New solution for the non-linear evolution Balitsky-Kovchegov equation

C. Contreras

Departamento de Física, Universidad Técnica Federico Santa María
Chile

In collaboration with
G. Levin and R . Meneses

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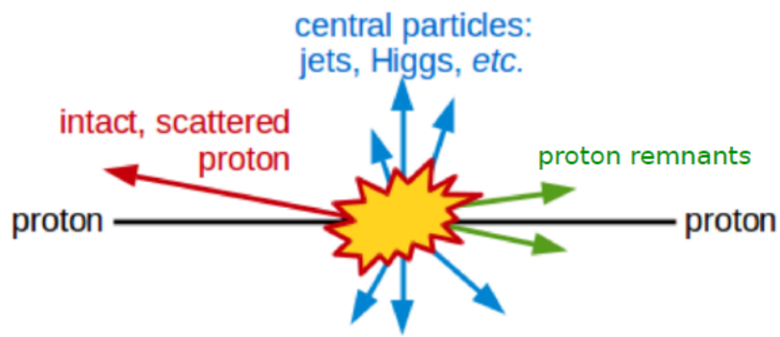
Outline

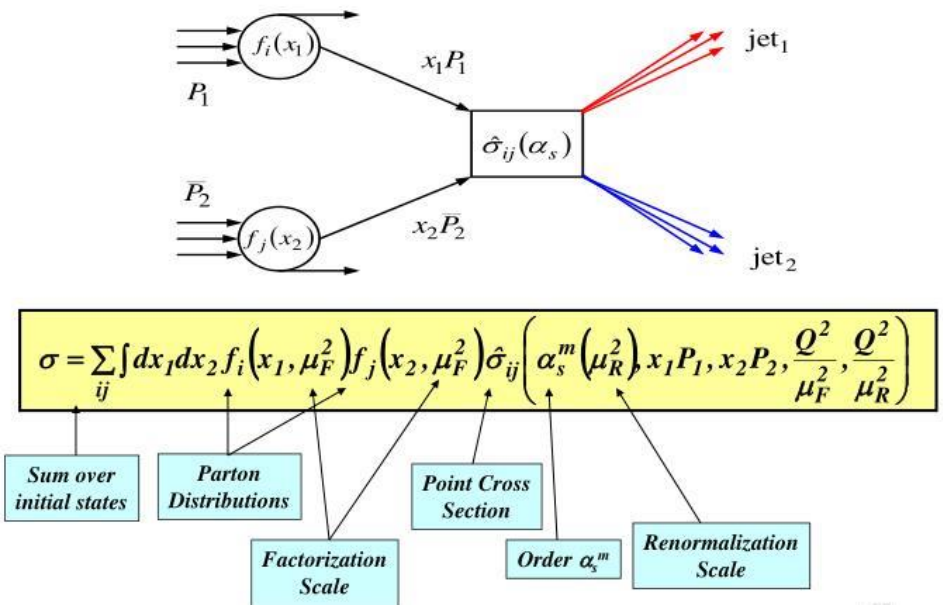
- Introduction
- BK and Geometric Scaling
- BK solutions
- Homotopic approach
- Summary and Outlook

Phys. Rev. D **107 (2023) 9, 094030**
Phys. Rev. D **106 (2022) 3, 034011**
Phys. Rev. D **104 (2021) 11, 116020**
Eur. Phys. J. C **79 (2019) 10, 842**
Eur. Phys. J. C **78 (2018) 6, 475**

Proton-Proton Collition

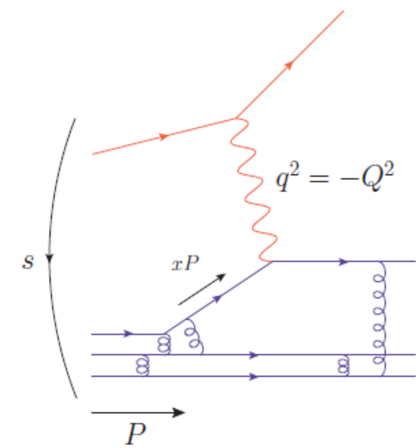


Partonic content of Proton



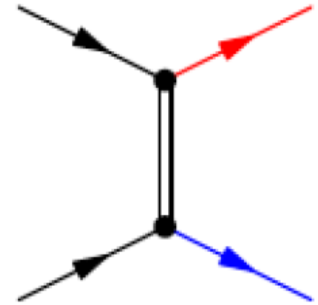
$$\sigma = \sum_{ij} \int dx_1 dx_2 f_i\left(x_1, \mu_F^2\right) f_j\left(x_2, \mu_F^2\right) \hat{\sigma}_{ij}\left(\alpha_s^m\left(\mu_R^2\right), x_1P_1, x_2P_2, \frac{Q^2}{\mu_F^2}, \frac{Q^2}{\mu_R^2}\right)$$

DIS Partonic content of Proton



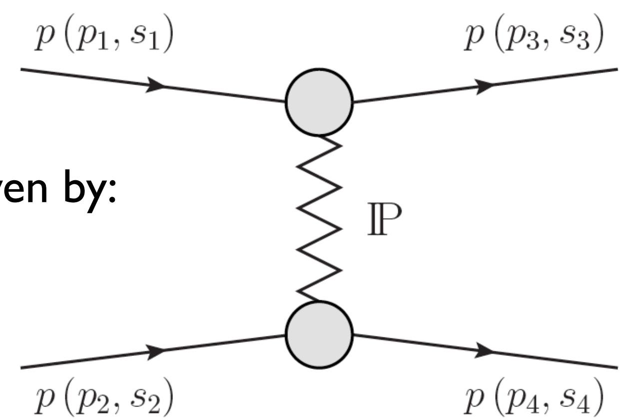
Electron-proton collision (parton model)

Diffractive Processes in pp at the LHC



- Elastic interaction $pp \rightarrow pp$ *Soft Diffractive LHC*
- Interaction explained by the t cannel exchange of a colorless object
- Two Gluon interaction → Pomeron (Pomeranchuk 1956)

Regge theory: (Gribov 1986)

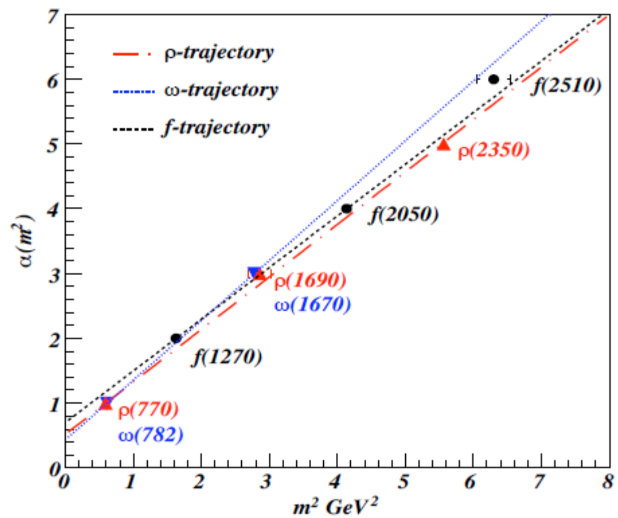


total Cross section, considering the optical theorem is given by:

$$\sigma_T = A_i s^{\alpha_i(0)-1}$$

Elastic Scatering/diffraction

Reggeon trajectory

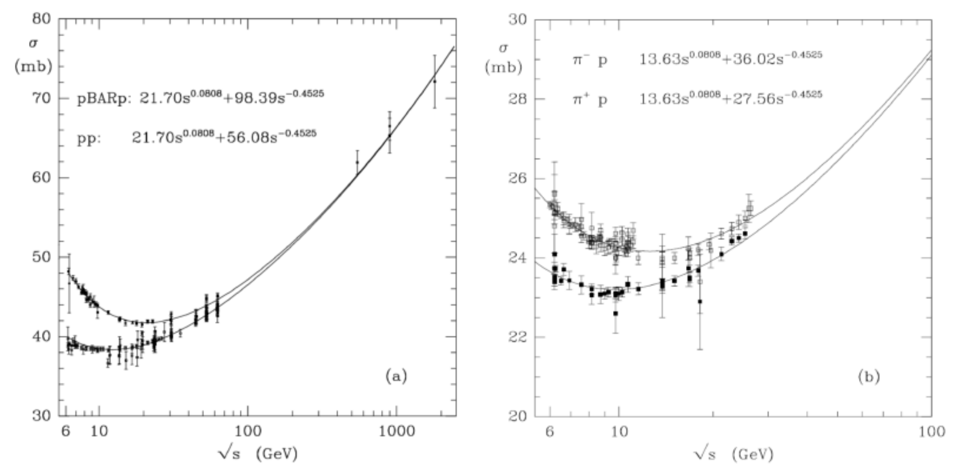


Regge trajectories are almost straight lines and in standard Regge theory they are parameterized by $\alpha(t) = \alpha_0 + \alpha' \cdot t$

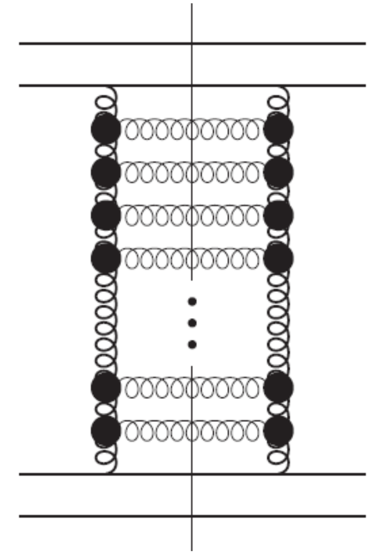
$\mu = \alpha_0 - 1 = \text{intercept}$
and α' slope

$\mu = \alpha_0 - 1 = 0.08$
and
$\alpha' = 0.25\ GeV^{-2}$

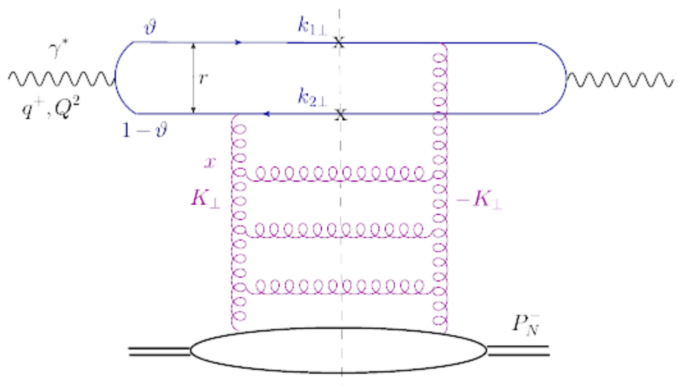
A. Donnachie and Landshoff : 1992

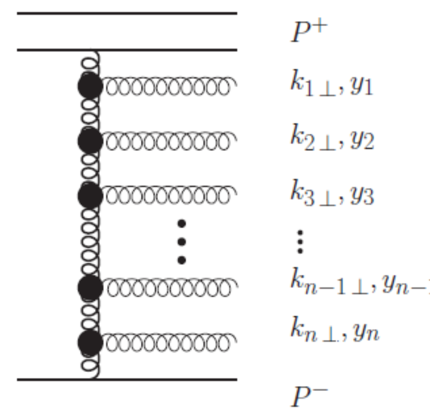


- pQCD:
- using QCD the contribution to the scattering cross section is Two Gluon interaction



2 Gluon generate Ladder Hard/BFKL Pomeron

- 



- **Elastic interaction $p\bar{p} \rightarrow p\bar{p}$ *Soft Diffractive LHC***

- Interaction explained by the Exchange of a colorless object between the $p\bar{p}$
- Pomeron + Odderon

- Pomeron and Odderon correspond to positive and negative C parity:

Pomeron is made of two gluons which leads to a +1 parity

Odderon is made of 3 gluons corresponding to a −1 parity

Lukaszuk, Nicolescu, LNC 8 (1973) 405

Scattering amplitudes can be written as:

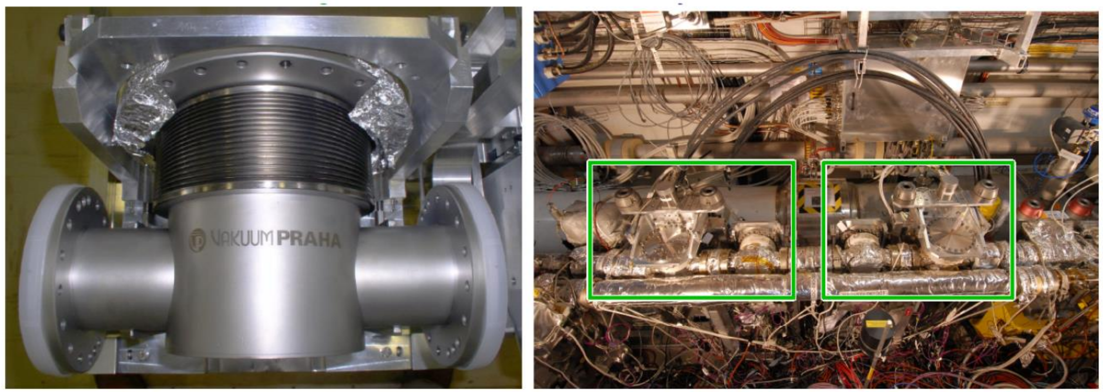
App = Even + Odd and A$p\bar{p}$ = Even − Odd

From the equations above, it is clear that observing a difference between pp and $p\bar{p}$ way to observe the odderon

For $p\bar{p} \rightarrow p\bar{p}$

- We use special detectors to detect intact protons/anti-protons called Roman Pots
- Roman pots installed on both sides of CMS at about 220 m from the interaction point
- TOTEM/D0 result Odderon evidence TOTEM, D0, PRL 127 (2021) 6, 062003

Roman Pot detectors at the LHC



Thank
Talk C. Rayon
Low x 2023

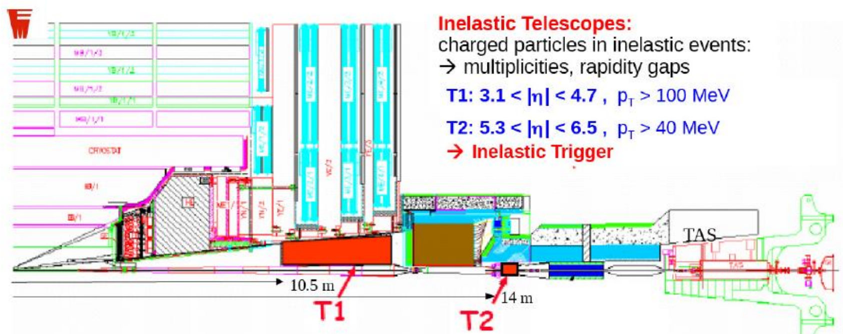
Forward coverage in CMS-TOTEM

Inelastic Telescopes:
charged particles in inelastic events:
→ multiplicities, rapidity gaps

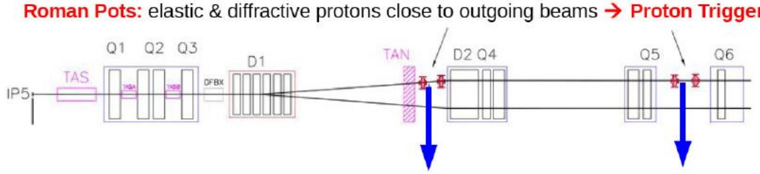
T1: 3.1 < |η| < 4.7, p_T > 100 MeV

T2: 5.3 < |η| < 6.5, p_T > 40 MeV

→ Inelastic Trigger



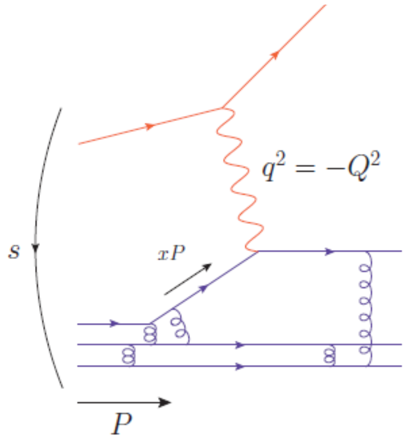
Roman Pots: elastic & diffractive protons close to outgoing beams **→ Proton Trigger**

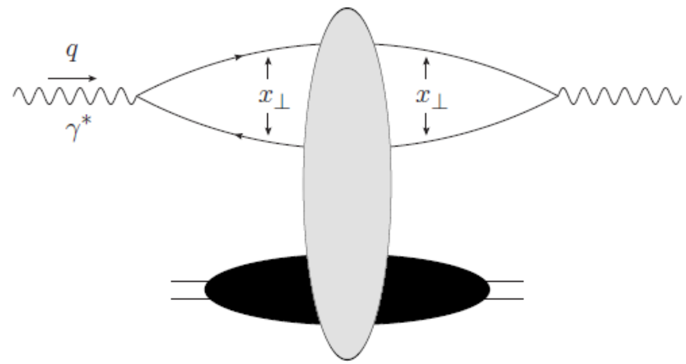


Roman Pot stations in the LHC tunnel (before LS1)

RP (147 m) **RP (220m)**

Mueller Dipole Approximation





- HERA data represented the most direct way of probing that the virtual photon fluctuates into a $\bar{q}q$ pair long before the scattering
- the $\bar{q}q$ color dipole acts as a probe of the gluon distribution at small x
- the dipole transverse size r is preserved by the scattering
- Photon wave function $\Psi^{\gamma}{}_{q\bar{q}}\ (\mathrm{r}, \mathrm{z}, \mathrm{Q}^2)$ in dipole approximation (known to NLO light cone

- $N(r_1, Y, b)$ is the dipole-Hadron scattering Amplitud

$Scattering\ amplitud\ S(r_T, Y, b) = 1 - N(r_T, Y, b) \quad where\ Y\ = \ln\left(\frac{1}{x}\right) = \ln s$ rapidity

- $\sigma_{T,L}(Y, Q^2) = 2\int d^2r \int d^2b \int dz | |\Psi_{\mathrm{NL,T}}(\mathrm{r}, \mathrm{z}, \mathrm{Q}^2)|^2 N(r, Y, b)$

Quantum Chromodynamics at high Energy, Kovchegov and Levin Cambridge 2012

Mueller 2002

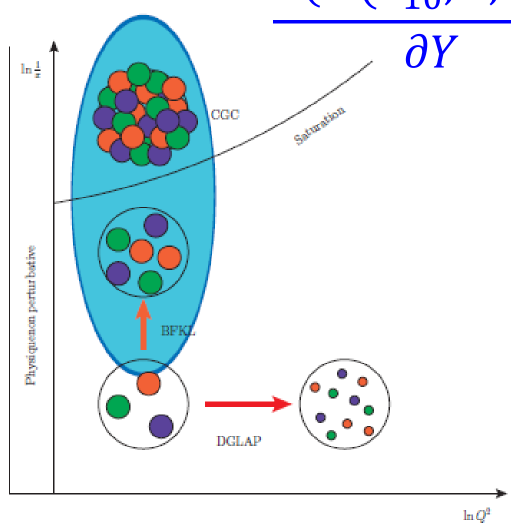
BK and BFKL equation

QCD at small $x_{\mathrm{Bj}} \sim Q^2/s$

Evolution in x $BFKL\ equation$
Small x, high Rapidity Y energy. High density. N(r, Y, b)

Regge limit: $Q^2 \ll s$

$$\frac{\partial(N(x_{10},b,Y))}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{02}^2}{x_{12}^2\,x_{01}^2}(N(x_{12},b,Y) + N(x_{20},b,Y) - (N(x_{10},b,Y))$$



There is problem with this linear equation and one need to introduce non linear term: Saturation region

$$\frac{\bar{\alpha}}{2\,\pi}\int d^2x_2 \frac{x_{02}^2}{x_{12}^2\,x_{01}^2}(N(x_{12},b,Y) * N(x_{20},b,Y)$$

Evalution in Q^2:
$DGLAP\ equation$ Better resolution in the partons
Dokshitzer–Gribov–Lipatov–Altarelli–Parisi

Rigorous treatment requires solution of b-dependent BK equation (or JIMWLK CGC) but is complicated

Rezaeian and Schmidt *Phys. Rev.D* 88 (2013)

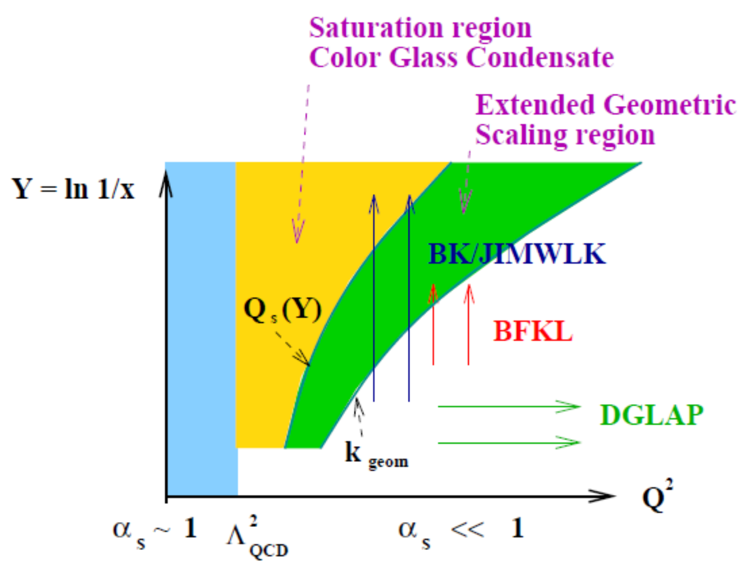
BFKL and BK equation: Transition to saturation and Geometric Scaling

Stasto, Golec-Biernat and Kwiecinski Phys. Rev. Lett. 86 (2001) 596.

have shown that the HERA data on DIS at low x , functions of two independent variables — the photon virtuality Q^2 and the Bjorken variable x , are consistent with scaling in terms of the variable τ

Saturation momentum:

$$Q_s^2 = Q_{s0}^2 e^{\lambda Y} = Q_{s0}^2 \left(\frac{x_0}{x}\right)^{\lambda} \rightarrow \tau = Q^2/Q_0^2 \left(\frac{x}{x_0}\right)^{\lambda} \quad \lambda = 0.3 - 0.4$$



is to show that the scaling region for the various distribution functions is in fact much larger than the saturation region

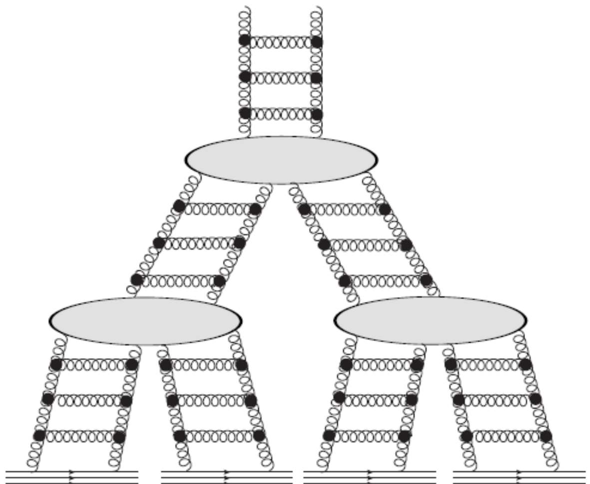
Balistky Phys. Rev. D75, 014001 (2007)
I. Balitsky, Nucl. Phys. B463, 99 (1996)
Kovchgovlancu, Itakura, and Larry McLerran arXiv 0203.137
Jalilian-Marian-Iancu-McLerran-Weigert-Leonidov-Kovner (JIMWLK)

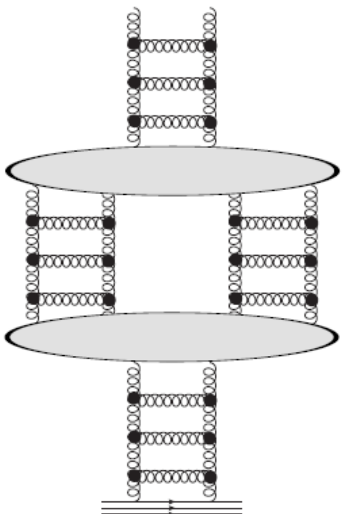
BFKL equation: Balitsky, Fadin, Kuraev and Lipatov

The solution of the BFKL equation grows like a power of center of mass energy s, therefore violating the unitarity bound at very high energies.

To this energy we need to consider that Pomeron can splits into two pomerons, just like in the fan diagram, but then the two pomerons merge back into one Pomeron: Non linear term BK contributions

Unitary need to consider the pomeron loop diagrams In other words the graphs containing pomerons not only splitting, but also merging together





BK evolution

$$\frac{\partial(N(x_{10},b,Y)}{\partial Y} = \frac{\bar{\alpha}}{2\,\pi}\int d^2x_2\,\frac{x_{02}^2}{x_{12}^2\,x_{01}^2}\,(N(x_{12},b,Y) + N(x_{20},b,Y) - N(x_{10},b,Y) + N(x_{12},b,Y) * N(x_{20},b,Y))$$

in principle need to solve the fully impact parameter dependent Balitsky Kovchegov (BK) equation

this work: local approximation → b becomes an external parameter

Initial condition at Y_0 is McLerran -Venugopalan model MV:

$$\mathrm{N}(r_\perp,b,Y) = 1 - \exp\left[-\frac{r_\perp^2 Q_{s0}^2}{4}\ln\left(\frac{1}{r_\perp\Lambda}+e\right)\right]$$

Kowalski, Lappi, Marquet, Venugopalan, PRC 78 (2008) 045201
Lappi, M¨antysaari, PRD 88 (2013) 114020

where initial saturation scales:

$$Q_{s0}^2 = \begin{cases} Q_{s0}^2 & \text{for proton} \\ \mathrm{A}^{1/3}\times Q_{s0}^2 & \text{for nucleus} \end{cases}$$

Analytical Solutions to the BK equation

- satisfying the initial condition given by the BFKL Pomeron
- satisfying the Geometric Scaling inside the saturation region, Non-perturbative

. Unfortunately finding an exact analytical solution seems difficult task, its is non–linear.
In the saturation regime the series diverges, but allows us to construct an asymptotic solution by analytical continuation

Perturbations approach

- First BFKL Pomeron Lipatov solution (1986)

$$\overline{\alpha}_s \equiv \frac{\alpha_s N_c}{\pi}.$$

Eigenfunctions of the Casimir operators of conformal algebra

$$E^{n,\nu}(\rho_0,\rho_1) = \left(\frac{\rho_{01}}{\rho_0\,\rho_1}\right)^{\frac{1+n}{2}+i\nu}\left(\frac{\rho_{01}^*}{\rho_0^*\,\rho_1^*}\right)^{\frac{1-n}{2}+i\nu}$$

$$\chi(\mathrm{n},\nu) = 2\psi(1) - \psi\left(\frac{1+|n|}{2} + i\,\nu\right) - \psi\left(\frac{1+|n|}{2} - i\,\nu\right);$$

$$\psi(z) = \frac{d\ln\Gamma(z)}{dz}$$ Digamma Function

$$N(x_T, Y) = \int_{-\infty}^{\infty} d\nu\; e^{2\overline{\alpha}_s\,\chi(0,\nu)\,Y}\,(x_T Q_{s0})^{1+2\,i\,\nu}\,C_\nu$$

$\gamma = \frac{1}{2} + i\,\nu$ Dominant contribution to high Y is

$$C_\nu \equiv \tilde{C}_\nu\,2^{-2\,i\,\nu}\,\frac{\Gamma\left(\frac{1-2\,i\,\nu}{2}\right)}{\Gamma\left(\frac{1+2\,i\,\nu}{2}\right)}.$$

$$\chi(0,\gamma) = 2\psi(1) - \psi(\gamma) - \psi(\gamma - 1) = \chi(\gamma)$$

Perturbations approach

Y. Kovchegov arXiv 9905214

Perturbative solution and it is CV structure functions are outside of the saturation region.

$$\frac{\partial \tilde{N}_1(k,Y)}{\partial Y} = \frac{2\alpha N_c}{\pi}\chi\left(-\frac{\partial}{\partial \ln k}\right)\tilde{N}_1(k,Y), \qquad \tilde{N}_1(k,Y) = \int \frac{d\lambda}{2\pi i}\exp\left[\frac{2\alpha N_c}{\pi}Y\chi(-\lambda)\right]\left(\frac{k}{\Lambda}\right)^{\lambda} C_\lambda.$$

We show that as energy increases the scattering cross section of the quark–antiquark pair of a fixed transverse separation on a hadron or nucleus given by the solution of BK equation inside of the saturation region unitarizes

We have to admit that in order to construct a solution BK in coordinate space, N(x⊥, Y), and the corresponding structure function F2 one has to have a better knowledge of the momentum space solution inside the saturation region.

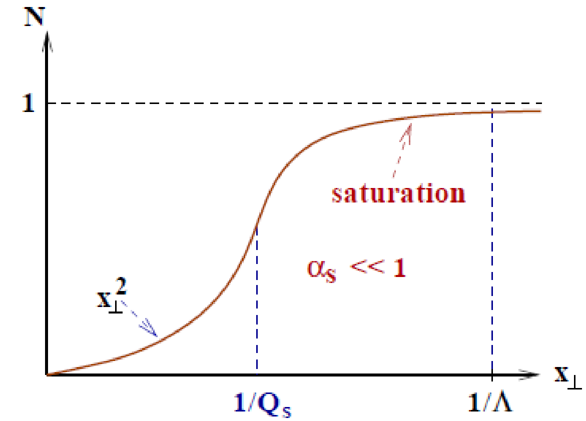
A solution of BK would probably be very helpful in determining exact values of N(x⊥, Y) and F2 at intermediately large rapidities

$$\frac{\partial \tilde{N}_2(k,Y)}{\partial Y} = \frac{2\alpha N_c}{\pi}\chi\left(-\frac{\partial}{\partial \ln k}\right)\tilde{N}_2(k,Y) - \frac{\alpha N_c}{\pi}\tilde{N}_1(k,Y)^2,$$

Motika and Sadzikowski arXiv 230602118 Hight Twist corrections

Preliminary results:

The dipole-nucleus scattering amplitude N is plotted (schematically) as a function of x⊥.



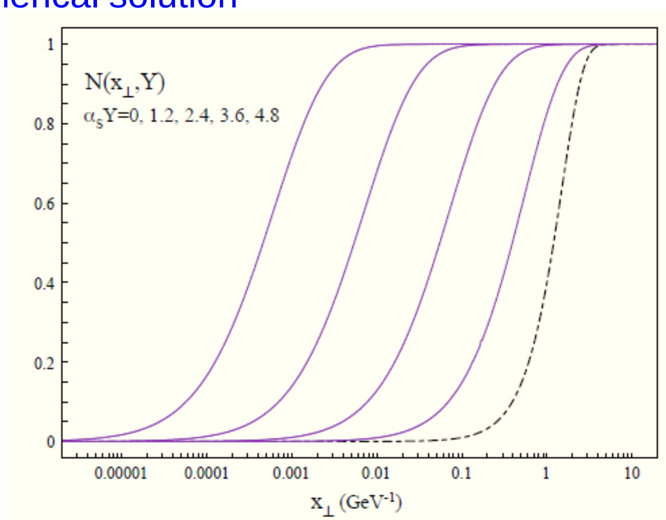
- One can see that, at small $x_\perp \ll \frac{1}{Q_s}$, we have N ~ 0
- This result is natural, since in the zero-size dipole the color charges of the quark and the anti-quark cancel each other, leading to disappearance of the interactions with the target. This effect is known as color transparency

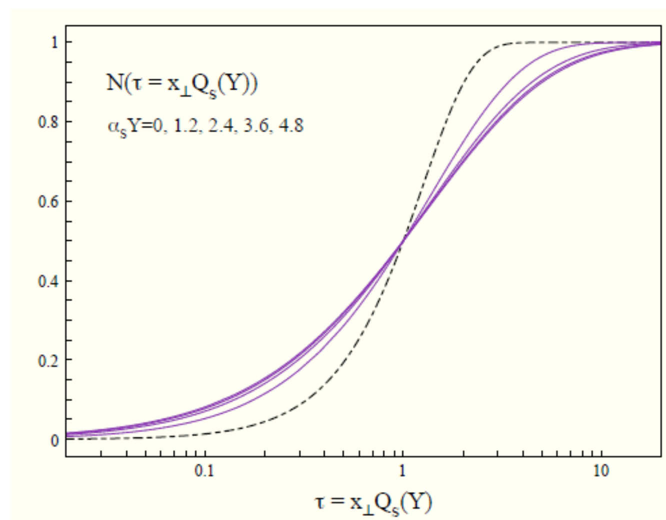
- at large dipole sizes $x_\perp > \frac{1}{Q_s}$, the growth stops and the amplitude levels off (saturates) at N = 1.

- (Kopeliovichet al 1981)

- The transition happens at around $x_\perp \sim \frac{1}{Q_s}$.

- Numerical solution





NonPerturbations approach

Levin Tuchin Solutions inside the saturation region GS (2000)

$N(r_\perp, b, Y) = 1 - \Delta(r_\perp, Y, b)$ and $r_\perp \gg \frac{1}{Q_s(Y)}$ we can find

$$\frac{\partial \Delta(r_\perp, Y, b)}{\partial Y} = -\bar{\alpha}\, z\, \Delta(r_\perp, Y, b)$$

Where $z = \ln r^2 Q_s^2(Y) = \bar{\alpha}\,\kappa\, Y + \ln r^2 Q_s^2(Y_0) = \bar{\alpha}\,\kappa\, Y + \xi_0$

The solution es $N(r, b, Y) = 1 - \Delta_0 e^{-\frac{z^2}{2\kappa}}$

Our Proposal Homotopic Solution

- One can see that a numerical solution of the BK equation does not allow us to introduce explicit dependence on Qs.
- we need to have the analytical solution in which we can see explicitly the dependence of the scattering amplitude on Qs(Y; b).
- we suggest a procedure to find the solution to the BK equation as a sequence

of iterations, based on homotopy approach

Homotopy Method: $\mathcal{L}[u] \; + \; \mathcal{N}_{\mathcal{L}}[u] = 0 \rightarrow \mathcal{K}[p,u] = \mathcal{L}[u] \; + p\, \mathcal{N}_{\mathcal{L}}[u] = 0$

$\mathcal{L}[u]$ *linear part* $\mathcal{N}_{\mathcal{L}}[u]$=Integral differential Operator

$$u_p\left(Y,\boldsymbol{x}_{10},\boldsymbol{b}\right) \;=\; u_0\left(Y,\boldsymbol{x}_{10},\boldsymbol{b}\right) \;+\; p\,u_1\left(Y,\boldsymbol{x}_{10},\boldsymbol{b}\right) + p^2\,u_3\left(Y,\boldsymbol{x}_{10},\boldsymbol{b}\right) \;+\; \dots$$

$\mathcal{L}[u_0] = 0$ *give de solution from linear part and*

u_1 *from the nonlinear part* *arXiv 2204.10111*

BK in momentum representation

- Transformation to Momentum space

$$N\left(r^2,b,Y\right) \;=\; r^2\int\frac{d^2k_\perp}{2\pi}\,e^{i\boldsymbol{k}_\perp\cdot\boldsymbol{r}}\,\tilde{N}\left(k_\perp,b,Y\right) \;=\; r^2\int_0^\infty k_\perp dk_\perp J_0\left(rk_\perp\right)\,\tilde{N}\left(k_\perp,b,Y\right)$$

$$\tilde{N}(k_\perp,b,\delta\tilde{Y}) = \int_0^\infty r dr\frac{J_0(rk_\perp)}{r^2}N(r,b,\delta\tilde{Y}) \qquad \frac{\partial\tilde{N}(k_\perp,b,\delta\tilde{Y})}{\partial k_\perp} = -\int_0^\infty drJ_1(rk_\perp)N(r,b,\delta\tilde{Y})$$

- BK Momentum space

$$\frac{\partial\tilde{N}\left(k_\perp,b,Y\right)}{\partial Y} \;=\; \bar{\alpha}_S\left\{\chi\left(-\frac{\partial}{\partial\tilde{\xi}}\right)\tilde{N}\left(k_\perp,b,Y\right) \;-\; \tilde{N}^2\left(k_\perp,b,Y\right)\right\}$$

$$\tilde{Z} = \ln(Q_s^2(Y,b)/k^2) = \bar{\alpha}\,\kappa\,(Y-Y_A) + \ln(Q_s^2(Y_A,b)/k^2) = \bar{\alpha}\,\kappa\,(Y-Y_A) + \tilde{\xi}_0$$

$$\frac{\partial^2\tilde{N}\left(k_\perp,b,Y\right)}{\partial Y\,\partial\tilde{\xi}} \;=\; \bar{\alpha}_S\left\{\chi_0\left(-\frac{\partial}{\partial\tilde{\xi}}\right)\frac{\partial\tilde{N}\left(k_\perp,b,Y\right)}{\partial\,\tilde{\xi}} \;+\; \tilde{N}\left(k_\perp,b,Y\right) \;-\; 2\frac{\partial\tilde{N}\left(k_\perp,b,Y\right)}{\partial\,\tilde{\xi}}\tilde{N}\left(k_\perp,b,Y\right)\right\}$$

$$\chi_0(\gamma) \;=\; \chi(\gamma) \;-\; \frac{1}{\gamma} \qquad\qquad \delta\tilde{Y} = \overline{\alpha}_S\left(Y-Y_A\right)$$

$$\frac{\partial\tilde{N}\left(k_\perp,b,Y\right)}{\partial\,\tilde{z}} \;=\; \frac{1}{2} + M\left(\tilde{z},b,\delta\tilde{Y}\right) \quad \text{or} \quad \tilde{N}\left(\tilde{z},b,\delta\tilde{Y}\right) \;=\; \frac{1}{2}\tilde{z} \;+\; \int_0^{\tilde{z}} d\tilde{z}' M\left(\tilde{z}',b,Y\right) \;+\; \Phi\left(\delta\tilde{Y}\right)$$

- Transformation to Momentum space

$$\frac{\partial M\left(\tilde{z},b,\delta\tilde{Y}\right)}{\partial\delta\tilde{Y}} + \kappa\frac{\partial M\left(\tilde{z},b,\delta\tilde{Y}\right)}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right) M\left(\tilde{z},b,\delta\tilde{Y}\right) - \tilde{z}\, M\left(\tilde{z},b,\delta\tilde{Y}\right) - 2\,\Phi\left(\delta\tilde{Y}\right) M\left(\tilde{z},\delta\tilde{Y}\right) - \frac{\partial A}{\partial\tilde{z}}$$

$$A(\tilde{z},b,\delta\tilde{Y}) = \left(\int_0^{\tilde{z}} M(z',b,\delta\tilde{Y})dz'\right)^2$$

- BK Momentum space

$$M(\tilde{z},b,\delta\tilde{Y}) = k_\perp\int_0^\infty dr J_1(rk_\perp)\left\{\frac{N(r,b,\delta\tilde{Y}) - 1}{2}\right\}$$

- BK Momentum space Homotopic structure

$$\mathscr{L}[F] = \frac{\partial F}{\partial\tilde{Y}} + \kappa\frac{\partial F}{\partial\tilde{z}} - \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right) F + \tilde{z}F + 2\,\Phi\left(\delta\tilde{Y}\right) F\,, \quad \mathscr{N}_{\mathscr{L}}[F] = \frac{\partial}{\partial\tilde{z}}\left(\int_0^{\tilde{z}} F dz'\right)^2$$

$$F \equiv M_p(\tilde{z},b,\delta\tilde{Y}) = \sum_{n=0}^{\infty} M_n(\tilde{z},b,\delta\tilde{Y})\, p^n$$

- Homotopic structure to order n

$$\frac{\partial M_n}{\partial\tilde{Y}} + \kappa\frac{\partial M_n}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right) M_n - (\tilde{z} + \zeta)\, M_n - 2\sum_{j=0}^{n-1}\Phi_{n-j}\left(\delta\tilde{Y}\right) M_j - \frac{\partial A_{n-1}}{\partial\tilde{z}}$$

- Homotopic structure order p=0 in Saturation Region

$$z = \ln(Q_s^2(Y,b)r^2) = \bar{\alpha}\,\kappa\,(Y - Y_A) + \ln(Q_s^2(Y_A,b)r^2) = \xi_0^A + \xi_0$$

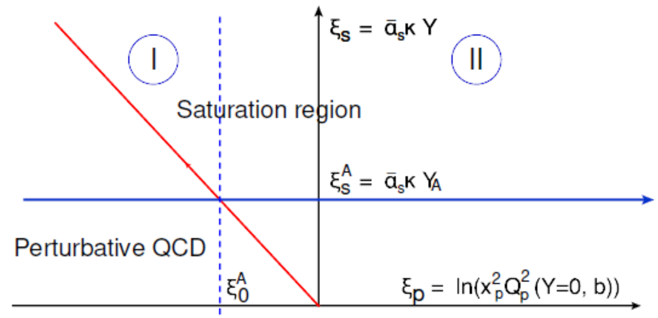


FIG. 1: Saturation region of QCD. The critical line (z=0) is shown in red. The initial condition for scattering with the dilute system of partons (with proton) is given at $\xi_s = 0$. For heavy nuclei the initial conditions are placed at $Y_A = (1/3)\ln A \gg 1$, where A is the number of nucleon in a nucleus. The line, where they are given, is shown in blue.

$$M_0\left(\tilde{z},\delta\tilde{Y}\right) = \begin{cases} M_0^I\left(\tilde{z}\right) & \text{for } -\kappa\delta\tilde{Y} < \tilde{\xi} < \tilde{\xi}_0^A; \\ M_0^{II}\left(\tilde{z},\delta\tilde{Y}\right) & \text{for } \tilde{\xi} > \tilde{\xi}_0^A; \end{cases}$$

$$\frac{\partial M_0}{\partial\,\delta\tilde{Y}} + \kappa\frac{\partial M_0}{\partial\tilde{z}} = \chi_0\left(\frac{\partial}{\partial\tilde{z}}\right)M_0 - (\tilde{z}+\zeta)M_0$$

$$\kappa\frac{dM_0\left(\tilde{z},b\right)}{d\tilde{z}} = \chi_0\left(\frac{d}{d\tilde{z}}\right)M_0\left(\tilde{z},b\right) - \left(\tilde{z}+\zeta\right)M_0\left(\tilde{z},b\right)$$

- Initial condition in Saturation Region

$$\text{region } I: \quad M_0^I\left(\tilde{z} = \tilde{\xi}_0^A, b\right) = \phi_0(b) - \frac{1}{2} = M_0 \qquad \phi_0(b) \ll \frac{1}{2};$$

$$\text{region } II: \quad M_0^{II}(\tilde{z},\delta\tilde{Y}=0;b) = -\frac{1}{2}\left(1-\exp(-\exp(-\tilde{\xi}))\right), M_0 = -\frac{1}{2}\left(1-\exp(-\exp(-\tilde{\xi}_0^A))\right) \text{ for } \tilde{\xi} > \tilde{\xi}_0^A$$

- Geometric Scaling solution $M_0^I(\tilde{z})$

Mellin Transformations

$$M_0(\tilde{z}, b) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\gamma(\tilde{z}+\zeta)} m_0(\gamma, b)$$

$$M_0^I(\tilde{z}) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{\gamma(\delta\tilde{Y}+\zeta)} m(0) \left(\frac{k_\perp}{Q_s(Y_A, b)}\right)^{-2\gamma} \frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} e^{\kappa\gamma^2/2} e^{-2\psi(1)\gamma}$$

- Coordinate representation

$$N_0^I(z) = 1 + \frac{4m(0)}{\sqrt{2\pi\kappa}} \exp\left(-\frac{(z+\zeta-z_0)^2}{2\kappa}\right)$$

$$z_0 = 2(\psi(1) + \ln 2)$$

- Saturation solution $M_0^{II}(\tilde{z}, \delta\tilde{Y})$

$$M_0^{II}\left(\tilde{z}, \delta\tilde{Y}\right) = \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{(\tilde{z}+\zeta)\gamma} m_0^{II}\left(\gamma, \delta\tilde{Y}\right)$$

$$M_0^{II}(\tilde{z}, b) = f_1\left(\delta\tilde{Y}\right) \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} u_1^{-\gamma} \left(-\frac{1}{2}\frac{\Gamma(1+\gamma)}{\Gamma(1-\gamma)} \Gamma\left(-(\gamma+\delta\tilde{Y})\right)\right)$$

$$u_1 = \frac{k_\perp^2}{Q_s^2(\delta\tilde{Y}, b)}.$$

$$f_1\left(\delta\tilde{Y}\right) = \exp\left(-\frac{1}{2}\kappa(\delta\tilde{Y})^2 - \zeta\delta\tilde{Y} + 2\psi(1)\,\delta\tilde{Y}\right)$$

$$N_0^{II}(r, \delta\tilde{Y}) = 1 - f_1(\delta\tilde{Y}) \left(\frac{r^2 Q_s^2(Y_A, b)}{4}\right)^{-\delta\tilde{Y}} \exp\left(-\frac{r^2 Q_s^2(Y_A, b)}{4}\right)$$

- Matching $\xi = \xi_0^A$

$$\left.\frac{dN_0(z)}{dz}\right|_{z=0} = \bar{\gamma}\, N_0$$

$$N_0^I\left(z = \kappa\delta\tilde{Y} + \xi_0^A\right) \;=\; N_0^{II}(\xi = \xi_0^A, z = \kappa\delta\tilde{Y} + \xi_0^A)$$

- Coordinate representation

$$N_0^{II}(r, b, \delta\tilde{Y}) = 1 - \exp\left\{-\frac{\Phi(z,\xi,b)}{2\kappa}\right\}$$

$$\Phi(z,\xi,b) = (z + \zeta - z_0)^2 - (\xi + \zeta - z_0)^2 + \kappa\frac{e^{\xi}}{2}$$

$$N_0^{II}(\xi, z) = 1 - G_0(\xi)\exp\left\{-\frac{(z+\zeta-z_0)^2}{2\kappa}\right\} \qquad G_0(\xi) = \exp\left\{\frac{(\xi+\zeta-z_0)^2}{2\kappa} - \frac{e^{\xi}}{4}\right\}$$

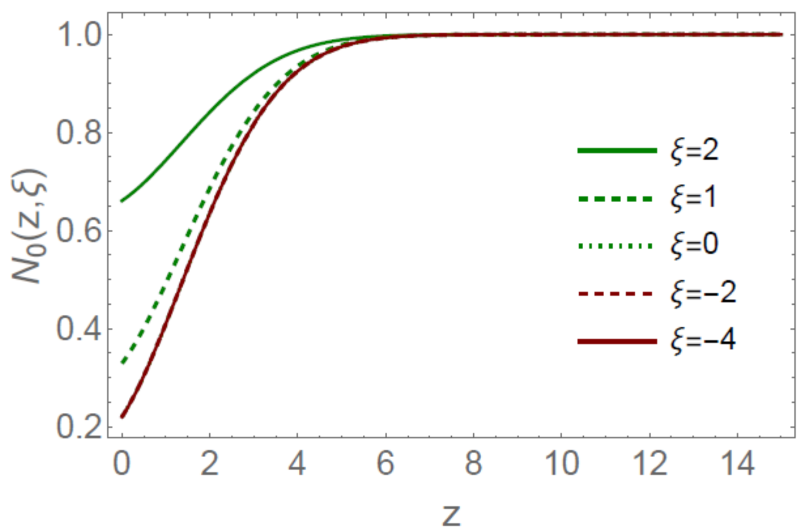


Fig. 2-a

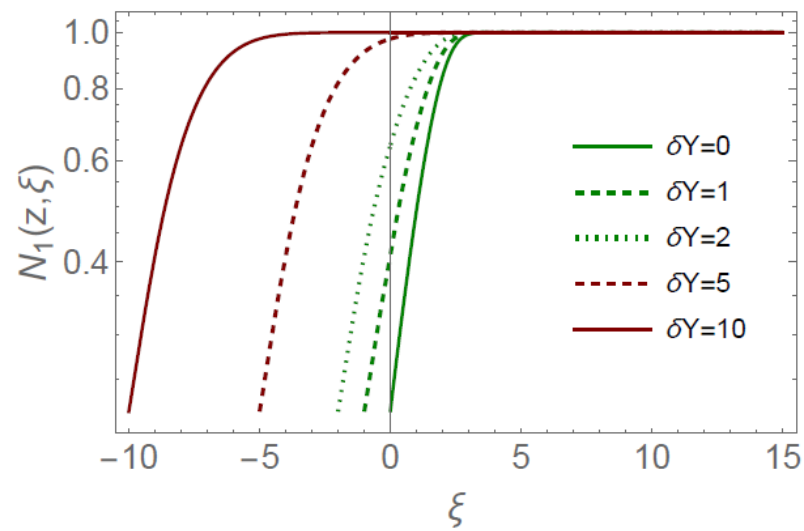


Fig. 2-b

FIG. 2: Fig. 2-a: $N_0(z,\xi)$ versus z at fixed values of $\tilde{\xi}$. Fig. 2-b: $N_0(z,\xi)$ versus ξ at fixed $\delta\tilde{Y}$ $(z = \kappa\delta\tilde{Y} + \xi)$. $N_0 = 0.22$; $\bar{\gamma} = 0.63$, $\zeta - z_0 = 0.867$, $C_1 = 0.842$, $\xi_0^A = 0$.

- Homotopic structure order p = 1 in Saturation Region

$$\frac{\partial M_1}{\partial \delta\tilde{Y}} + \kappa \frac{\partial M_1}{\partial \tilde{z}} = \chi_0\left(\frac{\partial}{\partial \tilde{z}}\right) M_1 - (\tilde{z}+\zeta) M_1 - 2\,\Phi_1\left(\delta\tilde{Y}\right) M_0 - \frac{\partial A_0}{\partial \tilde{z}}$$

$$A_0(\tilde{z}, \delta\tilde{Y}, b) = \left(\int_0^{\tilde{z}} M_0(z', \delta\tilde{Y}) dz'\right)^2$$

- Initial condition in Saturation Region

$$\text{region}\, I:\ M_1(\tilde{z} = \tilde{\xi}_0^A) = 0; \quad \text{region}\, II:\ M_1(\tilde{z}, b, \delta\tilde{Y} = 0) = 0.$$

$$N_1^I(z) = C_1\left(N_0^I(z) - 1\right) + \Delta_1^I(z) = C_1\left(N_0^I(z) - 1\right) + \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{d\gamma}{2\pi i} e^{(z+\zeta-z_0)\gamma} m(0) e^{\frac{1}{2}\kappa\gamma^2} \int_0^{\gamma} d\gamma' \frac{a_0(\gamma')}{m_0(\gamma')}$$

$$N_1^I(z) \sim -\frac{c_1^I(\zeta)\, m(0)}{2\kappa\sqrt{2\pi\kappa}} \left(z - z_0^A\right) \Delta_{01}(z)$$

$$\frac{\partial M_1}{\partial \delta \tilde{Y}} + \kappa \frac{\partial M_1}{\partial \tilde{z}} = \chi_0 \left(\frac{\partial}{\partial \tilde{z}} \right) M_1 - (\tilde{z} + \zeta) M_1 - 2\,\Phi_1 \left(\delta \tilde{Y} \right) M_0 - \frac{\partial A_0}{\partial \tilde{z}}$$

$$A_0(\tilde{z}, \delta\tilde{Y}, b) = \left(\int_0^{\tilde{z}} M_0(z', \delta\tilde{Y}) dz' \right)^2$$

- Initial condition in Saturation Region

$$\text{region } I: \; M_1(\tilde{z} = \tilde{\xi}_0^A) = 0; \quad \text{region } II: \; M_1(\tilde{z}, b, \delta\tilde{Y} = 0) = 0.$$

$$N_1^{II}(z, \delta\tilde{Y}) = h(\delta Y) \exp\left(-\frac{1}{2}(z + \zeta - z_0)\delta\tilde{Y}\right) \exp\left(-\frac{1}{2}(\xi + \zeta - z_0)\delta\tilde{Y}\right) \exp\left(-\frac{r^2 Q_s^2(Y_A, b)}{4}\right)(1 + o(1))$$

$$h(\delta\tilde{Y}) = -\int_0^{\delta\tilde{Y}} d\delta\tilde{Y}' \tilde{\Phi}_1^{II}(\delta\tilde{Y}').$$

- Numerical Estimates

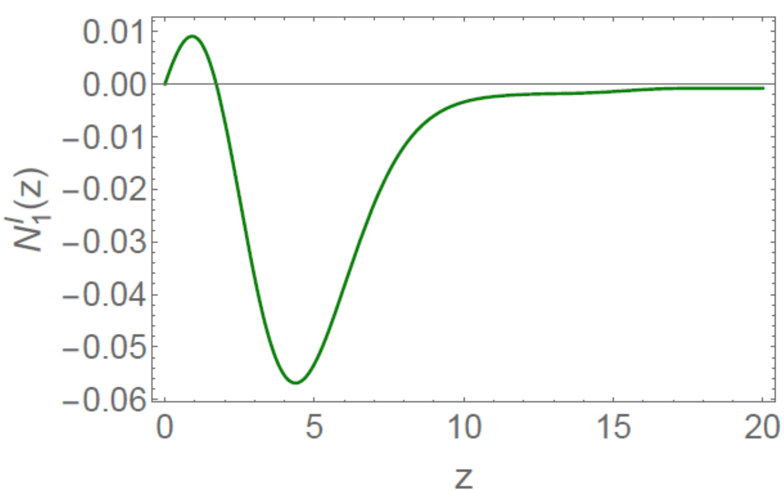


Fig. 3-a

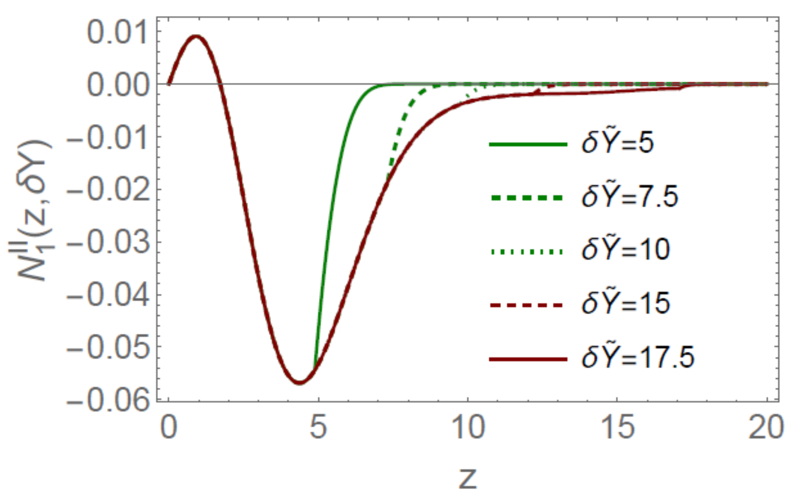


Fig. 3-b

FIG. 3: Fig. 3-a: $N_1^I(z)$ versus z. Fig. 3-b: $N_1(z, \delta Y)$ versus z at fixed $\delta\tilde{Y} = \bar{\alpha}_S(Y - Y_A)$. The value of $\xi_0^A = 0$.

Summary and outlook

we developed the homotopy approach for solving the non-linear evolution Balitsky-Kovchegov equation

First, we solved the linearized version of the BK equation in the momentum space deep in the saturation region. We found that this solution has the geometric scaling behavior for $\xi < \xi_0^A$

For $\xi > \xi_0^A$ we observe the violation of the geometric scaling behaviour in the saturation region.

This solution satisfies the boundary and initial conditions which are given perturbative QCD approach for $r\,Q_s < 1$ and by McLerran-Venugopalan for $Y = Y_A$.

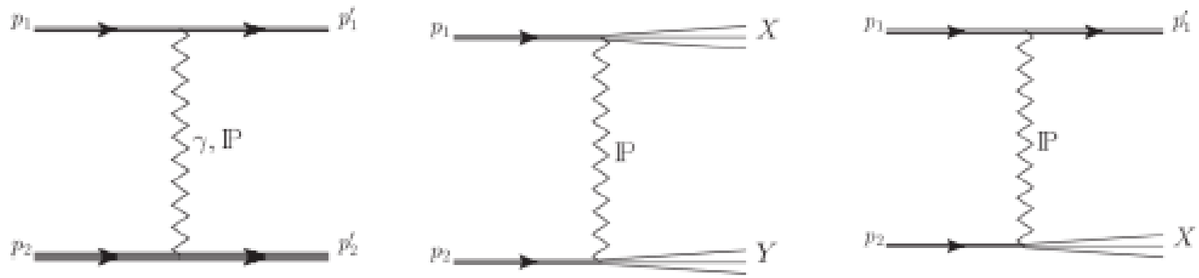
Finally in our approach, we have to take into account the remaining part of the non-linear correction that have not been included in the linearized form of BK equation. It turns out that these corrections are rather small indicating that our procedure gives a self consistent way to account them

We believe that this method of finding solution, which allow us to treat the most essential part of the scattering amplitude analytically.

The numerical part of the calculations is expressed through well converged integrals and can be easily estimated.

We found a way to study the BK equation in Momentum or coordinated representation, which can be extended to another process.

Expanding of our approach to other SD/DD processes



Elastic scattering

Single/double diffractive dissociation

BK´s for Pomerons and Odderons

$$\frac{\partial \mathcal{N}(\boldsymbol{r}_\perp, \boldsymbol{b}_\perp)}{\partial Y} = \frac{\alpha_S N_c}{2\pi^2} \int_{r_{1\perp}} \frac{\boldsymbol{r}_\perp^2}{\boldsymbol{r}_{1\perp}^2 \boldsymbol{r}_{2\perp}^2} \Big[\mathcal{N}(\boldsymbol{r}_{1\perp}, \boldsymbol{b}_\perp) + \mathcal{N}(\boldsymbol{r}_{2\perp}, \boldsymbol{b}_\perp) - \mathcal{N}(\boldsymbol{r}_\perp, \boldsymbol{b}_\perp)$$

$$+ \mathcal{N}(\boldsymbol{r}_{1\perp}, \boldsymbol{b}_\perp)\mathcal{N}(\boldsymbol{r}_{2\perp}, \boldsymbol{b}_\perp) - \mathcal{O}(\boldsymbol{r}_{1\perp}, \boldsymbol{b}_\perp)\mathcal{O}(\boldsymbol{r}_{2\perp}, \boldsymbol{b}_\perp) \Big]$$

$$\frac{\partial \mathcal{O}(\boldsymbol{r}_\perp, \boldsymbol{b}_\perp)}{\partial Y} = \frac{\alpha_S N_c}{2\pi^2} \int_{r_{1\perp}} \frac{\boldsymbol{r}_\perp^2}{\boldsymbol{r}_{1\perp}^2 \boldsymbol{r}_{2\perp}^2} \Big[\mathcal{O}(\boldsymbol{r}_{1\perp}, \boldsymbol{b}_\perp) + \mathcal{O}(\boldsymbol{r}_{2\perp}, \boldsymbol{b}_\perp) - \mathcal{O}(\boldsymbol{r}_\perp, \boldsymbol{b}_\perp)$$

$$- \mathcal{N}(\boldsymbol{r}_{1\perp}, \boldsymbol{b}_\perp)\mathcal{O}(\boldsymbol{r}_{2\perp}, \boldsymbol{b}_\perp) - \mathcal{O}(\boldsymbol{r}_{1\perp}, \boldsymbol{b}_\perp)\mathcal{N}(\boldsymbol{r}_{2\perp}, \boldsymbol{b}_\perp) \Big]$$