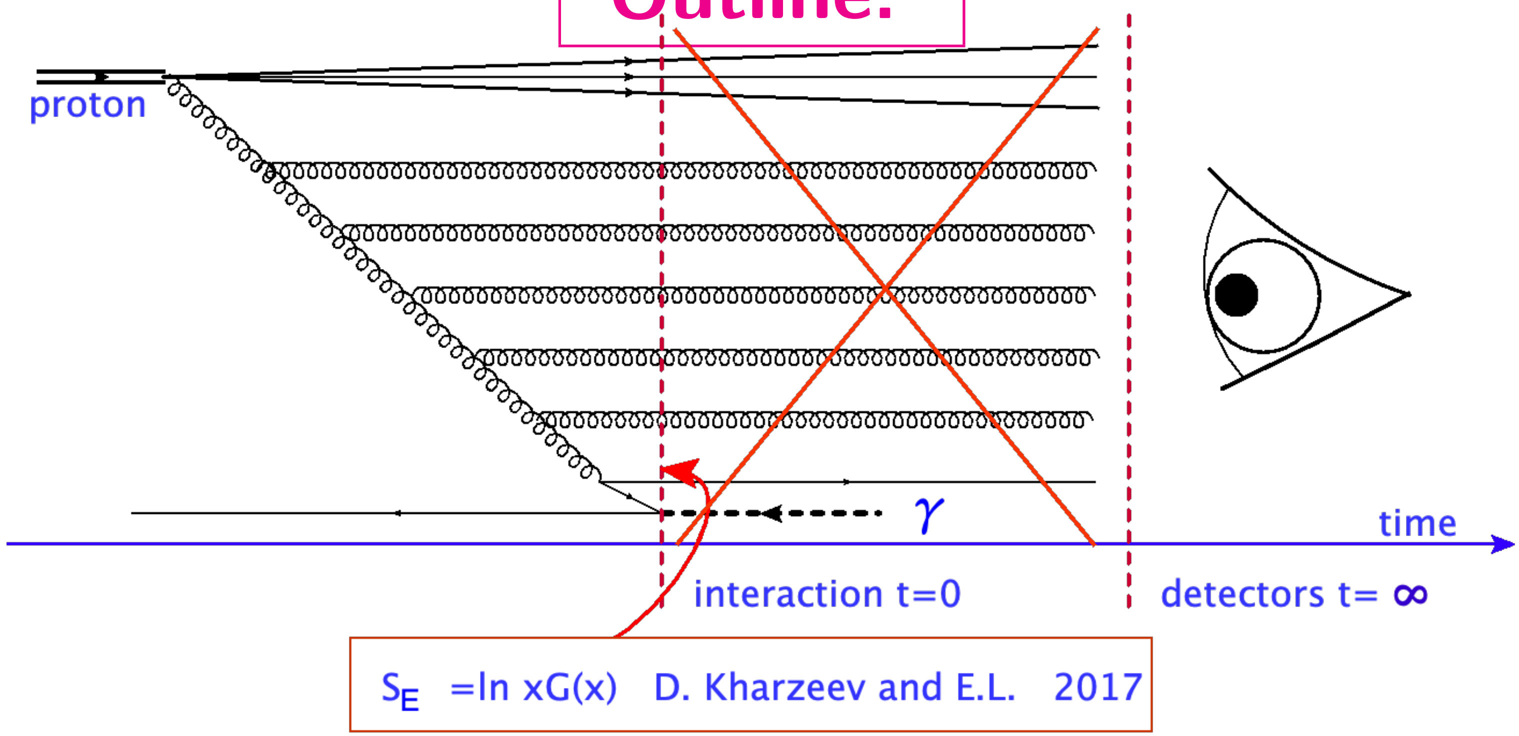

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Production processes in zero dimension models

$$S(Y) \;=\; \sum_{n,m} e^{-\gamma\, n\, m} P_n\,(Y - Y_0)\; P_m\,(Y_0)$$ A.H.Mueller & G.Salam (1996)

- $\gamma \rightarrow$ scattering amplitude of two dipoles;
- $P_n(Y) \rightarrow$ probability to find n dipoles at rapidity Y;
- $P^{BFKL}_n(Y) \;=\; e^{-\Delta\, Y}\left(1 - e^{-\Delta\, Y}\right)^{n-1};\; M_k = k!\, e^{\Delta\, Y}\left(e^{\Delta\, Y} \;-\; 1\right)^{k-1};$
- Independence on $Y_0 \rightarrow$ t-channel unitarity;
- $xG = \sum_n n\, P^{BFKL}_n\,(Y) = e^{\Delta\, Y};$
- $S_E = \sum_n \ln\left(P^{BFKL}_n\,(Y)\right)\; P^{BFKL}_n\,(Y) \xrightarrow{Y \gg 1} \Delta\, Y = \ln\left(xG(x)\right);$

Dipole-dipole scattering amplitude for the BFKL cascade

$$S(Y) = \sum_{k=0}^{\infty} \frac{(-\gamma)^k}{k!} \underbrace{\sum_n n^k P_n^{BFKL}(Y - Y_0)}_{<n^k>} \underbrace{\sum_m m^k P_n^{BFKL}(Y_0)}_{<m^k>}$$

$$\boxed{N = e^{\Delta Y}} = \sum_{k=0}^{\infty} (-\gamma)^k \; k! \left(\underbrace{e^{\Delta (Y - Y_0)} \, e^{\Delta Y_0}}_{N} \right)^k = \frac{1}{\gamma N} \, e^{\frac{1}{2\gamma N}} \Gamma\left(0, \frac{1}{2N\gamma}\right)$$

$$S(Y) \rightarrow \begin{cases} 1 - \gamma e^{\Delta Y} + 2\gamma^2 e^{2\Delta Y} + \dots & \text{for small } Y; \\ \frac{\ln\left(\gamma e^{\Delta Y}\right)}{\gamma e^{\Delta Y}} & \text{for large } Y; \end{cases}$$

$$S^{DIS}_{BK}(Y) \rightarrow \begin{cases} 1 - \gamma e^{\Delta Y} + \gamma^2 e^{2\Delta Y} + \dots & \text{for small } Y; \\ \frac{1}{\gamma e^{\Delta Y}} & \text{for large } Y; \end{cases}$$

Production processes

• **The imaginary part of the Pomeron exchange gives the cross section of produced $\Delta\,Y$ gluons, where Δ is the Pomeron intercept:**

$$Im\,G_{I\!P}\left(Y\right)\;=\;\sigma_{\bar{n}=\Delta\,Y}\;\;\text{with Poisson distribution}$$

• **The AGK cutting rules give the way to calculate the imaginary part of the amplitude with exchange of n-Pomerons through $Im\,G_{I\!P}\left(Y\right)$.**

$$\sigma_n\left(Y\right)\;=\;\sum_k \underbrace{\sigma^{AGK}_k\left(Y\right)}_{\propto\,(ImG_{I\!P})^k}\;\underbrace{\frac{\left(\Delta\,k\,Y\right)^n}{n!}e^{-\Delta\,k\,Y}}_{\text{Poisson distribution}}\;\;\to\;\sigma^{AGK}_{k=n/(\Delta\,Y)}(Y)$$

Multiplicity distribution in zero dimension models

- $$\sigma_k^{AGK}(Y) = \frac{1}{2\,\gamma\,N}\,k!\,U\left(k+1,1,\frac{1}{2\,\gamma\,N}\right)$$

- $$\sigma_k^{AGK}(Y) \rightarrow \begin{cases} \frac{1}{\gamma\,N}\,K_0\left(2\sqrt{\frac{k+1}{2\gamma\,N}}\right) & \text{for} \quad 2\gamma\,N\,k\, > \, 1; \\ k!\,(2\,\gamma\,N)^k & \text{for} \;\; 2\gamma\,N\,k\, < \, 1; \end{cases}$$

Entropy of produced gluons

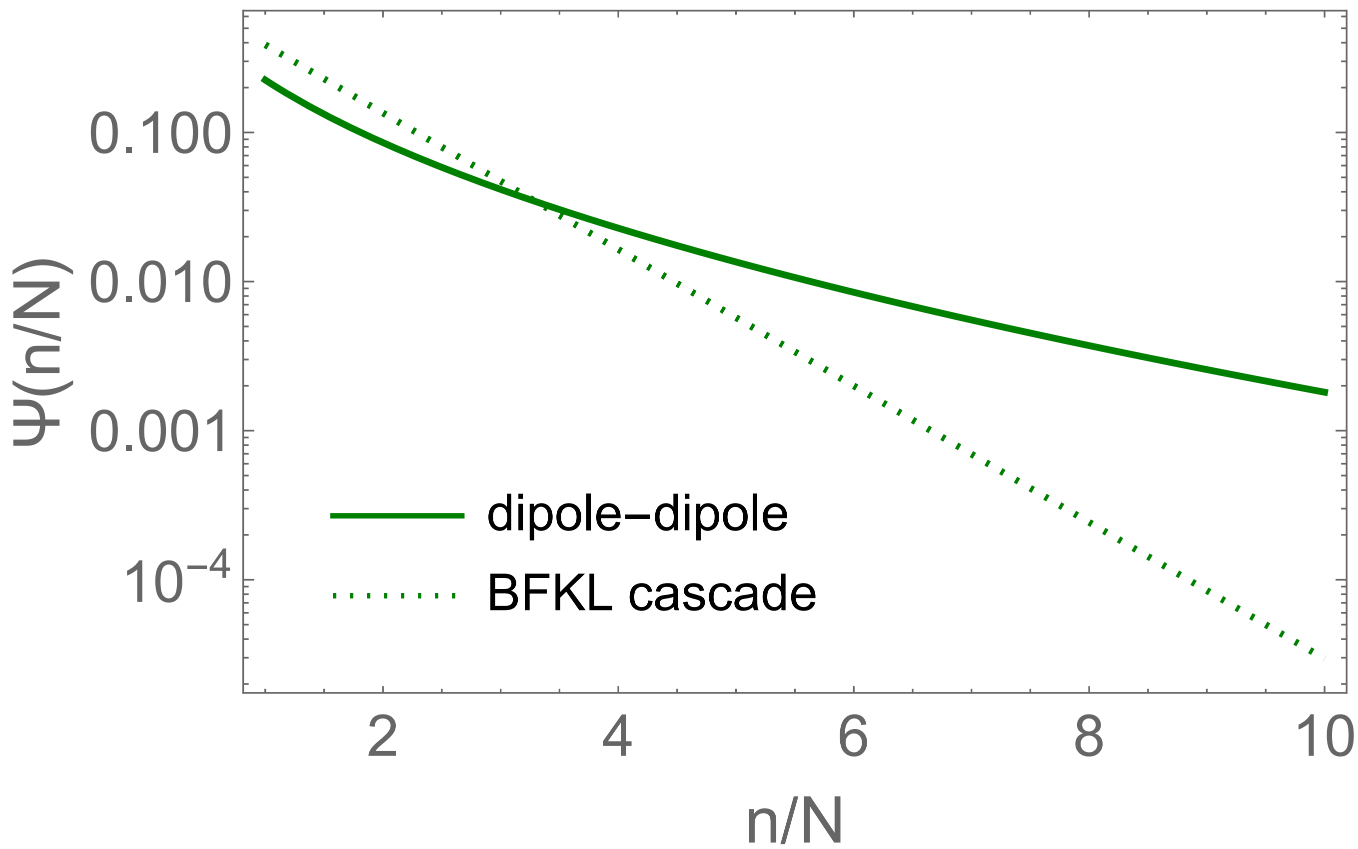
- $$\mathcal{P}_n\left(Y\right) \;=\; \frac{\sigma^{AGK}_{k=n/(\Delta\,Y)}(Y)}{\sum_k \sigma^{AGK}_{k=n/(\Delta\,Y)}(Y)}$$

- $$S_E\left(Y\right) \;=\; \sum_n \ln\left(\mathcal{P}_n\left(Y\right)\right)\;\mathcal{P}_n\left(Y\right)$$

- $$S_E \;=\; \ln\left(\underbrace{2\gamma\,N\left(Y\right)}_{xG(x)}\right) \;-\; \underbrace{\int_0^\infty \eta\,d\eta\,\ln[K_0\left(\eta\right)]\;K_0\left(\eta\right)}_{\approx 1.5}$$

with $\eta = 2\sqrt{\frac{n+1}{2\,\gamma\,N}}$

$$\mathcal{P}_n(Y) \;=\; \frac{1}{N}\Psi\left(\frac{n}{N}\right) \quad \to \textbf{ KNO scaling } \quad N \equiv <|n|>$$

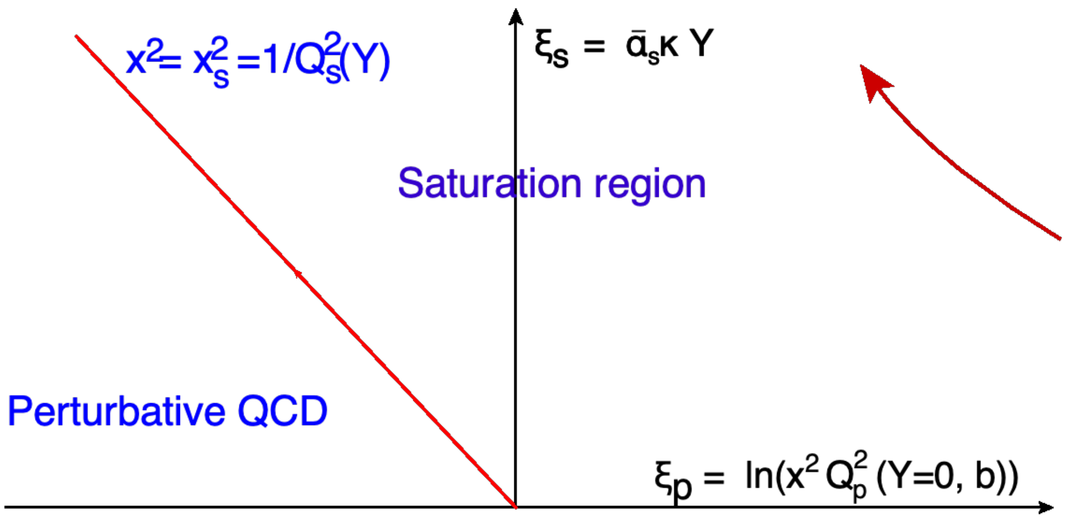


DIS in QCD

Balitsky-Kovchegov non-linear equation: $x_{01} \rightarrow x_{02} + x_{12}$

$$\frac{\partial N_{01}}{\partial Y} = \bar{\alpha}_S \int \frac{d^2 x_{02}}{2\pi} \underbrace{\frac{x_{01}^2}{x_{02}^2\, x_{12}^2}}_{K(x_{01},x_{02})} \left\{ N_{02} + N_{12} - N_{02} N_{12} - N_{01} \right\}$$

where $N_{ik} = N\left(Y, \vec{x}_{ik}, \vec{b}\right)$.



$$N^{\rm DIS}\left(z = \ln\left(x^2 Q_s^2\left(Y,b\right)\right)\right) =$$

$$1 - C(z)\exp\left(-\frac{z^2}{2\kappa}\right)$$

E.L. & K. Tuchin (2000)

$$\kappa = \chi\left(\gamma_{cr}\right)/\left(1-\gamma_{cr}\right);$$

$$\chi\left(\gamma_{cr}\right)/\left(1-\gamma_{cr}\right) = \chi'\left(\gamma_{cr}\right)$$

$$\frac{\partial \Delta_{01}}{\partial Y} = \bar{\alpha}_S \int_{1/Q_s(Y)} \frac{d^2 x_{02}}{2\pi} \frac{x_{01}^2}{x_{02}^2\, x_{12}^2} \left\{ -\Delta_{01} + \mathcal{O}\left(\Delta^2\right) \right\}; \qquad N_{ik} = 1 - \Delta_{ik}$$

Generating functional for production

BFKL cascade:

$$Z_0\left(Y;\{u\}\right)\;\equiv\;\sum_{n=1}\int\underbrace{P_n\left(Y;r_1,\ldots,r_n\right)}_{probability\ to\ find\ dipoles}\;\prod_{i=1}^{n}u(r_i)d^2r_i;$$

Initial and boundary conditions: $Z_0\left(Y=0;\{u\}\right)\;=\;u(r);\,Z_0\left(Y;\{u\}\right)|_{u=1}\;=\;1;$

$$\frac{dZ_0\left(Y;\{u\}\right)}{dY}=\frac{\bar{\alpha}_S}{2\pi}\int d^2r\;d^2r'\;K(r,r')\left\{u(r')\,u(|\vec{r'}-\vec{r}|)-u(r)\right\}\frac{\delta Z_0\left(Y;\{u\}\right)}{\delta u(r)}$$

Production process: (A. Kormilitzin, E.L. & A.Prygarin(2008))

$$Z\left(Y;\{u\},\{v\}\right)=\sum_{n,m}\int P^m_n\left(y;\underbrace{r_1,\ldots,r_n}_{t=0};\underbrace{r_1,\ldots,r_m}_{t=\infty}\right)\prod_{i=1}^{n}u(r_i)\prod_{k=1}^{m}v(r_k)d^2r_i\,d^2r_k$$

$$\frac{\partial Z\left(Y;\{u\},\{\zeta\}\right)}{\partial Y}\;=\;\frac{\bar{\alpha}_S}{2\pi}\int d^2r_2K\left(r_{10},r_{12}\right)\Bigg\{\left(u(r_{12})\,u(r_{02})\;-\;u(r_{10})\right)\frac{\delta Z}{\delta u(r_{10})}$$

$$+\;\left(\zeta(r_{12})\,\zeta(r_{02})\;-\;\zeta(r_{10})\right)\frac{\delta Z}{\delta\zeta(r_{10})}\Bigg\}\qquad\zeta=2u(r)-v(r)$$

Initial conditions:

$$Z\left(Y=0;\{u\},\{v\}\right) \;=\; v(r)$$

Two boundary conditions:

$$(1)\; Z\left(Y;\{u\},\{v\}\right)|_{u=1,v=1}=1;\;\; (2)\; Z\left(Y;\{u\},\{v\}\right)|_{v=2u-1}=2Z_0\left(Y;\{u\}\right)-1;$$

(2) comes from unitarity:

$$2N\left(Y,r,b\right) \;=\; \sigma_{sd}\left(Y,r,b\right) \;+\; \sigma_{in}\left(Y,r,b\right)$$

The master equation for σ_n

$$M\left(Y; r, b\right) = \sum_{n=1,m=0}^{\infty} \frac{(-1)^{n+m+1}}{n!\,m!}\left\{\int \prod_{i=1}^{n} d^2 r_i \int \prod_{k=1}^{m} d^2 r_k\, \gamma(r_i)\,\gamma_{in}(r_k) \frac{\delta^2 Z\left(Y; \{u\}, \{v\}\right)}{\delta u(r_i)\delta v(r_k)}\Big|_{u(r)=1, v(r)=1}\right\}$$

where $u(r) = 1 - \gamma(r)$ and $v(r) = 1 - \gamma_{in}(r)$.

$$S^{DIS} = \sum_n e^{-\gamma\, n} P_n\left(Y\right) = \sum \frac{(-\gamma)^k}{k!} \underbrace{\sum_n n^k P_n\left(Y\right)}_{moment\ \ M_k} = \sum \frac{(-\gamma)^k}{k!} \frac{d^k Z(u, Y)}{du^k}\Big|_{u=1}$$

- $$\frac{\partial M(Y; r_{10})}{\partial Y} = \frac{\bar{\alpha}_S}{2\pi} \int d^2 r_2\, K\left(r_{10}, r_{12}\right) \Big\{ M(Y; r_{12}) + M(Y; r_{20}) - M(Y; r_{10})$$

$$+ M(Y; r_{12}, b) M(Y; r_{20}) - 2\, M(Y; r_{12}) N_{02} - 2 N_{12} M(Y; r_{20}) + 2 N_{12} N_{20} \Big\}$$

- $$\sigma_n\left(Y, r; b\right) = \frac{1}{n!} \prod_{i=1}^{n} \gamma_{in}(r_i) \left(\frac{\delta}{\delta \gamma_{in}(r_i)} M\left(Y; \{\gamma\}, \{\gamma_{in}\}\right)\right)\Big|_{\gamma_{in}=0}$$

σ_1

$$\frac{\partial \sigma_1(Y; r_{01}, b)}{\partial Y} = -\left(\frac{\bar{\alpha}_S}{2\pi} \int_{\substack{r_{02} \gg 1/Q_s(Y) \\ r_{12} \gg 1/Q_s(Y)}} d^2 r_2 K\left(r_{01}, r_{12}\right)\right)\left\{\sigma_1\left(Y, r_{01}, b\right) - \right.$$

$$\left.\underbrace{\sigma_1\left(Y, r_{02}, b\right)\ \Delta_{in}\left(Y, r_{02}, b\right)\ - \sigma_1\left(Y, r_{12}, b\right)\ \Delta_{in}\left(Y, r_{02}, b\right)}_{\sim \Delta \ll 1}\right\} = -\, z\, \sigma_1\left(Y, r_{01}, b\right)$$

$\sigma_{sd} = 1 - \Delta_{sd}$; $N = 1 - \Delta_0$; $\Delta_{in} = 2\Delta_0 - \Delta_{sd}$;

- $\kappa \frac{d\, \sigma_1(z)}{d\, z} = -\, z\, \sigma_1(z)$ **with the solution** $\sigma_1(z) = C(z) \exp\left(-\frac{z^2}{2\, \kappa}\right)$

Momentum representation:

- $\sigma_1(\tilde{z}) = C(\tilde{z}) \exp\left(-\frac{\tilde{z}^2}{2\, \kappa}\right)$; $\tilde{z} = \ln \frac{Q_s^2(Y)}{k_T^2} = \kappa Y + \tilde{\xi}$; $\tilde{\xi} = \ln\left(\frac{Q_s^2(Y = 0)}{k_T^2}\right)$

σ_2 and σ_n :

$$\frac{\partial \sigma_2(Y;r_{01},b)}{\partial Y} = -\left(\frac{\bar{\alpha}_S}{2\pi}\int_{\substack{r_{02}\gg 1/Q_s(Y)\\ r_{12}\gg 1/Q_s(Y)}} d^2 r_2\, K\left(r_{01},r_{12}\right)\right)\sigma_2\left(Y,r_{01},b\right)$$

$$+\frac{\bar{\alpha}_S}{2\pi}\int d^2 r_2\, K\left(r_{01},r_{12}\right)\left\{\underbrace{\sigma_1\left(Y,r_{02},b\right)\,\sigma_1\left(Y,r_{12},b\right)}_{to\ satisfy\ \sigma_2=0\ at\ Y=0}+\mathcal{O}\left(\Delta_{in}\right)\right\}$$

Momentum representation:

- $$\kappa\frac{d\sigma_2(\tilde{z})}{d\tilde{z}} = -\,\tilde{z}\sigma_2(\tilde{z})+\sigma_1^2\left(\tilde{z}\right);\qquad \boxed{\sigma_2\left(\tilde{z}\right)\ =\ \sigma_1\left(\tilde{z}\right)\int_0^{\tilde{z}}\sigma_1\left(\tilde{z}'\right)\frac{d\tilde{z}'}{\kappa}};$$

$\sigma_n\left(\tilde{z}\right)$:
$$\sigma_n\left(\tilde{z}\right)\ =\sigma_1\left(\tilde{z}\right)\left(\int_0^{\tilde{z}}\sigma_1\left(\tilde{z}'\right)\ \frac{d\tilde{z}'}{\kappa}\right)^{n-1}$$

Assumption:

$$\int_0^{\infty} C\left(\tilde{z}'\right)\frac{d\tilde{z}'}{\kappa}\ =\ 1\ \rightarrow\ \sigma_{in}\left(\tilde{z}\right)\ =\ \sum_{n=1}^{\infty}\sigma_n\left(\tilde{z}\right)\propto\ \tilde{z}$$

Non-perturbative corrections:

Physics observable: $\int d^2b\, \sigma_n(\tilde z(b)) \;\to\; \infty$

Non-perturbative corrections for large b: $Q_s^2\left(\tilde z(b)\right) \to Q_s^2\left(\tilde z(b)\right) \underbrace{e^{-\mu\, b}}_{non-pert.}$

$$p_n = \frac{\sigma_n\left(\tilde z(b=0)\right)}{\sigma_{in}\left(\tilde z(b=0)\right)} = \begin{cases} \frac{2\,\kappa}{\tilde z^3(b=0)} \frac{\left(\tilde z(b=0) - \sqrt{2\,\kappa\;\ln(n-1)}\right)}{(n-1)} & \text{for}\;\; n-1 \leq \bar n; \\ \frac{2\,e\,\kappa^2}{\tilde z^4(b=0)}\frac{1}{\bar n}\frac{\bar n^2}{(n-1)^2}\;\exp\left(-\frac{n-1}{\bar n}\right) & \text{for}\;\; n-1 > \bar n \end{cases}$$

with $\bar n = A e^{\frac{\tilde z^2(b=0)}{2\,\kappa}}$.

Entropy:

$$S_E = -\sum_{k=1}^{\infty} p_k \ln p_k = -\int_1^{\bar n} dk\, p_k \ln p_k \;-\; \int_{\bar n}^{\infty} dk\, p_k \ln p_k = \boxed{0.3\,\frac{\tilde z^2(b=0)}{2\,\kappa}}$$

$$S_E(b) = \frac{\tilde z^2(b)}{2\,\kappa} = S_E \;\text{ of D.K. \& E.L. paper}$$

Conclusions:

- In zero dimension models the entropy for produced particles in scattering processes:

$$S_E \;=\; \ln\left(G(x,Q^2)\right)$$

- In QCD the entropy for produced gluons in DIS:

$$S_E \;=\; \underbrace{0.3}_{non-perturbative\ corrections} \;\underbrace{\frac{\tilde{z}^2(b=0)}{2\,\kappa}}_{perturbative\ QCD}$$