Teleparallel Scalar-Tensor Gravity Through Cosmological Dynamical Systems and It's Relevance to H0 Tension

Siddheshwar Kadam

Department of Mathematics
BITS-Pilani, Hyderabad Campus, India

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Outline of Presentation

- Teleparallel Equivalent of GR.
- Teleparallel Scalar-Tensor Gravity Through Cosmological Dynamical Systems.
- Scalar-Tensor Gravity Through Cosmological Dynamical Systems and It's Relevance to H0 Tension.
- Results and Discussions.

Teleparallel Equivalent of GR

• Three main cosmological phenomena which GR unables to describe

(i) Dark Matter

(ii) Inflation

(iii) Dark Energy

• The accelerating expansion of the Universe was first discovered in 1998 by the observations of Type Ia supernovae (SNe Ia)[1,2].

• A lot of cosmological observations, such as WMAP [3], SDSS [4], Chandra X-ray Observatory [5] etc., find that our universe is experiencing an accelerated expansion.

• There are essentially two approaches one could take when attempting to solve the dark matter, dark energy and inflation problems, that is either to modify matter part (by adding additional dark matter, dark energy component) or modify geometric part (to study modified theory of gravity).

[1] A.G Riess, *et al.*, *Astrons. J.*, **116**, 1009 (1998).
[2] S. Perlmutter, *Astrophys. J.* , **517**, 565 (1999).
[3] C. L. Bennett *et al.*, *Astrophys. J. Suppl.*, **148**, 1 (2003).
[4] M. Tegmark *et al.*, *Phys. Rev. D*, **69**, 103501 (2004).
[5] S. W. Allen, R. W. Schmidt, *et al.*, *Astron. Soc.*, **353**, 457 (2004).

Teleparallel Equivalent of GR

• To unify electromagnetism and gravitation, the first attempt to modify GR was made by H. Weyl in 1918[6].

• In the late 1920s Einstein himself attempted to unify electromagnetism and gravitation, using the mathematical structure of teleparallelism.

• Even though teleparallel gravity is dynamically completely equivalent to general relativity, it has a very different physical interpretation.

• The modification in the geometrical part leads to several extended theories of gravity such as $f(T)$ gravity [7], $f(T,B)$ gravity [8], $f(T,T_G)$ gravity[9] , $f(T,\phi)$ gravity [10] and so on.

[6] H. Weyl, *Sitzungsber. Preuss. Akad. Wiss. Berlin (Math. Phys.)*,**1918**, 465 (1918).
[7] N. Tamanini, C. G. Bohmer, *Phys. Rev. D*, **86**, 044009 (2012).
[8] S. Bahamonde, C. G. Bohmer, and M. Wright, Phys. Rev. D, **92**,104042 (2015).
[9] G. Kofinas, E. N. Saridakis, *Phys. Rev. D***90**, 084045 (2014).
[10] M. Gonzalez-Espinoza, G. Otalora, *Eur. Phys. J. C.*, **81**, 480 (2021).

Teleparallel scalar-tensor gravity through cosmological dynamical systems

• In spite of their great success, ΛCDM and general relativity (GR) are plagued with many shortcomings. The value of the cosmological constant [11], the nature of dark matter and dark energy etc.

• One of the most important and simplest modifications to the GR was suggested by Brans and Dicke in 1961[12].

• Horndeski gravity is the most general scalar tensor theory[13], its current form, in curved space-time, was given by Deffayet, Deser, and Esposito-Farese.[14]

• The minimally coupled scalar field contributes to the energy density of the Universe in the form of dynamical vacuum energy.

• The major success of scalar field models is their capability to offer a valid alternative explanation of the smallness of the present vacuum energy density [15].

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[11] S. Weinberg, *Rev. Mod. Phys.*, **61**, 1 (1989).
[12] C. Brans and R. H. Dicke, *Phys. Rev.* **124**, 925 (1961).
[13] G. W. Horndeski, *Int. J. Theor. Phys.* **10**, 363 (1974).
[14] C. Deffayet, S. Deser, and G. Esposito-Farese, *Phys. Rev. D*, **80**, 064015 (2009).
[15] E. J. Copeland,M. Sami, S. Tsujikawa, *Dynamics of dark energy*, **arXiv:hep-th/0603057**.

Mathematical Formalism of Teleparallel Scalar-Tensor Gravity:

A flat isotropic and homogeneous Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2)\,, \tag{1}$$

Where $a(t)$ is the scale factor and the tetrad field can be described as follow,

$$e^A_\mu = (1, a(t), a(t), a(t))\,, \tag{2}$$

The Weitzenböck connection can be defined as,

$$\Gamma^\sigma_{\nu\mu} := E^\sigma_A \left(\partial_\mu e^A_\nu + \omega^A_{B\mu} e^B_\nu\right)\,, \tag{3}$$

The torsion tensor can be described as follow,

$$T^\sigma_{\mu\nu} := 2\Gamma^\sigma_{[\nu\mu]}\,, \tag{4}$$

By an appropriate combination of contractions of torsion tensors, a torsion scalar can be written as follow,

$$T := \frac{1}{4} T^\alpha{}_{\mu\nu} T_\alpha{}^{\mu\nu} + \frac{1}{2} T^\alpha{}_{\mu\nu} T^{\nu\mu}{}_\alpha - T^\alpha{}_{\mu\alpha} T^{\beta\mu}{}_\beta\,, \tag{5}$$

Mathematical Formalism of Teleparallel Scalar-Tensor Gravity:

Naturally, the regular Horndeski terms from curvature-based gravity also appear in this framework.

$$\begin{aligned}
\mathcal{L}_2 &:= G_2(\phi, X)\,, \\
\mathcal{L}_3 &:= G_3(\phi, X)\mathring{\Box}\phi\,, \\
\mathcal{L}_4 &:= G_4(\phi, X)\left(-T + B\right) + G_{4,X}(\phi, X)\left((\mathring{\Box}\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right)\,, \\
\mathcal{L}_5 &:= G_5(\phi, X)\mathring{G}_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}(\phi, X)\left((\mathring{\Box}\phi)^3 + 2\phi_{;\mu}\quad\nu\phi_{;\nu}\quad\alpha\phi_{;\alpha}\quad\mu - 3\phi_{;\mu\nu}\phi^{;\mu\nu}\,\mathring{\Box}\phi\right)\,,
\end{aligned} \tag{6}$$

Where the kinetic term is defined as $X := -\frac{1}{2}\partial^\mu\phi\partial_\mu\phi$. BDLS (Bahamonde Dialektopoulos Levi Said) theory simply adds the further Lagrangian component,

$$\mathcal{L}_{\rm Tele} := G_{\rm Tele}\left(\phi, X, T, T_{\rm ax}, T_{\rm vec}, I_2, J_1, J_3, J_5, J_6, J_8, J_{10}\right)\,. \tag{7}$$

This results in the BDLS action given by [16]

$$\mathcal{S}_{\rm BDLS} = \frac{1}{2\kappa^2}\int d^4x\, e\mathcal{L}_{\rm Tele} + \frac{1}{2\kappa^2}\sum_{i=2}^{5}\int d^4x\, e\mathcal{L}_i + \int d^4x\, e\mathcal{L}_{\rm m}\,, \tag{8}$$

[16] S. Bahamonde, K. F. Dialektopoulos, J. L. Said, *Phys. Rev D*, **100**, 064018 (2019).

Mathematical Formalism of Teleparallel Scalar-Tensor Gravity:

In this work, we consider the class of models in which,

$$G_2 = X - V(\phi)\,, \quad G_3 = 0 = G_5\,, \quad G_4 = 1/2\kappa^2\,, \tag{9}$$

We have studied the models as follow,

$$G_{\mathrm{Tele}_1} = X^\alpha T\,, \quad G_{\mathrm{Tele}_2} = X^\alpha I_2\,, \tag{10}$$

The torsion scalar and torsion contraction scalar invariant can be obtained as follow,

$$T = 6H^2\,, \quad I_2 = 3H\dot{\phi}\,. \tag{11}$$

Thus, we can write the effective Friedmann equations as

$$\frac{3}{\kappa^2}H^2 = \rho_{\mathrm{m}} + \rho_{\mathrm{r}} + X + V + 6H\dot{\phi}G_{\mathrm{Tele},I_2} + 12H^2 G_{\mathrm{Tele},T} + 2XG_{\mathrm{Tele},X} - G_{\mathrm{Tele}}\,, \tag{12}$$

$$-\frac{2}{\kappa^2}\dot{H} = \rho_{\mathrm{m}} + \frac{4}{3}\rho_{\mathrm{r}} + 2X + 3H\dot{\phi}G_{\mathrm{Tele},I_2} + 2XG_{\mathrm{Tele},X} - \frac{d}{dt}\left(4HG_{\mathrm{Tele},T} + \dot{\phi}G_{\mathrm{Tele},I_2}\right)\,, \tag{13}$$

The scalar field equation is given by,

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3\dot{\phi}\left(1 + G_{\mathrm{Tele},X}\right)\right] = -V'(\phi) - 9H^2 G_{\mathrm{Tele},I_2} + G_{\mathrm{Tele},\phi} - 3\frac{d}{dt}\left(HG_{\mathrm{Tele},I_2}\right)\,. \tag{14}$$

Model-I General α case :

The action for Model-I [17] is described as below,

$$S = \int d^4xe[P(\phi)X - V(\phi) - \frac{T}{2\kappa^2} + X^\alpha T] + S_m + Sr\,, \tag{15}$$

Following are the Friedmann equations with the equation of motion,

$$-p_r = -V(\phi) + \frac{T}{2\kappa^2} + P(\phi)X - X^\alpha T + \frac{2\dot{H}}{\kappa^2} - 4X^\alpha\dot{H} - 8\alpha X^\alpha H\frac{\ddot{\phi}}{\dot{\phi}}\,, \tag{16}$$

$$\rho_m + \rho_r = \frac{T}{2\kappa^2} - V(\phi) - p(\phi)X - X^\alpha T - 2\alpha X^\alpha T\,, \tag{17}$$

$$0 = V^{'}(\phi) + 3HP(\phi)\dot{\phi} + P^{'}(\phi)X + \frac{2X^\alpha\alpha T}{\dot{\phi}}\left(3H + \frac{2\dot{H}}{H}\right) + \ddot{\phi}[P(\phi) - \alpha X^{\alpha-1}T + \alpha^2X^{\alpha-2}T]\,. \tag{18}$$

[17] S. Bahamonde, K. F. Dialektopoulos, J. L. Said, *Phys. Rev D*, **100**, 064018 (2019).

Model-I General α case :

The equations of pressure and energy density for effective dark energy can be defined as,

$$\rho_{de} = X^\alpha T + 2\alpha X^\alpha T + V(\phi) + P(\phi)X \,,$$
$$p_{de} = -V(\phi) + P(\phi)X - X^\alpha T - 4X^\alpha \dot{H} - \frac{8\alpha X^\alpha H\ddot{\phi}}{\dot{\phi}} \,. \tag{19}$$

In this the set of dimensionless variables are [18],

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H} \,, \quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H} \,, \quad u = 2X^\alpha\kappa^2 \,,$$
$$\rho = \frac{\kappa\sqrt{\rho_r}}{\sqrt{3}H} \,, \quad \lambda = \frac{-V^{'}(\phi)}{\kappa V(\phi)} \,, \quad \Gamma = \frac{V(\phi)V^{''}(\phi)}{V'(\phi)^2} \,, \tag{20}$$

These dimensionless variables satisfy the constraint equation as follow,

$$x^2 + y^2 + \rho^2 + (1+2\alpha)u + \Omega_m = 1 \,. \tag{21}$$

$$\frac{\dot{H}}{H^2} = \frac{x^2\left(\rho^2 + 3(\alpha(2\alpha-5)-1)u - 3y^2 + 3\right) - 2\sqrt{6}\alpha\lambda u x y^2 - \alpha u\left(3(6\alpha-1)u + (2\alpha-1)\left(-\rho^2+3y^2-3\right)\right) + 3x^4}{2(u-1)x^2 - 2\alpha u(2\alpha(u+1)+u-1)} \tag{22}$$

[18] M. Gonzalez-Espinoza, G. Otalora, *Eur. Phys. J. C.*, **81**, 480 (2021).

Dynamical System :

The dynamical system in this case is as follow,

$$
\begin{aligned}
\frac{dx}{dN} &= \frac{x\left(-x^2\left(\rho^2+3(\alpha(2\alpha+5)+1)u-3y^2-3\right)\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} \\
&\quad -\frac{\alpha u x\left(2\alpha\left(\rho^2+3\right)+\rho^2+(6\alpha+3)u-3(2\alpha+1)y^2-3\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} \\
&\quad +\frac{\sqrt{6}\lambda x y^2(2\alpha u+u-1)-3x^4}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)}\,, \\
\frac{dy}{dN} &= \frac{-y\left(x^2\left(\rho^2+\left(6\alpha^2+9\alpha-3\right)u-3y^2+3\right)-2\sqrt{6}\alpha\lambda u x y^2+3x^4\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} \\
&\quad +\frac{\alpha u(-y)\left((6\alpha+3)u-(2\alpha-1)\left(-\rho^2+3y^2-3\right)\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)}-y\sqrt{\frac{3}{2}}\lambda x\,, \\
\frac{du}{dN} &= \frac{\alpha u\left(2\alpha u\left(\rho^2+3x^2-3y^2\right)+(u-1)x\left(6x-\sqrt{6}\lambda y^2\right)\right)}{\alpha u(2\alpha(u+1)+u-1)-(u-1)x^2}\,, \\
\frac{d\rho}{dN} &= \frac{\rho\left(-x^2\left(\rho^2+6\alpha^2 u+9\alpha u+u-3y^2-1\right)+2\sqrt{6}\alpha\lambda u x y^2-3x^4\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)} \\
&\quad +\frac{\alpha\rho u\left(2\alpha u+u+(2\alpha-1)\left(-\rho^2+3y^2+1\right)\right)}{2(u-1)x^2-2\alpha u(2\alpha(u+1)+u-1)}\,, \\
\frac{d\lambda}{dN} &= \sqrt{6}(\Gamma-1)\lambda^2 x\,.
\end{aligned}
\tag{23}
$$

We will now on wards focus on the exponential potential $V(\phi)=V_0 e^{-\lambda\kappa\phi}$.

Critical Points:

Table 1: Critical Points for Dynamical System Corresponding to Model-I, for General α.

Critical Point	x_c	y_c	u_c	ρ_c	(q)	ω_{tot}
A, $2\alpha^2\tau^2 + \alpha\tau^2 + 2\alpha^2\tau - \alpha\tau \neq 0$	0	0	τ	0	$\frac{1}{2}$	0
B	1	0	0	0	2	1
C	-1	0	0	0	2	1
D, in this case $\lambda = 0$	$\zeta, (\alpha-1)^2\alpha\zeta \neq \left(\alpha^2-1\right)\zeta^3$	$\frac{\sqrt{(\alpha+1)\zeta^2+\alpha}}{\sqrt{\alpha}}$	$-\frac{\zeta^2}{\alpha}$	0	-1	-1
E, in this case $\lambda = 0$	$\zeta, (\alpha-1)^2\alpha\zeta \neq \left(\alpha^2-1\right)\zeta^3$	- $\frac{\sqrt{(\alpha+1)\zeta^2+\alpha}}{\sqrt{\alpha}}$	$-\frac{\zeta^2}{\alpha}$	0	-1	-1
F	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\lambda^2}}$	0	0	$\frac{1}{2}$	0
G	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$-\sqrt{\frac{3}{2}}\sqrt{\frac{1}{\lambda^2}}$	0	0	$\frac{1}{2}$	0
H	$\frac{\lambda}{\sqrt{6}}$	$\sqrt{1-\frac{\lambda^2}{6}}$	0	0	$\frac{1}{2}\left(\lambda^2-2\right)$	$-1+\frac{\lambda^2}{3}$
I	$\frac{\lambda}{\sqrt{6}}$	$-\sqrt{1-\frac{\lambda^2}{6}}$	0	0	$\frac{1}{2}\left(\lambda^2-2\right)$	$-1+\frac{\lambda^2}{3}$
J, in this case $\lambda = 2$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{1}{3}}$	0	0	1	$\frac{1}{3}$
K, in this case $\lambda = 2$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{1}{3}}$	0	0	1	$\frac{1}{3}$
L	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\chi, \chi - 1 \neq 0$	0	$\frac{1}{2}$	0
M	$\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$-\frac{\sqrt{\frac{3}{2}}}{\lambda}$	$\chi, \chi - 1 \neq 0$	0	$\frac{1}{2}$	0
$N, \alpha = 0$	$\delta, \delta \neq 0$	0	$1-\delta^2$	0	2	1

Stability of Critical Points:

Table 2: Eigenvalues and Stability of Eigenvalue at Corresponding Critical Points Corresponding to Model-I General α

Name of Critical Point	Corresponding Eigenvalues	Stability
A	$\{\frac{3}{2}, \frac{3}{2}, -\frac{1}{2}, 0\}$	Unstable
B	$\left\{3, 1, -6\alpha, \frac{1}{2}\left(6-\sqrt{6}\lambda\right)\right\}$	Unstable
C	$\left\{3, 1, -6\alpha, \frac{1}{2}\left(6+\sqrt{6}\lambda\right)\right\}$	Unstable
D	$\{0, -3, -3, -2\}$	Stable
E	$\{0, -3, -3, -2\}$	Stable
F	$\left\{-\frac{1}{2}, -3\alpha, \frac{3\left(-\lambda^2-\sqrt{24\lambda^2-7\lambda^4}\right)}{4\lambda^2}, \frac{3\left(\sqrt{24\lambda^2-7\lambda^4}-\lambda^2\right)}{4\lambda^2}\right\}$	Stable for $\alpha > 0$ $\wedge \left(-2\sqrt{\frac{6}{7}} \leq \lambda < -\sqrt{3} \vee \sqrt{3} < \lambda \leq 2\sqrt{\frac{6}{7}}\right)$
G	$\left\{-\frac{1}{2}, -3\alpha, \frac{3\left(-\lambda^2-\sqrt{24\lambda^2-7\lambda^4}\right)}{4\lambda^2}, \frac{3\left(\sqrt{24\lambda^2-7\lambda^4}-\lambda^2\right)}{4\lambda^2}\right\}$	Stable for $\alpha > 0$ $\wedge \left(-2\sqrt{\frac{6}{7}} \leq \lambda < -\sqrt{3} \vee \sqrt{3} < \lambda \leq 2\sqrt{\frac{6}{7}}\right)$
H	$\left\{-\alpha\lambda^2, \frac{1}{2}\left(\lambda^2-6\right), \frac{1}{2}\left(\lambda^2-4\right), \lambda^2-3\right\}$	Stable for $\alpha > 0 \wedge \left(-\sqrt{3} < \lambda < 0 \vee 0 < \lambda < \sqrt{3}\right)$
I	$\left\{-\alpha\lambda^2, \frac{1}{2}\left(\lambda^2-6\right), \frac{1}{2}\left(\lambda^2-4\right), \lambda^2-3\right\}$	Stable for $\alpha > 0 \wedge \left(-\sqrt{3} < \lambda < 0 \vee 0 < \lambda < \sqrt{3}\right)$
J	$\{-1, 1, 0, -4\alpha\}$	Unstable
K	$\{-1, 1, 0, -4\alpha\}$	Unstable
L	$\left\{0, -\frac{1}{2}, \frac{3}{4}\left(\frac{\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{\lambda^2(\chi-1)}-1\right), -\frac{3\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{4\lambda^2(\chi-1)}-\frac{3}{4}\right\}$	Stable for $\lambda \in \mathbb{R} \wedge \lambda \neq 0 \wedge \frac{7\lambda^2-24}{7\lambda^2} \leq \chi < \frac{\lambda^2-3}{\lambda^2}$
M	$\left\{0, -\frac{1}{2}, \frac{3}{4}\left(\frac{\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{\lambda^2(\chi-1)}-1\right), -\frac{3\sqrt{-\lambda^2(\chi-1)\left(7\lambda^2(\chi-1)+24\right)}}{4\lambda^2(\chi-1)}-\frac{3}{4}\right\}$	Stable for $\lambda \in \mathbb{R} \wedge \lambda \neq 0 \wedge \frac{7\lambda^2-24}{7\lambda^2} \leq \chi < \frac{\lambda^2-3}{\lambda^2}$
N	$\left\{0, 1, 3, \frac{1}{2}\left(6-\sqrt{6}\delta\lambda\right)\right\}$	Unstable

Phase Portrait Diagrams:

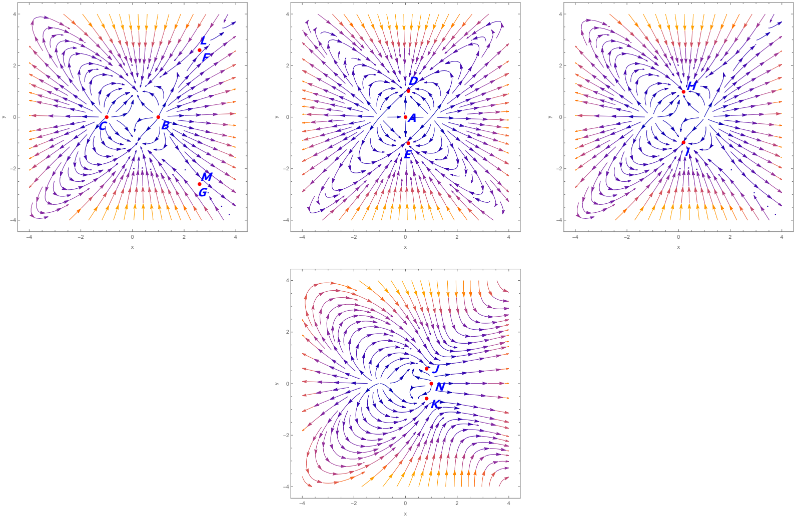

Figure 1: For upper left plot, $u = 0$, $\rho = 0$, $\lambda = \sqrt{\frac{2}{9}}$. The upper middle plot is for $u = 0$, $\rho = 0$, $\zeta = \frac{1}{9}$, $\tau = 1$, $\alpha = 1$, upper left stream plot $u = 0$, $\rho = 0$, $\lambda = \sqrt{\frac{2}{9}}$, lower stream plot $u = 0$, $\rho = 0$, $\delta = 1$.

Argument for further study :

- The proposed autonomous systems can be restrictive over a dynamical variable, which may contains information related to the dynamics on the H0 value [19].
- We wish to describe the treatment of our non-linear autonomous system by the study of hyperbolic as well as nonhyperbolic critical points and can discuss an interesting phenomenological feature in regards to H0 tension.
- To study H0 tension, we will employed a H(z) observational sample from Cosmic Chronometers plus BAO estimates.

[19] G. A. Rave-Franco, C. Escamilla-Rivera, J. L. Said, *Phys. Rev. D*, **103**, 084017, (2021).

Results and Discussions

- In this work we have explored the cosmological dynamics of dark energy through the prism of scalar-torsion gravity in the context of power-law couplings of torsion scalar and the kinetic term. Assuming a flat homogeneous and isotropic background solution, dynamical systems are analysed to determine the number and nature of the critical points.

- The Friedmann equations and the Klein-Gordon equation directly lead to a set of autonomous equations for each of the models under investigation. These are then used in each case to derive the critical points of the particular cosmologies from which we can expose the model behaviour using the dynamical analysis in the parameter phase space. The stability of critical points using the eigenvalues of the Jacobean matrix of the system is used to express whether these positions in the cosmic evolution are stable or not.

- In this model we utilize the dynamical variables together with the equations of motion, are then used to derive the system of autonomous equations which express the behaviour of the model in phase space. These first order equations of motion of the dynamical variables are represented as derivatives with respect to $N = \ln a$, which shows the behaviour of the system in a more direct way.

Publication details

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Thank you