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Tensions with cosmological singularities: Should we try to avoid their appearance?

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Based on:

Roberto Casadio, Iberê Kuntz and
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Absence of covariant singularities in pure gravity,
International Journal of Modern Physics D 31, 2150130 (2022),

Covariant singularities in quantum field theory and quantum gravity,
Nuclear Physics B 971, 115496 (2021),

Covariant singularities: A brief review,
Modern Physics Letters A 37, 2230007 (2022).

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Introduction and Motivations

- Appearance of singularities is one of the most important phenomena in General Relativity and its generalizations and modifications.
- The singularities were first discovered in such simple geometries as those of Friedmann and Schwarzschild and later their general character was established (Penrose, Hawking).
- The investigation of the oscillatory approach to the cosmological singularity (Belinsky, Khalatnikov, Lifshitz) known also as Mixmaster universe (Misner) has opened the way to the birth of a new branch of the mathematical physics chaotic cosmology and hyperbolic Kac-Moody algebras (Damour, Henneaux, Nicolai).

Introduction and Motivations

- Should we try to avoid the singularities and to construct the models without them ?
- There are attempts to construct the histories of the universe without singularities (e.g. Rubakov).
- One can construct also regular black holes (e.g. Simpson, Visser).
- However, one can try to cross a singularity!
- Sometimes one can suggest and justify a prescription to match the geometry and matter field configurations in the regions separated by a singularity. This can be called singularity crossing.

Introduction and Motivations

- In the case of soft or sudden singularities, the curvature is divergent but the Christoffel symbols are finite. The geodesics are well defined and the geometry can be reconstructed.
- The crossing of the Big Bang - Big Crunch singularities looks more counterintuitive.
- However, it can be sometimes described by using the reparametrization of fields, including the metric.
- One can say that to do this, it is necessary to resort to one of two ideas, or a combination thereof.
- One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant non-singular.

Introduction and Motivations

- Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge.
- The application of these ideas looks in a way as an craftsman work.
- Our goal is to develop a general formalism to distinguish "dangerous" and "non-dangerous" singularities, considering the field variable space of the model under consideration.

Spacetime singularities

- There exist the singularities in the metric connected with the unhappy choice of the coordinates.
- Such singularities are called "coordinate singularities".
- Some of them are trivial like the singularity in the origin of the spherical coordinate system of the flat space $r = 0$. It is removed by the transition to the Cartesian coordinates.
- The coordinate singularity at the horizon in the Schwarzschild metric is much more involved:

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

One can eliminate it by the transition to the Kruskal coordinates, but the horizon is physically significant. One can cross it only in one direction.

- Mathematically, there exists no parameter which smoothly connects the above change of coordinates to the identity. Such changes of coordinates are called "large".
- In the center $r = 0$ one has the singularity of the Kretschmann invariant

$$R_{ijmn}R^{ijmn},$$

which cannot be eliminated by the coordinate change.
- In the Friedmann universe

$$ds^2 = dt^2 - a^2(t)dl^2$$

the Ricci scalar R diverges at $t = 0$.
- What can we do with these singularities?

Examples of spacetime singularities removable by field redefinitions

Hawking -Turok instanton

$$ds^2 = d\sigma^2 + b^2(\sigma)\left[d\chi^2 + \sin^2(\chi)\, d\Omega^2\right],$$

$$b(\sigma) \approx \begin{cases} \sigma \;, & \text{for} \quad \sigma \sim 0 \;, \\ (\sigma_f - \sigma)^{1/3} \;, & \text{for} \quad \sigma \sim \sigma_f \;, \end{cases}$$

$$\phi(\sigma) \approx \begin{cases} \frac{1}{2}\,\sigma^2 \;, & \text{for} \quad \sigma \sim 0 \;, \\ -\sqrt{\frac{2}{3}}\ln(\sigma_f - \sigma), & \text{for} \quad \sigma \sim \sigma_f \;. \end{cases}$$

On Wick rotating χ, one obtains an open universe.

The Ricci scalar is

$$R \sim \frac{1}{(\sigma_f - \sigma)^2} \ ,$$

there is a spacetime singularity at $\sigma = \sigma_f$.
Changing spacetime coordinates to $d\bar{\sigma} = b^{-1}\, d\sigma$, followed by a Weyl transformation $\bar{g}_{\mu\nu} = b^{-2}\, g_{\mu\nu}$ gives us a non-singular geometry.
Let us make another Weyl transformation

$$\bar{g}_{\mu\nu} = \Omega^2\, \tilde{g}_{\mu\nu},$$

$$\Omega = 1 + \beta\, e^{-\alpha \sqrt{2/3}\, \phi},$$

where α and β are free parameters.

Introducing a new, canonically normalized scalar field

$$d\tilde{\phi}^2 = 6\, e^{-\sqrt{2/3}\,\phi}\,\Omega^2\,\frac{\partial \ln \Omega}{\partial \phi}\left(\sqrt{\frac{2}{3}} - \frac{\partial \ln \Omega}{\partial \phi}\right) d\phi^2,$$

we come to the situation when both the geometry and the scalar field are regular.

Flat Friedmann universe with a scalar field

$$ds^2 = dt^2 - a^2(t)dl^2.$$

In such a universe there is a Big Bang - Big Crunch singularity. One can prescribe the rules for its crossing making conformal transformations between the Einstein and Jordan frames, combined with the transformation of the scalar field, which leaves it canonically normalized.

$$U_0 R \leftrightarrow U(\phi)R.$$

Field space and singularities

- When the spacetime singularities can be removed by a reparametrization of the field variables?
- Our hypothesis: when the geometry of the space of the field variables is non-singular.
- The field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory.
- This approach requires introducing a local metric G in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with G.
- G is actually determined by the kinetic part of the action and its dimension depends on the field content of the latter.

Geometry of field space for pure gravity

For pure gravity theories there is a unique one-parameter family of field-space metrics

$$G_{ab} = G_{AB}\,\delta(x, x')\ ,$$

where

$$G_{AB} = \frac{1}{2}\left(g_{\mu\rho}\,g_{\sigma\nu} + g_{\mu\sigma}\,g_{\rho\nu} + c\,g_{\mu\nu}\,g_{\rho\sigma}\right)$$

called DeWitt super-metric. It involves a dimensionless parameter c.

Following Vilkovisky and DeWitt, we introduce also the Christoffel symbols, covariant derivatives and curvature tensor in the field space.
For the DeWitt functional metric, the Ricci scalar is

$$\mathcal{R} = \frac{n}{4} - \frac{n^2}{8} - \frac{n^3}{8},$$

where n is the dimensionality of the spaceime.
We shall define the functional Kretschmann scalar of the underlying field space $\mathcal{S}$ as

$$\mathcal{K} = \mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}.$$

Rather cumbersome calculations give

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right).$$

This shows that $\mathcal{K}$ is smooth for any spacetime metric g in any spacetime dimension n.
Besides, $\mathcal{K}$ does not depend on the DeWitt parameter c.
Therefore, every theory of pure gravity is free of curvature singularities in the field space $\mathcal{G}$.

Quantum effective action and topological classification of functional singularities

At some field configurations the quantum effective action and the corresponding path integral can become ill-defined. These configurations can correspond to the appearance of the gravitational singularities.

It is somewhat surprising that both the functional Kretschmann scalar and the path-integral measure remain regular in four spacetime dimensions for the DeWitt metric. This suggests that $n = 4$ stands at a special place from the perspective of the geometry of field space.

Let us introduce the functional

$$\psi[\varphi] = e^{i\,\Gamma[\varphi]}.$$

We shall call $\psi[\varphi]$ the functional order parameter because ψ plays the analogous role of an order parameter in the theory of phase transitions in ordered media or cosmology.
The field space $\mathcal{M}$ can be thought of as the ordered medium itself, whereas functional singularities correspond to topological defects.
The functional order parameter ψ defines the map

$$\psi : \mathcal{M} \to \mathbb{S}^1,$$

from the field space to the unit circle, the latter playing the role of the order parameter space.

The singularities can be characterized by the fundamental group (first homotopy group).
Since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$, the homotopy classes are labeled by the winding number $\mathcal{W}$.
A functional singularity exists whenever $\mathcal{W} \neq 0$.

We have considered a flat Friedmann universe filled with a massless scalar field.
There is the singularity of the Big Bang - Big Crunch type.
This singularity can be eliminated by a field reparametrization.
Direct (while tricky) calculation shows that in this case the winding number is equal to zero.

Conclusions

- We have proposed to investigate singularities in the field space rather than in spacetime.
- Existing examples show that certain singularities in spacetime can be removed by field redefinitions albeit being non-removable under change of coordinates.
- Finding field redefinitions that can eliminate singularities is not always feasible in practice.
- The promising approach is to calculate curvature invariants in field space.
- We showed that the Kretschmann scalar of the DeWitt functional metric turns out to be free of singularities.
- The fact that a singularity is removable in field space does not imply that there is no interesting physics occurring around it.

- We have introduced a topological classification of the functional singularities based on the notion of the effective action of the theory.
- We have given an example that the topological triviality confirms the fact that in a particular model with a spacetime (curvature) singularity, one can describe the passage through this singularity, using a field reparametrization.