

The Effective Theory of Gravity and Dynamical Vacuum Energy

Emil Mottola*

Department of Physics and Astronomy,

University of New Mexico

Albuquerque NM 87131

Gravity and general relativity are considered as an Effective Field Theory (EFT) at low energies and macroscopic distances. The effective action of the conformal trace anomaly of light or massless quantum fields has significant effects on macroscopic scales, due to associated light cone singularities that are not captured by an expansion in local curvature invariants. A compact local form for the Wess-Zumino effective action of the conformal anomaly and stress tensor is given, requiring the introduction of a new light scalar field, which it is argued should be included in the low energy effective action for gravity. This scalar *conformalon* couples to the conformal part of the spacetime metric and allows the effective value of the vacuum energy, described as a condensate of a 4-form abelian gauge field, to change in space and time. This is achieved by the extension of the conformal anomaly to a general Einstein-Cartan space and the identification of the vacuum energy with a certain torsional contribution to the fermion anomaly. The resulting EFT of vacuum energy replaces the fixed constant Λ of classical gravity, and its apparently unnaturally large sensitivity to UV physics, with a dynamical condensate whose ground state value in empty flat space is $\Lambda_{\text{eff}} = 0$ identically. By allowing Λ_{eff} to vary rapidly near a black hole horizon, the proposed EFT of dynamical vacuum energy also provides an effective Lagrangian framework for gravitational condensate stars, as the final state of complete gravitational collapse consistent with quantum theory. The possible consequences of dynamical vacuum dark energy for cosmology, the cosmic coincidence problem, and the role of conformal invariance for other fine tuning issues in the Standard Model are discussed.

* mottola.emil@gmail.com, emottola@unm.edu

Contents

The Quantum Vacuum and Vacuum Energy in Black Holes and Cosmology	2
Relevance of the Conformal Anomaly to Macroscopic Gravity	4
The Effective Action of the Conformal Anomaly and Conformal Scalar	8
The Cosmological Term as a 4-Form Gauge Field	13
The Trace Anomaly for Fermions: Torsion and the Source for F	16
The Effective Theory of Gravity in the Absence of Massless Fermions	20
The Effective Theory of Gravity in the Presence of Massless Fermions	23
Gravitational Vacuum Condensate Stars	26
Discussion and Outlook: Vacuum Energy as a Dynamical Condensate	30
References	35
Metrics, Tetrads, Differential Forms, and Hodge Star Dual	39
The Chern-Simons 3-Form and Topological Current of the Anomaly	41
Topological and Torsional Susceptibility of the Gravitational Vacuum	44

I. The Quantum Vacuum and Vacuum Energy in Black Holes and Cosmology

Observations of type Ia supernovae (SN) at moderately large redshifts indicate that the expansion of the universe is accelerating [1, 2]. This is possible in classical general relativity (GR) only if the dominant energy of the universe has an effective mean eq. of state satisfying $\rho + 3p < 0$, *i.e.* assuming positive energy density $\rho > 0$, it must have negative pressure. As the cosmological term Λ enters Einstein's eqs. as a constant with $p_\Lambda = -\rho_\Lambda = -\Lambda/8\pi G_N$ pervading all space, the SN observations taken at face value imply a Λ value of [3]

$$\Lambda_{SN} = \Omega_\Lambda \times 3 \left(\frac{H_0}{c} \right)^2 \simeq \left(\frac{\Omega_\Lambda}{0.70} \right) \left(\frac{H_0}{70 \text{ km/sec/Mpc}} \right)^2 \left(\frac{3.1 \times 10^{-122}}{L_{Pl}^2} \right) \quad (1.1)$$

when expressed in terms of the present Hubble expansion rate of $H_0 \simeq 70$ km/sec/Mpc, or the microscopic Planck length $L_{Pl} = \sqrt{\hbar G_N/c^3} \simeq 1.616 \times 10^{-33}$ cm. respectively. Thus some $\Omega_\Lambda \simeq 70\%$ of the energy in the present universe is in the form of Λ dark energy, and is the principal component of the current Λ CDM model of cosmology.

The contrast in (1.1) between the dimensionless value of the cosmological term Ω_Λ , of order unity in cosmological Hubble units, but of order 10^{-122} in microscopic Planck units, is striking. From the time of W. Pauli it has been thought that Λ is related to the zero-point energy density of the vacuum in quantum field theory (QFT), in which it appears as an ultraviolet (UV) divergent sum over all field modes [4–7]. If Λ is such a UV sensitive quantity, and the short distance cutoff is of order of L_{Pl} , then the value of Λ_{SN} in Planck units represents the most severe scale hierarchy problem in all of physics, clashing with expectations of ‘naturalness’ developed over several decades of successful application of Effective Field Theory (EFT) methods [8–10].

On the other hand, if one adopts the EFT hypothesis that macroscopic gravity and the value of Λ at cosmological scales should be decoupled from and not require detailed knowledge of extreme UV physics, then (1.1) suggests instead that the EFT of gravity is incomplete at *low energies*, and one or more additional EFT degrees of freedom are needed to account for a vacuum energy naturally of order of the Hubble scale.

If the EFT of gravity relates Λ_{eff} to the cosmological Hubble scale $3H_0^2$ rather than the microscopic Planck scale L_{Pl}^{-2} , the further implication is that Λ_{eff} would have to become a *dynamical* quantity, *i.e.* dependent upon the content and evolution of the universe, as the Hubble ‘constant’ itself is [6]. Related to both possibilities of additional low energy gravitational degrees of freedom other than the

metric of classical GR and of Λ_{eff} becoming dynamical as a result, it is worth noting that current cosmological models already require at least one additional scalar (inflaton) field of unknown origin, in order to generate the present small ($\sim 10^{-5}$) CMB anisotropies during a very early epoch of cosmic inflation [11]. This epoch is assumed to have been dominated by a much larger effective Λ_{eff} vacuum energy, that is supposed to have dynamically ‘relaxed’ to its present much smaller value.

Indications that the EFT of gravity may require some additional degree(s) of freedom relevant at macroscopic scales come also from the quite different domain of black hole (BH) physics. EFT methods in gravity for BHs have been put into question by both the extreme blueshifting of energy scales in the presence of horizons, invalidating the EFT assumption of decoupling of short distance from long distance physics [12], and by the BH ‘information paradox’ [13–20], and apparent conflict with unitary evolution it entails [21]. The various forms of this paradox arise from ascribing an enormous entropy to a BH, equal to 1/4 its horizon area A_H in Planck units,

$$S_{BH} = k_B \frac{A_H}{4 L_{Pl}^2} \simeq 1.1 \times 10^{77} k_B \left(\frac{M}{M_\odot} \right)^2 \quad (1.2)$$

despite the assumed classical nature of the BH horizon as a mathematical causal boundary only, with no independent degrees of freedom of its own. Significant quantum effects on the macroscopic scale of the BH horizon are also in apparent conflict with the usual EFT approach to gravity, which relies on an expansion in local curvature invariants [22–24], since these can yield only negligibly small corrections for large BHs with small local curvatures at their horizons. In fact, the enormous entropy (1.2) and Hawking effect upon which it is predicated rely crucially upon the specification of the quantum vacuum state, which (as always in quantum theory), requires *non-local* boundary conditions, that are not determined just by the local curvature [25–27].

In previous work it has been noted that large quantum effects on the horizon follow quite generally from the stress tensor of the conformal anomaly in both BH and cosmological spacetimes [28–30]. The importance of the conformal anomaly in the near horizon behavior of the stress tensor is a consequence of the conformal scaling behavior of the metric near the horizon and the extreme blueshifting of local frequencies, which renders all finite mass scales irrelevant there [30]. It is just this extreme blueshifting of frequencies, hence energies, that can lead to effects not accounted for in local EFT expansions based on the assumption of decoupling and strict separation of scales. State dependent quantum vacuum entanglement and polarization effects are contained in the effective action of the conformal anomaly, which by its nature spans multiple scales.

That the physics of BHs and vacuum energy are related is inherent also in the resolution of the BH information paradox by the formation of a gravitational vacuum condensate star with interior Λ_{eff} eq. of state $p = -\rho$ [31, 32]. This Λ_{eff} eq. of state in the interior of a *gravastar*, collapsed to its gravitational radius $r_M = 2GM/c^2$ prevents further collapse to a BH singularity for the same reason that it causes the Hubble expansion of the universe to accelerate, namely by defocusing (rather than the usual focusing) of worldline geodesics, avoiding the classical singularity theorems [33–35]. The localized formation of such a $p = -\rho$ gravitational vacuum condensate within an ultra compact star can occur only if there is at least one additional degree of freedom in the low energy EFT of gravity, whose variation allows Λ_{eff} to change abruptly at or near $r = r_M$. This turns the BH horizon from a mathematical surface to a physical phase boundary layer with a positive surface tension [36–38].

The purpose of this paper is to propose and develop the EFT of low energy gravity, deduced from general principles of QFT in curved space and the conformal anomaly, in which finite dynamical vacuum energy is consistently described as a scalar vacuum condensate. The description of vacuum energy by the scalar dual to a 4-form abelian field strength requires $\Lambda_{\text{eff}} \geq 0$ with $\Lambda_{\text{eff}} = 0$ the unique value of lowest energy in flat space, independently of UV physics. When the 4-form field strength is identified with a certain function of torsion dictated by the conformal anomaly of massless (or light) fermions in a general Einstein-Cartan space in Sec. V, it and hence Λ_{eff} become spacetime dependent. The observational implications of this EFT extension of classical GR and identification of the relevant low energy degrees of freedom describing Λ_{eff} as a dynamical gravitational vacuum condensate for both BH physics and cosmological vacuum dark energy can then be discussed.

The metric and curvature conventions of the paper are those of MTW [39], while the definitions and conventions for tetrads and differential forms used are reviewed in Appendix A. Two additional appendices B and C are devoted to the topological aspects of the Euler density, associated Chern-Simons 3-form and physical interpretation of the new constant \varkappa introduced in the EFT, as a topological or torsional susceptibility of the gravitational vacuum.

II. Relevance of the Conformal Anomaly to Macroscopic Gravity

If one takes as the basic building block of a gravitational theory the spacetime metric $g_{\mu\nu}(x)$, with the requirements that the field eqs. must transform as tensor eqs. under general coordinate transformations, and be no higher than second order in derivatives of the metric, one arrives at the classical

theory of general relativity (GR). This is described by the classical action

$$S_{cl} = S_{EH} - \frac{\Lambda}{8\pi G_N} \int \sqrt{-g} d^4x = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} (R - 2\Lambda) \quad (2.1)$$

namely the Einstein-Hilbert (E-H) action involving the Ricci curvature scalar R , second order in derivatives of the metric or first order in derivatives of the symmetric Christoffel connection

$$\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} = \frac{1}{2} g^{\lambda\rho} (-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\nu\rho} + \partial_\nu g_{\mu\rho}), \quad (2.2)$$

together with the cosmological constant term Λ , involving no metric derivatives. In modern terms the requirement of an action S_{cl} composed of sums of all integrals of scalars that are invariant under general coordinate transformations (up to possible surface terms) and yield field eqs. no higher than second derivatives of the metric is just what is meant by a low energy EFT of gravity, since local invariant terms higher order in derivatives of the metric are negligible at low energies or long wavelengths. In (2.1) the constants G_N and Λ can be determined only by experiment or astronomical observations. At this purely classical level, if Λ is given by observations to be (1.1), there is no naturalness problem, for there is no other scale in the classical theory to which it can be compared.

As É. Cartan pointed out soon after the appearance of GR [40, 41], the most general setting of an affine geometry and differential manifold allows also for a non-zero torsion, which is described by an anti-symmetric part of the connection $\Gamma^\lambda_{[\mu\nu]} \neq 0$ that Einstein had assumed to be vanishing, as the simplest realization of the Equivalence Principle. An anti-symmetric part of $\Gamma^\lambda_{\mu\nu}$ drops out of the geodesic eq. for the worldlines of freely falling point particles in any case. For particles with internal spin, half-integral spin representations of the Lorentz group and in particular the quantum conformal anomaly of fermion fields, the contribution of torsion becomes relevant, as will be explained in Sec. V.

The problems of reconciling classical GR with QFT first appear by treating the stress-energy source $T^{\mu\nu}$ for Einstein's eqs. as completely classical, whereas $\hat{T}^{\mu\nu}$ is a UV divergent operator in QFT. Since matter and radiation in the Standard Model (SM) are certainly quantum in nature, replacing the quantum operator $\hat{T}^{\mu\nu}$ by its renormalized expectation value $\langle \hat{T}^{\mu\nu} \rangle$ in a semi-classical approximation amounts to two different assumptions, which should be recognized and distinguished at the outset.

The first assumption is that UV divergences of QFT are to be removed by counterterms involving up to dimension-four curvature invariants, such as $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$, $R_{\alpha\beta}R^{\alpha\beta}$, R^2 , by adding them to the effective action with finite renormalized coefficients, leaving the low energy EFT unchanged. These terms

$\sim (\partial^2 g_{\mu\nu})^2$, involve up to four derivatives of the metric, in contrast to the dimension-two E-H action. Since quantum theory introduces a new scale L_{Pl} , this standard renormalization procedure amounts to the assumption that such higher derivative terms may be important only on the corresponding Planck energy scale of $M_{Pl}c^2 = 1.221 \times 10^{19}$ GeV. Since this scale is so much higher than those generally encountered either in terrestrial accelerators or astrophysics, the reasonable assumption of an EFT approach is that Planck scale physics decouples from the low energy EFT, so that knowledge of the UV completion or full quantum theory is not needed to describe macroscopic gravitation.

Less often noted is a second critical assumption in the replacement of the stress tensor source in Einstein's eqs. by a finite renormalized $\langle \hat{T}^{\mu\nu} \rangle$, treated classically, namely that the operator $\hat{T}^{\mu\nu}$ is well approximated by its sharply peaked mean value. It is easily verified in QFT that even after renormalization there are quantum fluctuations from the mean, and *e.g.* $\langle \hat{T}^{\alpha\beta}(x) \hat{T}^{\mu\nu}(y) \rangle - \langle \hat{T}^{\alpha\beta}(x) \rangle \langle \hat{T}^{\mu\nu}(y) \rangle \neq 0$ at one-loop order [42]. Connected higher point correlation functions of this kind probe the polarization and entanglement properties of the quantum vacuum even at macroscopic scales. These quantum correlators exhibit operator product singularities as $x \rightarrow y$. All is well with these UV singularities since dimensional analysis, as well as explicit calculations show that these quantum correlations grow large as L_{Pl}^2/ℓ^2 relative to $\langle \hat{T}^{\mu\nu} \rangle$ only for metric variations on the length scale $\ell \lesssim L_{Pl}$. Neglecting these short distance correlations at scales $\ell \gg L_{Pl}$ reduces then to the first decoupling assumption of EFT.

There is however a second kinematic regime where the Lorentz invariant distance $(x - y)^2 \rightarrow 0$, even for $x \neq y$, and where two and higher point quantum correlation functions of $\hat{T}^{\mu\nu}$ can become large, namely on the light cone. Since light cones extend over arbitrarily large distances, lightlike correlations are *not* limited to the ultrashort L_{Pl} , but can lead to *macroscopic* quantum effects, in particular on null horizons [28], which can be relevant in both BH and cosmological spacetimes with positive Λ , such as de Sitter space. The two very different sorts of quantum effects, short distance UV vs. macroscopic lightlike correlations are distinct and require two quite different EFT treatments.

The short distance UV quantum corrections to GR are taken into account by adding to the action (2.1) of classical GR the expansion in ascending powers of higher derivatives of local invariants, divided by appropriate powers of the UV cutoff scale, expected to be the Planck scale M_{Pl} for gravity. This is the most common EFT approach [22–24]. As already mentioned, it amounts to the decoupling assumption common to all EFT approaches, based on the decoupling theorem of massive states in the UV from the low energy degrees of freedom [43].

On the other hand it has also been known for some time that QFT anomalies are not captured by such an expansion in higher order local invariants, nor are they suppressed by any UV scale. Anoma-

lies are associated instead with the fluctuations of massless fields which do not decouple, and which lead to $1/k^2$ poles in momentum space correlation functions, that grow large on the light cone $k^2 \rightarrow 0$ rather than the extreme UV regime $k^2 \sim M_{Pl}^2$. The prototype of this light cone pole is the Schwinger model of 1+1 dimensional massless electrodynamics and its chiral anomaly [44], which extends to the two-dimensional conformal anomaly in curved space [45, 46]. In 3+1 dimensional flat space, light cone poles are found in explicit calculations in the triangle anomaly diagrams of $\langle \hat{J}_5^\lambda \hat{J}^\alpha \hat{J}^\beta \rangle$, $\langle \hat{T}^{\mu\nu} \hat{J}^\alpha \hat{J}^\beta \rangle$ in massless QED₄ [29, 47], and in the stress tensor three-point correlator $\langle \hat{T}^{\alpha\beta} \hat{T}^{\gamma\lambda} \hat{T}^{\mu\nu} \rangle$ of a general conformal field theory (CFT), by solution of the conformal Ward Identities in momentum space [48, 49].

Quite contrary to the decoupling hypothesis, quantum anomalies lead instead to the principle of *anomaly matching* from UV to low energy EFT [50]. In the strong interactions, the chiral anomaly of the UV theory, QCD, survives to low energies, requiring a specific Wess-Zumino (WZ) addition to the low energy meson EFT [8, 51], which is not suppressed by any high energy scale, and is technically a marginally relevant operator in the infrared (IR). Indeed, as befits being associated with light cone singularities, the chiral anomaly has both UV and IR features. The successful prediction of the low energy $\pi^0 \rightarrow 2\gamma$ decay rate provides a window into the UV and evidence for the $SU(3)^{color}$ group and fractional charge assignments of quarks that helped establish QCD as the UV theory of the strong interactions [52–54].

For gravitational theory it is the conformal anomaly in the trace of the energy-momentum-stress tensor $\hat{T}^{\mu\nu}$ that is associated with $1/k^2$ anomaly poles in stress tensor correlators, such as $\langle \hat{T}^{\alpha\beta} \hat{T}^{\gamma\lambda} \hat{T}^{\mu\nu} \rangle$, implying the propagation of at least one additional light (*a priori* massless) degree of freedom in the low-energy EFT of macroscopic gravity, not accounted for in the classical action of GR (2.1), or an expansion in higher order local invariants.

A representative of the light cone singularities and massless pole associated with conformal anomalies is afforded by the effective action of two-dimensional gravity coupled to conformal matter [45, 55]

$$S_{\text{anom}}^{\text{NL}, 2\text{D}}[g] = -\frac{c_m}{96\pi} \int d^2x \sqrt{-g(x)} \int d^2y \sqrt{-g(y)} R(x) (\square^{-1})_{xy} R(y) \quad (2.3)$$

where $c_m = N_s + N_f$ is the central charge, given by the sum of the numbers of massless scalar and fermion fields, and $(\square^{-1})_{xy}$ denotes the Green's function inverse of the scalar wave operator. This exhibits the light cone pole of the 2D conformal anomaly, appearing already in the connected two-point function $\langle \hat{T}^{\alpha\beta}(x) \hat{T}^{\mu\nu}(y) \rangle$ of the underlying CFT. The massless scalar pole in (2.3) indicates that there is an additional scalar degree of freedom in 2D gravity coupled to conformal matter. This scalar

degree of freedom can be made explicit by expressing the non-local anomaly effective action (2.3) in the local form

$$S_{\mathcal{A}}^{2D}[g; \varphi] = -\frac{c_m}{96\pi} \int d^2x \sqrt{-g} \{g^{\mu\nu}(\nabla_\mu \varphi)(\nabla_\nu \varphi) - 2R\varphi\} \quad (2.4)$$

by the introduction of the scalar field φ describing a collective spin-0 degree of freedom, which is linearly coupled to R , and whose massless propagator gives rise to the light cone singularities of the underlying massless CFT [46]. Variation of (2.4) with respect to φ gives its eq. of motion $-\square\varphi = R$, which when solved for $\varphi = -\square^{-1}R$ and substituted back into (2.4) returns the non-local form of the effective action (2.3). At the same time variation of (2.4) with respect to the metric $g_{\mu\nu}$ yields the stress tensor $T_{\mathcal{A}}^{\mu\nu}[g; \varphi]$ whose trace is $-(c_m/24\pi)\square\varphi = c_m R/24\pi$ which is the 2D conformal anomaly. Taking two variations with respect to $g_{\alpha\beta}(x)$ and $g_{\mu\nu}(y)$ yields the connected two-point CFT correlator $\langle \hat{T}^{\alpha\beta}(x)\hat{T}^{\mu\nu}(y) \rangle$ which exhibits a massless $1/k^2$ light cone pole in flat space [46]. Clearly such a massless scalar degree of freedom affects the macroscopic behavior of 2D gravity [56], but cannot be described by a local action in curvature invariants alone, as is the very nature of an anomaly.

III. The Effective Action of the Conformal Anomaly and Conformal Scalar

The anomalous Ward identities for all higher point quantum correlation functions of the stress tensor containing anomalous light cone singularities can be derived by functional variation of the basic one-point expectation value of $\langle \hat{T}^{\mu\nu} \rangle$ in a general curved space background [49]. In $D = 4$ this mean value, defined and renormalized by any method that preserves its covariant conservation, results in it acquiring an anomalous trace in background gravitational and gauge fields, the general form of which is [57–59]

$$\langle \hat{T}^\mu{}_\mu \rangle \equiv g_{\mu\nu} \langle \hat{T}^{\mu\nu} \rangle = b C^2 + b' \left(E - \frac{2}{3} \square R \right) + \sum_i \beta_i \mathcal{L}_i \quad (3.1)$$

even if the underlying QFT is conformally invariant at tree level, and one might have expected this trace to vanish. In (3.1)

$$E = R_{\alpha\beta\gamma\lambda} R^{\alpha\beta\gamma\lambda} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2, \quad C^2 = R_{\alpha\beta\gamma\lambda} R^{\alpha\beta\gamma\lambda} - 2R_{\alpha\beta} R^{\alpha\beta} + \frac{1}{3} R^2 \quad (3.2)$$

are the Euler-Gauss-Bonnet invariant and the square of the Weyl conformal tensor respectively. The b, b', β_i coefficients in (3.1) are finite dimensionless coefficients (in units of \hbar) that depend only upon the number and spin of the massless conformal fields contributing to the anomaly, *i.e.*

$$b = \frac{1}{(4\pi)^2} \frac{1}{120} (N_s + 6N_f + 12N_v), \quad b' = -\frac{1}{(4\pi)^2} \frac{1}{360} (N_s + 11N_f + 62N_v) \quad (3.3)$$

where (N_s, N_f, N_v) are the number of free conformal scalar, Dirac fermion, and gauge vector fields respectively. Interactions of the massless or light QFT degrees of freedom are taken into account by the \mathcal{L}_i terms in (3.1), which denote invariant Lagrangians to which these fields are coupled, such as $\mathcal{L}_F = F_{\alpha\beta}F^{\alpha\beta}$ for light charged particles coupled to electromagnetism, or $\mathcal{L}_G = \text{tr}\{G_{\alpha\beta}G^{\alpha\beta}\}$ for light quarks coupled to the $SU(3)^{color}$ gluonic gauge fields of QCD. The β_i are proportional to the β -functions of these couplings.

All terms in (3.1) are dependent only upon the low energy QFT particle content, independently of a UV cutoff or the Planck scale. They are therefore independent of UV physics and with fixed matter content cannot be removed in any metric theory of gravity with a covariantly conserved stress tensor. Successive variations of (3.1) with respect to the arbitrary metric background yield the anomalous trace Ward identities that the stress tensor correlators must satisfy, even in flat space. Note that these variations are independent of the purely local counterterms needed to define a renormalized $\langle \hat{T}^\mu{}_\mu \rangle$.

To construct the effective action corresponding to (3.1) one notes that upon multiplying (3.1) by $\sqrt{-g}$, the various terms transform as

$$\sqrt{-g} C^2 \rightarrow \sqrt{-g} C^2 \quad (3.4a)$$

$$\sqrt{-g} \mathcal{L}_i \rightarrow \sqrt{-g} \mathcal{L}_i \quad (3.4b)$$

$$\sqrt{-g} \left(E - \frac{2}{3} \square R \right) \rightarrow \sqrt{-g} \left(E - \frac{2}{3} \square R \right) + 4 \sqrt{-g} \Delta_4 \sigma \quad (3.4c)$$

under the conformal variation of the metric $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$. Here

$$\Delta_4 \equiv \nabla_\mu \left(\nabla^\mu \nabla^\nu + 2R^{\mu\nu} - \frac{2}{3} R g^{\mu\nu} \right) \nabla_\nu = \square^2 + 2R^{\mu\nu} \nabla_\mu \nabla_\nu - \frac{2}{3} R \square + \frac{1}{3} (\nabla^\mu R) \nabla_\mu \quad (3.5)$$

is the (unique) fourth order scalar differential operator that is conformally covariant [60, 61]

$$\sqrt{-g} \Delta_4 \rightarrow \sqrt{-g} \Delta_4 \quad (3.6)$$

for arbitrary $\sigma(x)$. It is thus the four-dimensional analog of the second order wave operator \square which has the conformal property $\sqrt{-g} \square \rightarrow \sqrt{-g} \square$ in $D=2$.

Using these properties under conformal variations and the fact that the anomaly density

$$\mathcal{A} \equiv \sqrt{-g} \langle \hat{T}^\mu{}_\mu \rangle = \frac{\delta \mathcal{S}_{\text{anom}}}{\delta \sigma} = 2 g_{\mu\nu} \frac{\delta \mathcal{S}_{\text{anom}}}{\delta g_{\mu\nu}} \quad (3.7)$$

is the conformal variation of an effective action of the anomaly, it is straightforward to find an action

satisfying (3.7). The non-local form of this anomaly effective action is [61–64]

$$S_{\text{anom}}^{NL}[g] = \frac{1}{4} \int d^4x \sqrt{-g_x} \left(E - \frac{2}{3} \square R \right)_x \int d^4y \sqrt{-g_y} (\Delta_4^{-1})_{xy} \left\{ \frac{b'}{2} \left(E - \frac{2}{3} \square R \right) + b C^2 + \sum_i \beta_i \mathcal{L}_i \right\}_y \quad (3.8)$$

where $(\Delta_4^{-1})_{xy}$ denotes the Green's function inverse of the fourth order differential operator (3.5) between the spacetime points x and y , indicated by subscripts in (3.8), so that $\int d^4x (\sqrt{-g} \Delta_4)_x (\Delta_4^{-1})_{xy} = 1$, for all y . The non-local effective action (3.8) of the conformal anomaly in $D = 4$ is the analog of non-local effective action (2.3) in $D = 2$ spacetime dimensions. Note that due to the appearance of the fourth order curvature invariants in (3.1) and (3.8) of $D = 4$, a single light cone $1/k^2$ pole of the anomaly first appears for *three* variations of (3.8) with respect to the metric, in the 3-point correlation function $\langle \hat{T}^{\alpha\beta} \hat{T}^{\gamma\lambda} \hat{T}^{\mu\nu} \rangle$ in flat space. This has been checked explicitly for a general CFT in [49].

The anomaly effective action (3.8) is one term in the full one-particle irreducible (1PI) effective action obtained by integrating out all the matter/radiation fields in a fixed but arbitrary metric and background gauge fields. It is possible to classify all the terms in the full 1PI effective action of QFT into three general classes, so that the full 1PI action may be expressed as the sum [65]

$$S_{\text{1PI}}[g] = S_{\text{local}}[g] + S_{\text{anom}}^{NL}[g] + S_{\text{inv}}[g] \quad (3.9)$$

of (i) purely local, (ii) non-local anomalous, and (iii) non-local invariant under the action of the local Weyl transformation $g_{\mu\nu} \rightarrow e^{2\sigma} g_{\mu\nu}$. The classification of terms (3.9) for the possible quantum corrections to the effective action for gravity and the conformal anomaly effective action (3.8) applies in a general curved space background, and hence is considerably more general than expansions around flat space [66–68]. The local terms S_{local} are the ones usually considered in EFT approaches including the clearly IR relevant cosmological constant and E-H term of (2.1) which scale as $e^{4\sigma_0}$ and $e^{2\sigma_0}$ under global Weyl rescalings respectively, together with terms higher order in local curvature invariants, divided by some high energy UV energy scale, which scale as $e^{-2n\sigma_0}$ for $n \geq 0$. For $n > 0$ these local terms are irrelevant in the IR. The $n = 0$ terms are marginal and require special care.

The local R^2 and C^2 actions are neutral under global Weyl rescalings, while S_{anom}^{NL} of (3.8) is unique (up to the b, b', β_i coefficients and S_{inv}) in scaling *linearly* with σ_0 , *i.e.* logarithmically under the global rescalings of the metric and distance scales. Hence the anomaly terms (3.8) can grow to importance in the IR and are classified as marginally relevant. All other $n = 0$ terms satisfying $S_{\text{inv}}[e^{2\sigma} g] = S_{\text{inv}}[g]$ are neutral under all Weyl rescalings, and hence do not grow in the IR. While contributions to low energy gravity of the Weyl invariant terms $S_{\text{inv}}[g]$ cannot be excluded, the minimal assumption is to

take only the relevant $n < 0$ local terms of (2.1) and the logarithmic non-local (3.8) as the basis for an EFT treatment of gravity. One could add to (3.1) a $\square R$ term with an arbitrary coefficient, but since this term is the trace of the metric variation of a local R^2 action, it is classified with the local terms S_{local} and not as part of the true anomaly, which is *not* the trace of a variation of any local action [65].

Like the chiral anomaly in QCD, the conformal anomaly is intrinsically a non-local quantum effect and like (2.3), $S_{\text{anom}}^{NL}[g]$ is non-local in terms of the original metric and curvature variables. The non-local 1PI effective action of the conformal anomaly (3.1) in 4D can nevertheless be rendered in a compact local form [30, 61, 64, 69] by introducing an additional local scalar field φ and making the replacement

$$S_{\text{anom}}^{NL}[g] \rightarrow S_{\mathcal{A}}[g; \varphi] = \frac{b'}{2} \int d^4x \sqrt{-g} \left\{ -(\square\varphi)^2 + 2 \left(R^{\mu\nu} - \frac{1}{3} R g^{\mu\nu} \right) (\nabla_\mu\varphi) (\nabla_\nu\varphi) \right\} + \frac{1}{2} \int d^4x \mathcal{A} \varphi. \quad (3.10)$$

Since $S_{\mathcal{A}}[g; \varphi]$ is quadratic in φ with kinetic term $-(b'/2) \int d^4x \sqrt{-g} \varphi \Delta_4 \varphi$, variation of (3.10) with respect to φ gives the linear eq. $b' \sqrt{-g} \Delta_4 \varphi = \mathcal{A}/2$, which when solved for φ and substituted back into (3.10) returns the non-local form of the effective action (3.8), up to a Weyl invariant term which can be absorbed into $S_{\text{inv}}[g]$. The anomaly effective action (3.10) is the 4D analog of (2.4) and both are non-trivial solutions of the WZ consistency condition [63–65]

$$S_{\mathcal{A}}[e^{-2\sigma} g; \varphi] = S_{\mathcal{A}}[g; \varphi + 2\sigma] - S_{\mathcal{A}}[g; 2\sigma] \quad (3.11)$$

under Weyl conformal transformations. The scalar degree of freedom φ is therefore closely related (by the shift $\varphi \rightarrow \varphi + 2\sigma$) to the conformal factor of the spacetime metric to which it couples.

The general classification (3.9) does not preclude the possibility that S_{inv} might contain additional light cone singular terms relevant in the IR, implying additional low energy degrees of freedom besides φ . However only a single scalar φ is necessary to account for the known anomaly. Thus although it is possible to introduce additional low energy degrees of freedom, as was done in [28, 69], there is no general symmetry-based reason to do so, and the simplest possibility is the minimal one of $S_{\mathcal{A}}[g; \varphi]$, with φ the single scalar carrying all the conformal transformation properties of the WZ effective action and identity (3.11). Thus (3.10) is the minimal addition to the classical action (2.1) needed to take into account the light cone singularities of correlation functions of the stress tensor of quantum matter. Also although φ resembles the dilaton of string theory in some respects, it should be distinguished from it, because of the specific WZ identity (3.11) required by the Weyl cohomology of the anomaly [65], which reflects its quite different physical origin. The scalar φ is a *collective* mode,

composed of a quantum correlated pair of massless SM fields contributing to $\langle \hat{T}^{\mu\nu} \rangle$ [29, 30], similar to the Schwinger boson in QED₂, or a $U(1)^{ch}$ flavor singlet (η or η') meson in low energy QCD. For this reason the distinct term of *conformal* is reserved for φ .

As an explicit solution of the WZ consistency condition (3.11), the effective action (3.10) is a generating functional that reproduces all the anomalous conformal Ward identities of a CFT. Since φ has zero scaling dimension, and its propagator $(\Delta_4^{-1})_{xy} \sim \log(x-y)^2$ in position space, (3.10) scales logarithmically under global Weyl rescalings, and is a marginally IR relevant operator under finite size renormalization group scaling [70]. In $S_{\mathcal{A}}[g; \varphi]$ the b, b' coefficients are to be fixed by experiment, as they may receive contributions from the gravitational field(s) themselves [63].

As in 2D the WZ effective action of the anomaly $S_{\mathcal{A}}$ is not purely a local functional of higher order curvature invariants, unlike the higher dimensional quantum corrections to GR usually considered [22, 24, 71]. Being derived from non-local quantum fluctuations of massless fields, the effects of (3.10) need not be negligible on macroscopic scales far greater than L_{Pl} , and generally are relevant on null horizons, even when local curvatures are small there [28, 30, 72]. The massless excitations described by φ do not generally decouple, so they can have physically relevant effects at low energies, which must be studied on a case by case basis, and particularly in cases where naive EFT decoupling arguments would seem to fail. In fact the covariantly conserved stress tensor

$$T_{\mathcal{A}}^{\mu\nu}[g; \varphi] \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S_{\mathcal{A}}[g; \varphi] \quad (3.12)$$

derived from (3.10), whose trace is (3.1), generally grows without bound like $(r-r_M)^{-2}$, resp. $(r-r_H)^{-2}$, as either the Schwarzschild or de Sitter horizons at r_M or r_H are approached [28], *cf.* Sec. VIII.

The effective action $S_{\mathcal{A}}$ thus amounts to a specific addition to Einstein's GR, consistent with, and in fact required by first principles of QFT, general covariance, and the general form of the conformal anomaly (3.1). It is a relevant addition in both the mathematical and physical sense [65], capturing the macroscopic light cone singularities of anomalous correlation functions. It is therefore a necessary part of the low energy EFT of gravity, and should be added to (2.1) of classical GR, much as the WZ term must be added to the low energy meson theory to account for the chiral anomaly of QCD.

IV. The Cosmological Term as a 4-Form Gauge Field

The action (3.10) of the conformal anomaly is the first essential element in the EFT of gravity taking macroscopic quantum effects into account. Quite apart from and independent of the anomaly,

there is a second element that plays an essential role in the characterization of vacuum energy and resolution of the naturalness problem of Λ . This is the observation that the constant Λ term in (2.1) can be reformulated in terms of an abelian gauge theory as follows [73–76]. Let

$$F = \frac{1}{4!} F_{\alpha\beta\gamma\lambda} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\lambda \quad (4.1)$$

be a 4–form field strength which is the curl of a totally anti-symmetric 3-form gauge potential

$$F_{\alpha\beta\gamma\lambda} = 4 \partial_{[\alpha} A_{\beta\gamma\lambda]} = 4 \nabla_{[\alpha} A_{\beta\gamma\lambda]} = \nabla_\alpha A_{\beta\gamma\lambda} - \nabla_\beta A_{\alpha\gamma\lambda} + \nabla_\gamma A_{\alpha\beta\lambda} - \nabla_\lambda A_{\alpha\beta\gamma} \quad (4.2)$$

so that

$$F = dA, \quad A = \frac{1}{3!} A_{\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \quad (4.3)$$

and F is an exact 4-form. As a natural generalization of ordinary electromagnetism where $F = dA$ is an exact 2-form exterior derivative of the 1-form vector gauge potential $A = A_\mu dx^\mu$, let F be provided with the ‘Maxwell’ action

$$S_F = -\frac{1}{2\kappa^4} \int F \wedge \star F = -\frac{1}{48\kappa^4} \int d^4x \sqrt{-g} F_{\alpha\beta\gamma\lambda} F^{\alpha\beta\gamma\lambda} = \frac{1}{2\kappa^4} \int d^4x \sqrt{-g} \widetilde{F}^2 \quad (4.4)$$

where

$$\widetilde{F} \equiv \star F = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\lambda} F^{\alpha\beta\gamma\lambda}, \quad F_{\alpha\beta\gamma\lambda} = -\varepsilon_{\alpha\beta\gamma\lambda} \widetilde{F} \quad (4.5)$$

is the scalar Hodge star \star dual to F , *cf.* (A6), and κ is a free parameter whose significance as torsional or topological susceptibility of the gravitational vacuum is discussed in Appendix C.

Now the point is that when the rank of the D -form is matched to the number of $D = 4$ spacetime dimensions, the free ‘Maxwell’ theory (4.4) has two very special properties, namely: (i) F is constrained to be a constant, with no propagating degrees of freedom, and (ii) its stress tensor $T_F^{\mu\nu}$ is proportional to the metric $g^{\mu\nu}$, hence equivalent to a cosmological term. The simplest example of this is usual 2-form electrodynamics in $D = 2$ spacetime dimensions, where the classical Maxwell action is

$$-\frac{1}{4e^2} \int d^2x \sqrt{-g} F_{\alpha\beta} F^{\alpha\beta} = \frac{1}{2e^2} \int d^2x \sqrt{-g} \widetilde{F}^2 \quad (4.6)$$

with the dual $\widetilde{F} = \frac{1}{2} \varepsilon_{\alpha\beta} F^{\alpha\beta} = F^{01}$ the electric field in one spatial dimension. The stress tensor corresponding to (4.6) is $-g^{\mu\nu} \widetilde{F}^2 / 2e^2$, provided $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is taken to be independent of the

metric, with spacetime indices raised by $g^{\alpha\beta}$. Since the electric field \widetilde{F} is constrained by Maxwell's eqs. $\partial_\nu F^{\mu\nu} = j^\mu$ in $D = 2$, to be a spacetime constant of integration, *i.e.* $\widetilde{F} = \text{const.}$ in the absence of sources $j^\mu = 0$, the stress tensor of electric field energy, proportional to $g^{\mu\nu}$ is equivalent to a cosmological vacuum energy. Classical Maxwell theory contains no propagating degrees of freedom at all in one space plus one time dimension, in the absence of sources, and the constant \widetilde{F} simply parametrizes the energy of the vacuum. In $D = 2$ the electric charge e has mass dimension one, while in $D = 4$, the constant \varkappa also carries mass dimension one if the field strength tensor $F_{\alpha\beta\gamma\lambda}$ is mass dimension four.

The exactly analogous situation obtains in $D = 4$ for (4.1)-(4.5), and in fact may be generalized to any even D spacetime dimension [76]. The equivalence to Λ in $D = 4$ follows again the fact that the energy-momentum-stress tensor corresponding to (4.4)

$$T_F^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_F}{\delta g_{\mu\nu}} = -\frac{1}{4! \varkappa^4} \left(\frac{1}{2} g^{\mu\nu} F^{\alpha\beta\gamma\lambda} F_{\alpha\beta\gamma\lambda} - 4 F^{\mu\alpha\beta\gamma} F^\nu{}_{\alpha\beta\gamma} \right) = -\frac{1}{2\varkappa^4} g^{\mu\nu} \widetilde{F}^2 \quad (4.7)$$

is proportional to the metric tensor, if the convention that $F_{\alpha\beta\gamma\lambda}$ with all lower indices is independent of the metric is again adopted. Analogous to the $D = 2$ case that \widetilde{F} is a constant follows from the sourcefree 'Maxwell' eq. obtained by variation of (4.4) with respect to $A_{\alpha\beta\gamma}$, *viz.*

$$\nabla_\lambda F_{\alpha\beta\gamma}{}^\lambda = 0, \quad \text{for} \quad J^{\alpha\beta\gamma} = 0 \quad (4.8)$$

and $\partial_\lambda \widetilde{F} = 0$, so that $\widetilde{F} = \widetilde{F}_0$ is a spacetime constant—in the complete absence of any sources $J = 0$.

Hence (4.1)-(4.5), (4.7)-(4.8) are completely equivalent to a cosmological term in Einstein's eqs. in $D = 4$, with the identification

$$\Lambda_{\text{eff}} = \frac{4\pi G_N}{\varkappa^4} \widetilde{F}^2 \geq 0 \quad (4.9)$$

the effective (necessarily non-negative) cosmological constant term for \varkappa and $\widetilde{F} = \widetilde{F}_0$ real constants. In this way one can freely trade the cosmological constant Λ of classical GR for a new fundamental constant \varkappa of the low energy EFT, together with an integration constant of the constraint $\partial_\lambda \widetilde{F} = 0$.

It may seem at first sight that little has been gained by this trade of an equivalent reformulation of the Λ term as a 4-form gauge field at the classical level. However the free integration constant \widetilde{F}_0 can then be fixed by a classical global boundary condition in flat space, without any reference to quantum zero-point energy, UV divergences, or cutoffs. A vanishing Λ_{eff} corresponds instead to the vanishing of the sourcefree 'electric' field strength $\widetilde{F} = F^{0123}$ in infinite three dimensional empty flat space, analogous to the vanishing of $\widetilde{F} = F^{01}$ electric field of one space dimension in the absence of

sources, as simply the classical state of lowest energy in either case.

Moreover this setting of the value of the free constant $\widetilde{F} = \widetilde{F}_0$, which is *a priori* independent of geometry, to zero in empty flat space is *required* by the sourcefree Einstein's eqs.

$$\left[R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \right]_{\text{flat}} = 0 = -\Lambda_{\text{eff}} \Big|_{\text{flat}} \eta_{\mu\nu} \quad (4.10)$$

viewed as a low energy EFT. This already indicates that flat space QFT estimates of vacuum energy in any way dependent upon UV cutoffs or heavy mass scales are *inconsistent* with Einstein's eqs. QFT in flat space is sensitive only to *differences* in energy. Hence the absolute value of quantum zero point energy, and any dependence upon cutoffs or UV regularization schemes is arbitrary and of no physical significance, a point made before [77], but well worth emphasizing. The value of Λ or Λ_{eff} is significant only through its gravitational effects, and hence cannot be evaluated in isolation, but only within the context of a gravitational EFT, as in (4.10), where both sides can be evaluated *independently*, and Λ_{eff} is a *free* constant of integration, not a fixed parameter of the Lagrangian.

What has been gained then is that whereas in the usual treatment of Λ as a fixed parameter of the classical theory (2.1), Λ is seemingly dependent upon quantum zero point energies and sensitive to UV physics, treating Λ_{eff} as related instead to an integration constant of the classical 'Maxwell' eq. $\partial_\lambda \widetilde{F} = 0$ of a 4-form gauge field, the free constant $\widetilde{F} = \widetilde{F}_0$ is independent of UV physics. Initially independent also of spacetime curvature, its value is then uniquely determined by evaluating both sides of (4.10) in flat space. Thus if Λ_{eff} is given by (4.9), \widetilde{F} and Λ_{eff} necessarily vanish in the flat space limit of GR, if classical GR is to be a consistent approximation to the low energy EFT of gravity.

Since the condition $\widetilde{F}_0 = 0$ in flat space holds for any value of the EFT parameter \varkappa in $D=4$, which remains arbitrary, the condition on the value of the integration constant \widetilde{F}_0 at the minimum of energy does not involve any fine tuning of fundamental constants or naturalness problem in the EFT with the Λ term replaced by (4.4), any more than it does for setting the electric field strength $F^{01} = 0$ in $D=2$ classical Maxwell theory. Parameters of the EFT Lagrangian, such as G_N and \varkappa (or Λ), may be subject to UV divergences at higher loop order — and require UV regularization and renormalization — but the value of the minimum energy classical solution to the EFT Maxwell eqs. are not — and do not.

Thus the reformulation (4.9) of the cosmological constant Λ_{eff} in terms of $\widetilde{F} = \widetilde{F}_0$ and \varkappa shifts the consideration of cosmological vacuum energy away from the UV divergences of QFT to a macroscopic (IR) boundary condition solving the classical constraint eq. (4.8) and minimization of energy in flat space. Although very simple mathematically, and a completely equivalent parametrization of

the $\Lambda \rightarrow \Lambda_{\text{eff}}$ term in the classical Einstein eqs. in the absence of any sources for F , trading Λ for \widetilde{F} and a boundary condition through (4.9) is a significant step conceptually. For in addition to removing the fine tuning or naturalness problem of Λ , introducing an independent 4-form field F in place of constant Λ also allows for the introduction of sources in (4.8) that will enable F (and hence Λ_{eff}) to change, departing from its zero value in infinite sourcefree flat space in finite calculable ways, and eventually to become full-fledged dynamical variable of the low energy EFT in its own right.

V. The Trace Anomaly for Fermions: Torsion and the Source for F

In Sec. IV the 4-form field strength derived from a 3-form gauge potential was simply postulated as an independent degree of freedom, with the observation that it contributes to the EFT of low energy gravity in the same way as an effective vacuum energy and cosmological term according to (4.9). This has the advantage of reformulating the naturalness problem of a vanishing cosmological term as simply the solution of the sourcefree ‘Maxwell’ eq. (4.8) that minimizes the ‘electric’ energy in flat space, and is required by consistency of GR in its flat space limit (4.10). A more fundamental origin of F or any relation to the conformal anomaly of the previous Sec. III has yet to be established.

In this section both these gaps will be filled by taking the third and final step needed to construct the EFT for low energy gravity, by the extension of the conformal anomaly for fermions to a general Einstein-Cartan spacetime including torsion. Since the minimal coupling of fermions to geometry involves the $SO(3, 1)$ spin connection through which independent torsion dependence enters, this will allow both A and F of Sec. IV to be identified with certain functions of torsion, establishing their fundamental basis in terms of properties of spacetime geometry. This relation of A to torsion and the fermion conformal anomaly requires it to be varied independently of the metric, leading directly to a source current for the ‘Maxwell’ eq. (4.8), and hence a dynamical vacuum energy for which the conformal anomaly of massless fermions acts as source.

The torsion 2-form \mathcal{T}^a is defined by the second Cartan equation of structure [41, 78]

$$\mathcal{T}^a = de^a + \omega^a_b \wedge e^b = \frac{1}{2} T^a_{bc} e^b \wedge e^c = \frac{1}{2} T^a_{\mu\nu} dx^\mu \wedge dx^\nu. \quad (5.1)$$

This definition provides the unique algebraic solution for the connection, viz. [41]

$$\omega_{ab\mu} = v^{\nu}_a \eta_{bc} (\nabla_\mu e^c{}_\nu) - K_{abc} e^c{}_\mu = -\omega_{ba\mu} \quad (5.2)$$

which contains a Riemannian part in terms of the torsionless covariant derivative

$$\nabla_{\mu} e^c_{\nu} = \partial_{\mu} e^c_{\nu} - \Gamma^{\lambda}_{\mu\nu} e^c_{\lambda} \quad (5.3)$$

with respect to the Levi-Civita connection $\Gamma^{\lambda}_{\mu\nu}$ of (2.2), and a non-Riemannian part dependent upon the torsion,

$$K_{abc} = \frac{1}{2} (T_{abc} + T_{bca} - T_{cab}) \quad (5.4)$$

where (5.4) is called the contorsion tensor. The definitions and properties of vierbein/tetrad field e^a_{μ} and its inverse ν^{μ}_a used here are given by eqs. (A1)-(A3) of Appendix A.

It follows from (5.2) that if all components of torsion vanish, (5.1) with $T^a_{bc} = 0$ leads to the connection $\omega_{ab\mu}$ becoming the Levi-Civita connection, which in holonomic coordinates is just the usual symmetric Riemann-Christoffel symbol $\Gamma^{\lambda}_{\mu\nu}$ of (2.2), that is fully specified by the metric and its derivatives. Conversely, once non-vanishing torsion is admitted, the 24 independent components of the affine connection $\omega_{ab\mu}$ are in one-to-one correspondence with the 24 independent components of the contorsion and torsion T^a_{bc} tensors by (5.2). Then they can be varied, and indeed must be varied, independently of the metric $g_{\mu\nu}$. This independent variation of the connection is the basis of the first order or Palatini formalism of GR [39, 79], which in the case of the E-H action turns out to be equivalent to the usual metric variation, since it leads to $T^a_{bc} = 0$ in the absence of spin currents.

For more general actions, including that of the conformal anomaly, independent variation of the affine connection and the metric generally leads to different Euler-Lagrange eqs., so that the resulting Einstein-Cartan theory differs in general from standard torsionless GR. Also, although the introduction of torsion is a possibility that can be realized by non-minimal couplings of arbitrary spin fields to geometry [80], it is only with spinor representations of the Lorentz group that it is unavoidable, even under the hypothesis of minimal coupling. We are thus led to reconsider the conformal anomaly of massless fermions, but now in a general Einstein-Cartan space, allowed to possess non-zero torsion.

The covariant derivative of Dirac spinors ψ which defines their minimal coupling to geometry is [41, 78]

$$\nabla_{\mu} \psi = \left(\partial_{\mu} + \Gamma_{\mu} \right) \psi = \left(\partial_{\mu} + \frac{1}{8} \omega_{ab\mu} [\gamma^a, \gamma^b] \right) \psi \quad (5.5)$$

where the tangent space Dirac γ matrices satisfy $\gamma^a \gamma^b + \gamma^b \gamma^a = -2\eta^{ab} = 2 \text{diag} (+ - - -)$, and $\omega_{ab\mu}$ is the general metric compatible $SO(3, 1)$ spin connection (5.2). Owing to the anti-symmetry $T^a_{bc} = -T^a_{cb}$ and the anti-symmetry also under interchange of a and b in (5.5), the torsion dependent piece of the

Dirac Lagrangian $\frac{1}{2}\bar{\psi}\gamma^\mu(\partial_\mu + \Gamma_\mu)\psi - \frac{1}{2}[(\partial_\mu - \Gamma_\mu)\bar{\psi}]\gamma^\mu\psi$ is totally anti-symmetric in the γ matrices, and can be expressed

$$\left(\frac{1}{2}\right)\left(-\frac{K_{abc}}{8}\right)\left(\gamma^c[\gamma^a, \gamma^b] + [\gamma^a, \gamma^b]\gamma^c\right) = -\frac{1}{4}K_{abc}\gamma^{[a}\gamma^b\gamma^{c]} = -iW^d\gamma_d\gamma^5 \quad (5.6)$$

in terms of the axial vector field

$$W^d \equiv \frac{1}{4}\epsilon^{abcd}K_{abc} = \frac{1}{8}\epsilon^{abcd}T_{abc} \quad (5.7)$$

where $\gamma^{[a}\gamma^b\gamma^{c]} = i\epsilon^{abcd}\gamma_d\gamma^5$ with $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ has been used. Thus the Dirac Lagrangian minimally coupled to the spin connection depends on torsion, but only through its totally anti-symmetric part, which is dual to an axial vector field $A_5^\mu = W^\mu = g^{\mu\nu}e^a_\nu\eta_{ab}W^b$. This has just 4 independent components, instead of the 24 independent components of the full torsion tensor (5.1), or spin connection (5.2).

Thus the conformal anomaly for N_f massless minimally coupled Dirac fermions in an arbitrary Einstein-Cartan space with torsion reduces to that in the presence of the four component axial vector field (5.7), and can be computed by standard Schwinger-de Witt heat kernel techniques. This gives the additional torsion dependent contributions [80–82]

$$\langle\Delta\hat{T}^\alpha{}_\alpha\rangle_{\text{Dirac}} = \frac{N_f}{12\pi^2}\nabla_\mu\left\{\nabla^\mu(W_\nu W^\nu) + W^\mu(\nabla_\nu W^\nu) - (W^\nu\nabla_\nu)W^\mu\right\} - \frac{N_f}{24\pi^2}\left(\nabla_\mu W_\nu - \nabla_\nu W_\mu\right)^2 \quad (5.8)$$

to the trace of the fermion stress tensor, where here ∇_μ is the torsionless covariant derivative (5.3) and $W_\mu = e^a_\mu\eta_{ab}W^b$, $W^\mu = g^{\mu\nu}W_\nu$. Since scalar and massless vector gauge fields do not couple to torsion at all in minimal coupling [80], these torsion dependent additions to the conformal anomaly arise for minimally coupled fields only in the case of massless fermions in a general Einstein-Cartan space.

By the analogous steps to those in Sec. III, both terms in (5.8) contribute to the total conformal anomaly \mathcal{A} and thus lead in the effective action (3.10) to additional terms linear in φ .¹ The first term in (5.8) in the trace anomaly of massless fermions minimally coupled to torsion is a total derivative of the form $k_f\nabla_\mu V^\mu$ where

$$V^\mu \equiv \nabla^\mu(W_\nu W^\nu) + W^\mu(\nabla_\nu W^\nu) - W^\nu(\nabla_\nu W^\mu), \quad k_f \equiv \frac{N_f}{12\pi^2}. \quad (5.9)$$

¹ Note Added in Proof: There are also torsion dependent terms quadratic in φ in the effective action resulting from (5.8) cf. [68, 80, 82]. These additional terms, which do not affect the main line of argument, are omitted here for simplicity, and will be treated fully in a future publication.

Since $\nabla_\mu V^\mu$ is a total derivative, the $(\nabla_\mu V^\mu) \varphi$ term in the effective action of the fermion conformal anomaly can be integrated by parts, and then be written

$$\frac{k_f}{2} \int d^4x \sqrt{-g} (\nabla_\mu V^\mu) \varphi = -\frac{k_f}{2} \int d^4x \sqrt{-g} V^\mu (\nabla_\mu \varphi) = +\frac{k_f}{2 \times 3!} \int d^4x \sqrt{-g} A_{\alpha\beta\gamma} (\varepsilon^{\alpha\beta\gamma\mu} \partial_\mu \varphi) \quad (5.10)$$

up to a surface term which does not affect local variations and which may be taken to vanish for suitable boundary conditions. In (5.10) the identification

$$A \equiv -(\star V), \quad A_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\lambda} V^\lambda \quad \Rightarrow \quad V = -(\star A), \quad V_\mu = \frac{1}{3!} \varepsilon_{\mu\alpha\beta\gamma} A^{\alpha\beta\gamma} \quad (5.11)$$

has been made for the 3-form potential of Sec. IV, so that A is taken to be due entirely to the torsion dependent total derivative term in the fermion trace anomaly. The overall normalization of (5.11) is arbitrary since it can eventually be subsumed into the constant \varkappa appearing in (4.4).

By defining now the anti-symmetric 3-current

$$J^{\alpha\beta\gamma} \equiv \frac{k_f}{2} \varepsilon^{\alpha\beta\gamma\mu} \partial_\mu \varphi = \frac{N_f}{24\pi^2} \varepsilon^{\alpha\beta\gamma\mu} \partial_\mu \varphi \quad (5.12)$$

the term (5.10) in the effective action of the conformal anomaly for massless fermions coming from the first total derivative torsional term in (5.8) can be expressed in the form

$$S_{\text{int}}[\varphi; A] = \frac{1}{3!} \int d^4x \sqrt{-g} J^{\alpha\beta\gamma} A_{\alpha\beta\gamma} \quad (5.13)$$

analogous to a $\int J \cdot A$ interaction term of ordinary electromagnetism, but with the 3-form current (5.12) and gauge potential A of (5.11) describing the three-dimensional worldsheets of two-dimensional surfaces, instead of the usual 1-form $J^\mu A_\mu$ interaction of the electromagnetic current and potential, describing the worldlines of point particles [73, 76, 83]. With the identification (5.11) we also have

$$\widetilde{F} = \star dA = -(\star d\star V) = \nabla_\mu V^\mu \quad (5.14)$$

and the vacuum energy (4.7) is entirely attributable to a non-zero torsion through (5.7) and (5.9).

The analogy with electromagnetism is apt because the current (5.12) is covariantly conserved, *i.e.*

$$\nabla_\gamma J^{\alpha\beta\gamma} = \frac{k_f}{2} \frac{1}{\sqrt{-g}} \partial_\gamma (\sqrt{-g} \varepsilon^{\alpha\beta\gamma\mu} \partial_\mu \varphi) = \frac{k_f}{2} \varepsilon^{\alpha\beta\gamma\mu} \partial_\gamma \partial_\mu \varphi = 0 \quad (5.15)$$

since $\sqrt{-g} \varepsilon^{\alpha\beta\gamma\mu}$ is independent of the spacetime metric *cf.* (A13). Thus the $J \cdot A$ interaction (5.13) is invariant under the abelian gauge transformation

$$\delta_{\Theta} A = d\Theta \quad \delta_{\Theta} F = d^2\Theta = 0 \quad (5.16)$$

where Θ is an arbitrary 2-form, by another integration by parts. This means that (5.12) is a candidate source term for the ‘Maxwell’ eq. (4.8), consistent with abelian gauge invariance, in analogy with ordinary electromagnetism. Since A defined by (5.11) is directly related to torsion which is independent of the spacetime metric, it must be treated as an independent degree of freedom of the low energy EFT, to be varied independently of $g_{\mu\nu}$, which is just what is required to arrive at ‘Maxwell’ eqs. for F . The identifications of the 3-forms A and $F = dA$ in (5.11)-(5.14), and consistent source term F through (5.12)-(5.13) have the consequence that the vacuum energy defined by Λ_{eff} of (4.9) will change if and when φ does and $J^{\alpha\beta\gamma} \neq 0$, provided massless (or very light) fermions contribute to the conformal anomaly (5.8), and non-zero torsion is admitted in an extended Einstein-Cartan geometry.

VI. The Effective Theory of Gravity in the Absence of Massless Fermions

Assembling the several elements of the previous sections, two separate cases for the EFT of gravity must be distinguished, depending upon whether or not there are massless fermions to contribute to the torsional conformal anomaly (5.8). Since all SM fermions, including neutrinos, are expected to be massive, this distinction will depend on the energy and distance scales of the EFT application in comparison with the lightest fermion mass $m_\nu > 0$.

For energy scales $E \ll m_\nu$ or physical distance scales $\ell \gg m_\nu^{-1}$, assuming that there are no massless fermions coupling to torsion contributing to (5.8), the low energy EFT of gravity is

$$S_{\text{eff}}^{(I)}[g; \varphi; A] = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R + S_{\mathcal{A}}[g; \varphi] + S_F[A; g] \quad (\text{I}) \quad (6.1)$$

where $S_{\mathcal{A}}[g; \varphi]$ is given by (3.10) and $S_F[A; g]$ is given by (4.4), with (4.2) and (4.8). In this case (I) the 3-form gauge field A and 4-form field strength $F = dA$ are not coupled in any way to the anomaly effective action $S_{\mathcal{A}}[g; \varphi]$. Hence $\tilde{F} = \tilde{F}_0$ is sourcefree and constant, as in (4.8), and entirely equivalent to an effective cosmological term Λ_{eff} by (4.9), provided only that the 3-form potential can be varied independently of the metric. The identification of A by (5.11) in terms of the torsion dependent V certainly satisfies this condition, whether or not massless fermions are present.

With the condition that $\widetilde{F}_0 = 0$ in asymptotically flat space, the lowest energy ground state, \widetilde{F} and $S_F[A; g]$ then drop out entirely. In this case the EFT (I) is just classical GR with $\Lambda_{\text{eff}} = 0$ and with the addition of the conformal anomaly effective action $S_{\mathcal{A}}[g; \varphi]$. Alternately, $S_F[A; g]$ with non-zero constant \widetilde{F} may be retained to parametrize an arbitrary positive constant vacuum energy Λ_{eff} in some region(s) of space, its value to be determined by appropriate boundary conditions, as in the application to gravitational condensate stars of Sec. VIII.

The classical Euler-Lagrange eqs. following from variation of (6.1) are

$$\Delta_4 \varphi \equiv \nabla_\mu \left(\nabla^\mu \nabla^\nu + 2R^{\mu\nu} - \frac{2}{3} R g^{\mu\nu} \right) \nabla_\nu \varphi = \frac{1}{2} \left(E - \frac{2}{3} \square R \right) + \frac{1}{2b'} \left(bC^2 + \sum_i \beta_i \mathcal{L}_i \right) \quad (\text{I}) \quad (6.2)$$

for the conformal scalar φ , together with the semi-classical Einstein eq.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N \left(T_{F\mu\nu} + T_{\mathcal{A}\mu\nu}[g; \varphi] + T_{\mu\nu}^{\text{cl}} \right) \quad (\text{I}) \quad (6.3)$$

with $T_F^{\mu\nu}$ and Λ_{eff} given by (4.7) and (4.9), and with

$$T_{\mathcal{A}}^{\mu\nu}[g; \varphi] \equiv \frac{2}{\sqrt{-g}} \frac{\delta}{\delta g_{\mu\nu}} S_{\mathcal{A}}[g; \varphi] = b' E^{\mu\nu} + b C^{\mu\nu} + \sum_i \beta_i T^{(i)\mu\nu} \quad (6.4)$$

the stress tensor resulting from the metric variation of $S_{\mathcal{A}}[g; \varphi]$. The first contribution here is

$$\begin{aligned} E_{\mu\nu} = & -2 (\nabla_{(\mu} \varphi) (\nabla_{\nu)} \square \varphi) + 2 \nabla^\alpha [(\nabla_\alpha \varphi) (\nabla_\mu \nabla_\nu \varphi)] - \frac{2}{3} \nabla_\mu \nabla_\nu [(\nabla_\alpha \varphi) (\nabla^\alpha \varphi)] \\ & + \frac{2}{3} R_{\mu\nu} (\nabla_\alpha \varphi) (\nabla^\alpha \varphi) - 4 R^\alpha_{(\mu} [(\nabla_{\nu)} \varphi) (\nabla_\alpha \varphi)] + \frac{2}{3} R (\nabla_{(\mu} \varphi) (\nabla_{\nu)} \varphi) \\ & + \frac{1}{6} g_{\mu\nu} \left\{ -3 (\square \varphi)^2 + \square [(\nabla_\alpha \varphi) (\nabla^\alpha \varphi)] + 2 (3R^{\alpha\beta} - R g^{\alpha\beta}) (\nabla_\alpha \varphi) (\nabla_\beta \varphi) \right\} \end{aligned} \quad (6.5)$$

$$\begin{aligned} -\frac{2}{3} \nabla_\mu \nabla_\nu \square \varphi - 4 C_{\mu\nu}^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi - 4 R^\alpha_{(\mu} \nabla_{\nu)} \nabla_\alpha \varphi + \frac{8}{3} R_{\mu\nu} \square \varphi + \frac{4}{3} R \nabla_\mu \nabla_\nu \varphi - \frac{2}{3} (\nabla_{(\mu} R) \nabla_{\nu)} \varphi \\ + \frac{1}{3} g_{\mu\nu} \left[2 \square^2 \varphi + 6 R^{\alpha\beta} \nabla_\alpha \nabla_\beta \varphi - 4 R \square \varphi + (\nabla^\alpha R) \nabla_\alpha \varphi \right] \end{aligned} \quad (6.6)$$

which is the metric variation of all the b' terms in (3.10), both quadratic and linear in φ [28, 30, 64], while

$$C_{\mu\nu} \equiv -\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left\{ \int d^4 x \sqrt{-g} C^2 \varphi \right\} = -4 \nabla_\alpha \nabla_\beta (C_{(\mu\nu)}^{\alpha\beta} \varphi) - 2 C_{\mu\nu}^{\alpha\beta} R_{\alpha\beta} \varphi \quad (6.7a)$$

$$T_{\mu\nu}^{(i)} \equiv -\frac{1}{\sqrt{-g}} \frac{\delta}{\delta g^{\mu\nu}} \left\{ \int d^4 x \sqrt{-g} \mathcal{L}_i \varphi \right\} \quad (6.7b)$$

are the metric variations of the last two b and β_i terms in (3.10), both of which are only linear in φ .

In (6.3) a classical matter/radiation stress tensor $T_{\mu\nu}^{cl}$ independent of φ has been allowed as well. If any $\beta_i \mathcal{L}_i$ or $T_{\mu\nu}^{cl}$ terms are non-zero, the Euler-Lagrange eqs. of the fields (or fluid constitutive relations) upon which these additional degrees of freedom depend must be appended to (6.2) and (6.3), to close the system (6.2)-(6.7).

Note that the stress tensor $T_{\mathcal{A}}^{\mu\nu}[g; \varphi]$ derived from $S_{\mathcal{A}}[g; \varphi]$ with φ treated as a classical field, is the finite renormalized stress tensor of the underlying quantum conformal QFT, after all short distance UV divergences have been removed, with $\langle T^{\mu\nu} \rangle_{flat}$ in infinite flat space defined to be zero, by the consistency condition (4.10), where φ may be taken to vanish as well. If boundary conditions different from infinite empty flat space are considered, $T_F^{\mu\nu}$ and Λ_{eff} of (4.7) parametrize any finite vacuum energy in place of Λ by (4.7), and φ may be different from zero. For example the stress tensor (6.6) with $\varphi \propto z^2/a^2$, $\square^2 \varphi = 0$ and $\Lambda_{\text{eff}} = 0$ can account for the Casimir energy and force between two parallel plates a distance a apart in the z direction in flat space, when boundary conditions appropriate to that situation are imposed on φ [28].

The scalar-tensor theory, defined by (6.1)-(6.7), based on first principles of QFT and the conformal anomaly of SM fields, is quite distinct from other modifications of GR, such as Brans-Dicke theory, Hordenski theory or massive gravity. As the possible anomaly sources for φ in (6.2) are negligibly small in our local neighborhood, this effective theory easily passes the most constraining solar system tests, as well as the laboratory bounds on *ad hoc* modified gravity or other scalar-tensor theories [64, 84]. Being generally covariant, the theory described by (6.1)-(6.7) is consistent with the Weak Equivalence Principle (WEP) and local Lorentz invariance. The solutions of (6.2) propagate at c , the speed of both light and of gravitational waves, consistent with present gravitational wave observations.

There are nevertheless two situations where the effects of (6.4) are significant at macroscopic scales and lead to effects qualitatively different from the purely classical theory:

- (1) In the vicinity of horizons, where there are large local blueshifts and the light cone singularities of anomalies come to the fore, relevant for BHs and cosmology, *cf.* Sec. VIII;
- (2) When the source for φ in (6.2) is non-gravitational in origin and sufficiently strong, such as from the QCD trace anomaly \mathcal{L}_G in dense nuclear matter, where scalar gravitational waves may be generated in compact binary mergers and in the hot, dense early universe [64].

Since the eq. of motion (6.2) for φ involves the fourth order conformal Panietz-Riegert operator Δ_4 of (3.5) [60, 61], and typically, differential eqs. higher than second order possess negative energy and/or unstable solutions growing in time, a few additional comments are in order here. Note first that $S_{\mathcal{A}}$ and Δ_4 do not occur in isolation, but as part of the EFT of gravity, subject to the first class

constraints of diffeomorphism invariance. These constraints restrict the class of physically allowed solutions both classically and quantum mechanically. Since $S_{\mathcal{A}}$ is quadratic in φ , a simple case in which the specific effects of the higher derivative terms in the anomaly effective action can be studied is the exactly solvable limit of $G_N^{-1} \rightarrow 0$, where the E-H term is neglected, and on the product space $\mathbb{R} \times \mathbb{S}^3$, where E, C^2 and $\square R$ all vanish. The result of this analysis is that only a small subset of solutions survive the constraints of diffeomorphism invariance, and correspondingly, only a small subspace of physical states with positive norm, and no propagating modes whatsoever, survive quantization [85].

When $G_N^{-1} \neq 0$ and the E-H term of classical GR is added, the resulting EFT (6.1) becomes non-linear, but then can be studied in linearized perturbation theory around flat space. The mixing of the scalar φ with the conformal factor of the metric turns this constrained mode of classical GR into a propagating one. The result is that (6.1)-(6.6) predicts the existence of scalar gravitational waves of positive energy [64], *i.e.* a ‘breather’ mode polarization in addition to the two transverse, traceless gravitational wave modes of classical GR. The active linearized solutions of (6.2) are those with $\square^2 \varphi = 0$, but $\Upsilon = \square \varphi \neq 0$. In other words, the modes with $\square \varphi = 0$ decouple entirely, and the remaining solutions satisfy the second order eq. $\square \Upsilon = 0$, with positive energy. Thus in linearization around flat space, half of the solutions of the fourth order Δ_4 operator are eliminated by the constraints, and do not appear in the physical asymptotic states and S -matrix of the EFT (I). There is no instability in this case either, consistent with general results on the stability of flat space to quantum corrections [42, 86].

Since $S_{\mathcal{A}}$ is derived from the conformal anomaly of well-behaved SM fields, each with a unitary S -matrix in flat space, the EFT incorporating the anomaly would not be expected to lead to unphysical instabilities in weakly curved or asymptotically flat space at low energies, within its range of validity. While this interesting issue certainly deserves further investigation, the detailed studies of the EFT (I) of (6.1) to date are consistent with this physical expectation.

VII. The Effective Theory of Gravity in the Presence of Massless Fermions

On high enough energy or short enough distance scales such that one or more light fermions can be treated as massless, $\ell \ll m_v^{-1}$ the conformal anomaly of the torsional terms (5.8) for massless fermions in a general Einstein-Cartan geometry become active. With the identification (5.11) for the 3-form potential of Sec. IV, A must be varied independently of the metric tensor $g_{\mu\nu}$, satisfying the conditions needed to identify $F = dA$ with vacuum energy by (4.7). With this torsion dependent term separated off and expressed as the interaction (5.13), the total lowest order effective action for

massless fermions coupled to torsion is then¹

$$S_{\text{eff}}^{(\text{II})}[g; \varphi; A; W] = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R + S_{\mathcal{A}}[g; \varphi] + S_F[g; A] + S_{\text{int}}[\varphi; A] + S_W[g; \varphi; W] \quad (\text{II}) \quad (7.1)$$

in which the first three terms are the same as in (6.1), S_{int} is given by (5.13) and

$$S_W[g; \varphi; W] = -\frac{k_f}{4} \int d^4x \sqrt{-g} (\nabla_\mu W_\nu - \nabla_\nu W_\mu)^2 \varphi \quad (7.2)$$

is the additional torsion dependent term in the anomaly effective action for massless fermions arising from the last term of (5.8).

The independent variables of (7.1) are $(g_{\mu\nu}, \varphi, A_{\alpha\beta\gamma})$ and the transverse part of W_μ . Variation with respect to φ now yields

$$\Delta_4 \varphi = \frac{1}{2} \left(E - \frac{2}{3} \square R \right) + \frac{1}{2b'} \left(k_f \tilde{F} + bC^2 + \sum_i \beta_i \mathcal{L}_i \right) - \frac{k_f}{4b'} (\nabla_\mu W_\nu - \nabla_\nu W_\mu)^2 \quad (\text{II}) \quad (7.3)$$

instead of (6.2).

Since $\sqrt{-g} \varepsilon^{\alpha\beta\gamma\mu}$ appearing in (5.13) is in fact independent of the metric, *cf.* (A13), S_{int} makes no contribution to the stress tensor, and variation of (7.1) with respect to the metric gives the Einstein eqs.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N \left(T_{F\mu\nu} + T_{\mathcal{A}\mu\nu}[g; \varphi] + T_{\mu\nu}^{cl} + T_{W\mu\nu} \right) \quad (\text{II}) \quad (7.4)$$

in case (II), instead of (6.3), where $T_{W\mu\nu}$ is the stress tensor due to S_W of (7.2).

Now the ‘Maxwell’ action (4.4) and interaction term (5.13) may be varied independently with respect to $A_{\alpha\beta\gamma}$, with $g_{\mu\nu}$ and φ held fixed. This results in the ‘Maxwell’ eq.

$$\nabla_\lambda F^{\alpha\beta\gamma\lambda} = \varkappa^4 J^{\alpha\beta\gamma} = \frac{k_f \varkappa^4}{2} \varepsilon^{\alpha\beta\gamma\lambda} \partial_\lambda \varphi \quad (7.5)$$

with the source current (5.12). Upon taking its dual, with (4.5), this becomes

$$\partial_\lambda \left(\tilde{F} + \frac{\varkappa^4 k_f}{2} \varphi \right) = 0 \quad (7.6)$$

which is immediately solved by

$$\tilde{F} = -\frac{\varkappa^4 k_f}{2} \varphi + \tilde{F}_0 \quad (7.7)$$

in which \widetilde{F}_0 is a spacetime constant that can be fixed to vanish when $\varphi = 0$ in sourcefree empty flat space. Thus \widetilde{F} given by (7.7) is no longer a constant, since φ now satisfies (7.3) and will change, as will Λ_{eff} , when φ does. Note that in the absence of massless fermions or torsion the terms proportional to k_f vanish and (7.5) reduces to the sourcefree ‘Maxwell’ eq. of Sec. VI, case (I), with constant Λ_{eff} .

Since (7.7) determines the scalar \widetilde{F} , and the one gauge invariant part of the 3-form potential (5.11), while the torsionless Riemannian metric curvature terms are determined in principle by (7.4), the eqs. so far give just one condition (7.7) on the torsional term $\widetilde{F} = \nabla_\mu V^\mu$. Referring to the definition of V^μ in (5.9), the condition (7.7) on $\nabla_\mu V^\mu$ can be viewed as one non-linear (implicit) scalar relation on $W_\mu = W_\mu^\perp + \nabla_\mu U$. Thus keeping $(g_{\mu\nu}, \varphi, A)$ and hence also $F = dA$ and $\nabla_\mu V^\mu$ fixed keeps this scalar relation on the total W^μ fixed, but allows the 3 transverse components W_μ^\perp to be varied independently of $(g_{\mu\nu}, \varphi, A)$. Since $\nabla_\mu W_\nu - \nabla_\nu W_\mu = \nabla_\mu W_\nu^\perp - \nabla_\nu W_\mu^\perp$, the last torsional dependent term in (7.2) and (7.3) depends only upon this transverse part W_μ^\perp , which is as yet undetermined by the solution of the eqs. (7.3)-(7.7).

The independent variation of W_μ^\perp in (7.1) keeping $(A, g_{\mu\nu}, \varphi)$ fixed then gives the additional eq.

$$\nabla_\mu (\nabla^\mu W^{\perp\nu}) = 0 \quad (7.8)$$

which is consistently solved by $W^{\perp\nu} = 0$. In that case the last torsion dependent terms in (5.8), (7.1)-(7.4) also vanish. Setting $W^{\perp\nu} = 0$ and taking (7.7) with $\widetilde{F}_0 = 0$, thereby eliminating \widetilde{F} in favor of φ , (7.3) and (7.4) simplify and become

$$\Delta_4 \varphi + \frac{\kappa^4}{4b'} k_f^2 \varphi = \frac{1}{2} \left(E - \frac{2}{3} \square R \right) + \frac{1}{2b'} \left(bC^2 + \sum_i \beta_i \mathcal{L}_i \right) \quad (\text{II}) \quad (7.9)$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G_N \left\{ -\frac{\kappa^4}{8} k_f^2 \varphi^2 g_{\mu\nu} + b' E_{\mu\nu} + bC_{\mu\nu} + \sum_i \beta_i T_{\mu\nu}^{(i)} + T_{\mu\nu}^{cl} \right\} \quad (\text{II}) \quad (7.10)$$

respectively, with $E_{\mu\nu}, C_{\mu\nu}, T_{\mu\nu}^{(i)}$ given previously by (6.6) and (6.7).

Eqs. (7.9)-(7.10) are the simplified form¹ of the proposed EFT (II) of gravity for energies above the lightest fermion mass but much less than the Planck mass scale, dependent only upon the metric and the additional conformalon scalar field φ (and any additional matter fields). They differ from (6.2)-(6.3) only in the appearance of the terms proportional to k_f^2 that describe the spacetime dependent vacuum energy in the presence of massless (or light) fermions and torsion through (4.7), (5.7), (5.9), (5.14), and (7.7). The EFT (7.1) of the 4-form Maxwell field strength coupled to the current source

of the dual of the φ gradient through (5.13) thus becomes a consistent, effective theory of dynamical vacuum energy that can vary in general in both space and time depending on the anomaly driven current source $J^{\alpha\beta\gamma}$. Eqs. (7.6)-(7.7) show that in this case (II) the effective cosmological vacuum energy (4.9) becomes dynamical when the torsional conformal anomaly of massless fermions is taken into account, and can change if and when φ does.

In deciding which form of the EFT (6.1) or (7.1) to use, the critical question is whether there are effectively massless fermions contributing to the conformal anomaly at the energy or distance scale of interest. Since the lightest fermions are neutrinos and the lightest ν has a mass no greater than approximately 0.04 eV [87], assuming the lightest ν mass eigenstate is of this order, at energies lower than this, or distances and wavelengths larger than $5\mu\text{m}$, the EFT (I) of (6.2)-(6.3) applies, while at energies greater than that of the lightest ν mass, or distances smaller than approximately $5\mu\text{m}$, the EFT (II) of (7.9)-(7.10) applies.

At higher energies and shorter distances more SM particles, bosons or fermions, can be treated as massless, and the b' coefficient changes accordingly. Strictly speaking these threshold effects should be smoothly and continuously matched also from EFT (I) to EFT (II), but as the simplest approximation one can consider step functions for (I) to (II), as well as for the b' anomaly coefficient as energies exceed each light mass threshold. The b and b' anomaly coefficients also receive finite contributions from the conformal Δ_4 operator at one-loop order, while the general form of the anomaly effective action and EFT (6.1) or (7.1) at low energies remain unchanged [63].

VIII. Gravitational Vacuum Condensate Stars

In [31, 32, 36] it was proposed that the solution of the multiple BH paradoxes is that the final state of complete gravitational collapse is a gravitational condensate star rather than a BH. The proposed gravastar is a compact object with a physical surface of positive surface tension replacing the BH horizon, and a static region of de Sitter space with the eq. of state $p = -\rho$ replacing the singular interior of a BH. Because such an object is both horizonless and non-singular, with low entropy, it suffers from no information paradox, and is consistent with quantum unitary evolution. The EFT of the preceding Secs. VI-VII provides a fundamental first principles Lagrangian basis for this proposal.

The requirements for a gravitational condensate star to be realized in gravitational collapse are that quantum vacuum polarization effects can grow large in the vicinity of a BH horizon and induce a phase transition to a non-vanishing interior gravitational Bose-Einstein condensate (GBEC) with $p = -\rho$, equivalent to a non-zero Λ_{eff} . That the anomaly effective action $S_{\mathcal{A}}[g; \varphi]$ and stress tensor

(6.6)-(6.7) of Sec. VI can have substantial effects in spacetimes with horizons, satisfying this first part of the gravastar hypothesis may be seen in the case of the exterior static Schwarzschild spacetime

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{h(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \quad (8.1)$$

with

$$f(r) = h(r) = 1 - \frac{r_M}{r} = 1 - \frac{2GM}{c^2 r}. \quad (8.2)$$

The general solution to (6.2) for $\varphi = \varphi(r)$ and $\mathcal{L}_i = 0$ that is finite as $r \rightarrow \infty$ is readily found by quadratures for the Schwarzschild metric (8.1)-(8.2) to be [28, 30]

$$\frac{d\varphi_S}{dr} = \frac{c_s r_M}{r(r-r_M)} - \frac{2}{3r_M} \left(\frac{r}{r_M} + 1 + \frac{r_M}{r} \right) \ln \left(1 - \frac{r_M}{r} \right) - \frac{2}{3r_M} - \frac{1}{r} \quad (8.3a)$$

$$\varphi_S(r) = c_s \ln \left(1 - \frac{r_M}{r} \right) + \int_{r/r_M}^{\infty} dx \left\{ \frac{2}{3x} (x^2 + x + 1) \ln \left(1 - \frac{1}{x} \right) + \frac{2}{3} + \frac{1}{x} \right\} \quad (8.3b)$$

in terms of the dimensionless integration constant c_s . This solution has the limits

$$\varphi_S(r) \rightarrow \begin{cases} c_s \ln \left(1 - \frac{r_M}{r} \right) + c_1 - 2 \left(1 - \frac{r_M}{r} \right) \left[\ln \left(1 - \frac{r_M}{r} \right) + \frac{2}{3} \right] + \dots, & r \rightarrow r_M \\ - \left(c_s + \frac{11}{9} \right) \frac{r_M}{r} - \left(2c_H + \frac{13}{9} \right) \frac{r_M^2}{4r^4} + \dots, & r \rightarrow \infty \end{cases} \quad (8.4)$$

where the constant c_1 is the finite integral in (8.3b) evaluated at the lower limit $r = r_M$, $x = 1$. Substituting the solution (8.3) into the anomaly stress tensor (6.4)-(6.6), one finds

$$(T^\mu{}_\nu)_{\mathcal{A}} \rightarrow \frac{c_s^2}{6r_M^2} \frac{b'}{(r-r_M)^2} \text{diag}(-3, 1, 1, 1) \rightarrow \infty \quad \text{as} \quad r \rightarrow r_M \quad (8.5)$$

diverging on the Schwarzschild BH horizon for any $c_s \neq 0$.

This leading order $(r-r_M)^{-2}$ divergence of the stress tensor on the horizon may be understood from the kinematic blueshifting of local frequencies in (8.2) near the horizon according to

$$\omega_{\text{loc}}(r) = \frac{\omega_\infty}{\sqrt{f(r)}} \quad (8.6)$$

relative to that at $r = \infty$. The corresponding energy $\hbar\omega_{\text{loc}}(r)$ diverges as $r \rightarrow r_M$ and therefore becomes much greater than any finite mass scale. This is reflected in the fact that the wave eq. for a quantum field of arbitrary finite mass and spin becomes indistinguishable from that of a massless conformal

field in the horizon limit $r \rightarrow r_M$ [30], and hence the conformal anomaly effects come to the fore. Since the stress tensor is a dimension four, conformal weight four operator, it behaves generically as the fourth power of ω_{loc} in (8.5), *i.e.* $\propto f^{-2}$. Noting that φ is a scalar, as is the norm of the static Killing field $K = \partial_t$ of (8.1), which is $\sqrt{-K^\mu K_\mu} = \sqrt{-g_{tt}} = \sqrt{f(r)}$, the divergence of (8.5) depending on the inverse fourth power of this norm is also a coordinate invariant scalar, and observer independent.

The diverging behavior of the local stress tensor (8.5) shows that the anomaly stress tensor can become important near the horizon of a BH and even dominate the classical terms in the Einstein eqs. (6.3), the smallness of the curvature tensor there notwithstanding. Even with $c_s = 0$ in (8.3), which can be arranged by specific choice of the state of the underlying QFT, to remove the leading f^{-2} divergence in (8.5), there remain subleading divergences proportional to f^{-1} , $(\ln f)^2$ and $\ln f$. In fact, there is no solution of (6.2) in Schwarzschild spacetime with $\varphi = \varphi(r)$ only, corresponding to a fully Killing time t invariant and spherically symmetric quantum state, with a finite stress tensor at both regular singular points $r = r_M$ and $r = \infty$ of the differential eq. (6.2). This result, following simply and directly from the conformal anomaly effective action, confirms results of previous studies of the stress tensor expectation value in specific states in Schwarzschild spacetime [27].

The divergences on either the future or past BH horizon (but not both) can be cancelled by allowing linear time dependent solutions of (6.2), which give rise to a Hawking flux $\langle T^t_r \rangle$, such as in the Unruh states; or by relaxing the regularity condition at infinity which gives rise to a non-zero stress tensor there, as in the Hartle-Hawking state. This thermal state is both incompatible with asymptotically flat boundary conditions and unstable. Usually the *assumption* of regularity of the semi-classical $\langle T^\mu_\nu \rangle$ on the horizon is used to argue for the necessity of Hawking radiation flux $\langle T^t_r \rangle > 0$ [88]. However the converse is also true: If a *truly static* and stable asymptotically flat solution of the final state of gravitational collapse is sought with $\langle T^t_r \rangle = 0$, then quantum effects at the horizon, specifically due to the conformal anomaly $T_{\mathcal{A}}^{\mu\nu}$ cannot be neglected, and imply instead the breakdown of regularity there.

A similar behavior is observed in de Sitter spacetime, and indeed in any spherically symmetric static spacetime (8.1) with a horizon at which $-K^\mu K_\mu = f(r) \rightarrow 0$. With Λ positive, the static patch of de Sitter space is of this form with $f(r) \propto h(r) = 1 - H^2 r^2 = 1 - \Lambda r^2/3$. The general spherically symmetric static solution of (6.2) for $\varphi = \varphi(r)$ which is regular at the origin in this case is [28, 30]

$$\begin{aligned} \varphi_{dS}(r) &= \ln(1 - H^2 r^2) + c_0 + \frac{q}{2} \ln\left(\frac{1 - Hr}{1 + Hr}\right) + \frac{2c_H - 2 - q}{2Hr} \ln\left(\frac{1 - Hr}{1 + Hr}\right) \\ &\rightarrow \left[c_H + \left(c_H - 1 - \frac{q}{2} \right) (1 - Hr) + \dots \right] \ln(1 - Hr) + c' + \mathcal{O}(1 - Hr) \end{aligned} \quad (8.7)$$

where the constant $c' = c_0 + (2 - c_H) \ln 2$. Substituting this into the anomaly stress tensor (6.4) gives

$$(T^\mu_\nu)_{\mathcal{A}} \rightarrow \frac{2}{3} c_H^2 H^4 \frac{b'}{(1 - Hr)^2} \text{diag}(-3, 1, 1, 1) \rightarrow \infty \quad \text{as} \quad r \rightarrow r_H \equiv H^{-1} \quad (8.8)$$

which also diverges as f^{-2} for any $c_H \neq 0$ as the de Sitter static horizon is approached.

As in the Schwarzschild case this divergence can be removed if a thermal state is considered, but then only if the temperature is precisely matched to the Hawking temperature $T_H = H/2\pi$ associated with the horizon, which in the de Sitter case leads to the maximally $O(4, 1)$ symmetric state [89, 90]. However, this state is not a vacuum state of QFT and is unstable to particle pair creation, much as a uniform, constant electric field is [91–95], and for essentially the same reason. Due to the non-existence of a global static time by which positive and negative frequency (particle and anti-particle) solutions can be invariantly distinguished, a time independent Hamiltonian bounded from below and stable vacuum state cannot be defined. The horizon where $f = 0$ and the Killing vector of time translation ∂_t becomes null in either Schwarzschild or de Sitter space is the sign of this, so that Λ cannot be globally constant and positive everywhere in space in QFT. The conformal anomaly shows this through its sensitivity to lightlike correlations on the horizon and non-local boundary conditions on the quantum state. Only flat space with $\Lambda_{\text{eff}} = 0$ can be a candidate stable ground state of the semi-classical EFT, as the state of lowest energy with vanishing ‘electric’ field $\widetilde{F} = 0$, a global static Killing time and no horizon, as also required by the low energy EFT consistency condition (4.10).

With Λ_{eff} replaced by a dynamical condensate according to (4.9), vacuum energy can be non-zero only if localized in space, within the static patch $r < r_H = H^{-1}$ of de Sitter space. If one seeks a stable spherically symmetric static solution of the EFT, the light cone enhanced effects of the anomaly stress tensor (3.12) at both the Schwarzschild and de Sitter horizons, (8.5) and (8.8) respectively, should be taken into account, and the assumption that either horizon is a mathematical boundary only should be re-examined. In the gravastar proposal [31, 32], the BH horizon is the location of a physical surface phase boundary layer between two different phases characterized by different values of the vacuum energy Λ_{eff} , regarded as a gravitational condensate. On general thermodynamic grounds the Gibbs relation $\rho + p = sT + \mu n = 0$ implies that the eq. of state of a zero temperature condensate with no conserved particle number should be $p = -\rho$. An argument based on non-relativistic condensed matter analogs given in [96] reached a similar conclusion.

In [36] the gravitational condensate star was shown to follow directly from Schwarzschild’s constant density interior solution in the limit $r \rightarrow r_M$, with $f(r) = h(r)/4$ leading to equal and opposite

surface gravities of the surface at $r_M = r_H$. The surface tension of this physical boundary layer replacing the Schwarzschild and de Sitter horizons is determined from the δ -function discontinuity in gradients of the surface gravities there, and the First Law becomes a purely mechanical relation of a gravastar with this surface tension, and the relativistic analog of the Rayleigh surface tension of a fluid droplet.

The EFT of Sec. VII provides a mechanism and Lagrangian description for the second part of the gravitational condensate star proposal of [31, 32, 36], by allowing the value of Λ_{eff} to change from exterior to interior of the gravastar. The scalar \widetilde{F} , dual to the 4-form field strength of sections IV–V, is a classical coherent field that provides an explicit realization of a gravitational condensate Λ_{eff} interior. When coupled to the conformal scalar through the 3-form abelian current $J^{\alpha\beta\gamma}$, concentrated on a three-dimensional extended world tube of topology $\mathbb{R} \times \mathbb{S}^2$ where $\partial_\mu\varphi$ grows large, φ, \widetilde{F} and the condensate Λ_{eff} all change rapidly in the radial direction. This is precisely the appropriate description of a thin shell phase boundary layer of a gravastar with \mathbb{S}^2 spatial topology sweeping out a tube in spacetime.

The first form of the EFT in Sec. VI applies far from the surface localized near $r_M = 2GM/c^2$ or $r_H = H^{-1}$ in the exterior Schwarzschild or interior de Sitter spacetimes, where there are no massless fermions contributing to the conformal anomaly and \widetilde{F} is constant, so that Λ_{eff} is equal to zero or $3H^2$ respectively. However, when approaching sufficiently close to either the Schwarzschild or de Sitter horizons the blueshifting of local frequencies $\sim f^{-\frac{1}{2}}$ as in (8.6) implies that the lightest fermions may then be treated as massless and contributing to the conformal anomaly, activating the second form of the EFT in Sec. VII. This allows the gravitational condensate \widetilde{F} and Λ_{eff} to change rapidly but continuously in the surface layer from $3H^2$ in the interior to zero in the exterior. The physical thickness of this phase boundary surface layer will then be of the order of the Compton wavelength of the lightest fermion in the SM, $1/m_\nu$, whose observational effects on binary BH mergers, gravitational waves, ringdown and ‘echoes’ can be investigated in the EFT. If the interior condensate of \widetilde{F} is identified with torsion through (5.14), the propagation of fermions in the interior will be affected in possibly observable ways as well.

IX. Discussion and Outlook: Vacuum Energy as a Dynamical Condensate

In this paper an EFT of gravity has been proposed taking account of the most significant macroscopic light cone effects of the conformal anomaly of massless or light SM fields. In this EFT, in particular in the presence of massless fermions coupled to torsion, Λ is no longer a fundamental constant, whose value appears to be sensitive to ultra high energy physics, but rather a dynamical condensate

described by a classical 4-form field strength and 3-form abelian gauge potential of Secs. IV-V.

The EFT based on the conformal anomaly (3.1) introduces two relevant scalar degrees of freedom to low energy gravity beyond classical GR. This follows from the local form of the effective action of the anomaly $S_{\mathcal{A}}[g; \varphi]$ of (3.10) in terms of the scalar φ conformalon field, which satisfies a fourth order eq. of motion (6.2) or (7.9). Before the addition of $S_{\mathcal{A}}[g; \varphi]$, the 4-form gauge field (4.1) is constrained to be a constant, and the conformal factor of the metric in GR is constrained to be non-propagating by the classical diffeomorphism constraints of Einstein's eqs. Once $S_{\mathcal{A}}[g; \varphi]$ is added to the effective action, the conformal part of the metric mixes with one of the two φ modes, and gives rise to scalar gravitational waves [64] in the first form of the EFT (I) of Sec. VI.

At energies above the lightest fermion mass scale, when at least one species of fermion can be treated as massless, its spin connection coupled to torsion contributes a torsional component to the conformal anomaly (5.8). In that case \widetilde{F} , identified with torsion through (5.7), (5.9), (5.14) becomes sourced by a 3-form current J of (5.12), depending upon the gradient of the scalar conformalon field, and becomes dynamical in the EFT (II) of Sec. VII. It is this second scalar degree of freedom in $S_{\mathcal{A}}[g; \varphi]$ that allows (and requires) the vacuum energy to become a dynamical variable through (4.9) and (7.7).

Thus in the EFT (II) each of the two scalar conformalon degrees of freedom present in the anomaly action for φ mix with, and source the previously classically constrained conformal factor of $g_{\mu\nu}$ and \widetilde{F} , turning them both into full-fledged dynamical degrees of freedom. Assuming the SM, the only new free parameter of the EFT is the constant \varkappa , a kind of gravitational vacuum susceptibility *cf.* Appendix C, determining the coupling of the current J to the gauge field A , field strength F , and Λ_{eff} of (4.9).

For the longstanding problem of the cosmological term when QFT is coupled to gravity, in either form of the EFT the classical state of minimum energy is that of vanishing 4-form condensate: $\widetilde{F} = \widetilde{F}_0 = 0$. By the identification of Λ_{eff} in (4.9), this condition automatically sets the value of the cosmological term to zero in infinite flat Minkowski space. By simply allowing a flat Minkowski solution, this removes one oft-stated obstacle and ‘no-go theorem’ to solution of the ‘cosmological constant problem’ [4, 97]. Moreover $\Lambda_{\text{eff}} = 0$ is required by consistency with the classical or semi-classical Einstein's eqs. (4.10) or (6.3) in infinite flat Minkowski space, in the complete absence of all matter and radiation. The condition (4.10) is independent of UV physics or cutoffs, as simply the unique classical state of minimum energy with zero curvature and zero torsion, and the stable ground state of the EFT of low energy gravity.

Empty flat space with zero external fields is also the unique classical state where the total anomaly

$\mathcal{A} = 0$, so that the WZ anomaly action (3.10)-(3.11) has an additional global shift symmetry under $\varphi \rightarrow \varphi + \varphi_0$ in that state, which enhanced symmetry resembles the 't Hooft naturalness criterion [50].

The consistency condition of $\Lambda_{\text{eff}} = 0$ on the classical condensate in the low energy EFT of gravity in flat space indicates that estimates of vacuum energy as sensitive to UV physics are not applicable if GR, with or without the conformal anomaly addition, is to be a well-behaved EFT at macroscopic scales. Arguments or estimates of the cosmological term in flat space as proportional to the fourth power of a UV cutoff (which break Lorentz invariance) or the fourth power of all masses $\sum_i m_i^4 \ln m_i$ in QFT (as in dimensional regularization) have no physical meaning in the absence of gravitation, as well as being in conflict with observations even on non-cosmological, solar system scales [7].

Instead a way around the 'naturalness' problem of the Λ term is to extend the EFT of gravity beyond classical GR to contain the additional scalar conformal degree(s) of freedom inherent in the conformal anomaly effective action (3.10), and to replace the fixed parameter Λ of the classical theory by Λ_{eff} of (4.9) in terms of a 4-form gauge field. As a result of the 'Maxwell' eq. it satisfies, this abelian gauge field becomes either a free constant of integration, or a fully dynamical degree of freedom of low energy gravity by (5.14) and (7.5), when the conformal anomaly of massless fermions in the presence of spacetime torsion is considered. The setting of an integration constant to zero at the minimum of energy in flat space is a solution of the naturalness problem of the cosmological term that involves no fine tuning of any fixed parameters of the EFT Lagrangian (6.1) of macroscopic gravity.

At lowest order all fields in (6.1) or (7.1) are treated as classical. Quantum loop corrections are then to be computed by the usual EFT method of appending local terms as needed to absorb UV divergences in an expansion in powers of $1/M_{pl}^2$, maintaining the physical meaning of the constants of the lowest order EFT. Thus at one-loop order the energy of the vacuum will continue to be defined to be identically zero in flat space by the consistency condition (4.10) on the constant $\widetilde{F}_0 = 0$, hence $\Lambda_{\text{eff}} = 0$ in the absence of any sources, and with $\varphi = 0$. All formal divergences in $\langle \hat{T}^{\mu\nu} \rangle_{flat}$ in purely flat space, quartic or otherwise, are treated as without physical significance and removed by this consistency condition on \widetilde{F}_0 . There is no sensitivity of this free integration constant, or Λ_{eff} defined in terms of it by (4.9), on UV divergences or UV mass scales.

Following the usual logic of EFT, divergences of quantum loops require the introduction of additional local terms which are higher order in powers of the Riemann curvature tensor and its derivatives, divided by higher powers of a UV scale, presumably the Planck mass scale M_{pl} [22–24], which ultimately limits the range of applicability of the low energy EFT. These higher order effects and their renormalization should be defined so as not to disturb the meaning of the low energy parameters

at lowest order, such as G_N , or ground state boundary condition on the condensate, $\Lambda_{\text{eff}} = 0$ in flat space. Thus divergences in curved space (regulated by any covariant method) proportional to R are absorbed into the renormalized value of $1/G_N$, while logarithmic divergences require the introduction of counterterms proportional to the local R^2 and C^2 curvature invariants, including also the finite logarithmic running of their dimensionless couplings, if desired. These terms and local terms involving still higher numbers of derivatives are not treated as fundamental, but rather as suppressed at energy scales far below the Planck energy $M_{Pl}c^2$, remaining negligibly small at macroscopic distance scales much greater than L_{Pl} , as consistent with existing EFT results [22–24].

One may also turn the EFT logic around, to conjecture that the important role of the conformal anomaly and anomalies in general as windows into the UV, and exceptions to the usual decoupling hypothesis of EFT, may indicate that in the fundamental theory all masses vanish and conformal invariance is restored—broken perhaps only spontaneously, at asymptotically high energies. Speculations of this kind for resolution of the naturalness problem of the cosmological term and possible relation with that of the Higgs mass hierarchy have been advanced by a number of authors, *e.g.* [98–102]. Although no clearly successful complete theory has emerged from these speculations, the idea of fundamental conformal invariance and relation between these large hierarchy problems and the properties of the quantum vacuum rather than UV physics remains intriguing.

That both Λ and Higgs hierarchies may be resolvable only by the consistent inclusion of gravity receives some support from the EFT approach to vacuum energy and the cosmological ‘constant’ as a dynamical condensate proposed in this paper. The role of torsion in describing the condensate by (5.14) as relevant to resolving BH singularities was anticipated perhaps in [103]. Its relation to chiral gauge fields through (5.7), (5.9) and hence potentially chiral symmetry breaking and neutrino mass generation is also intriguing. The microscopic constituents of the gravitational Bose-Einstein condensate (GBEC) described by \widetilde{F} or its superfluid nature still remain to be elucidated [104, 105].

As a more immediate practical matter, the description of Λ_{eff} as a dynamical condensate of a 4-form gauge field in the EFT of Sec. VII makes possible calculations of vacuum energy in numerous applications, first and foremost for a gravitational condensate star interior and surface. For cosmology, the universe as the interior of a gravastar realizes the hypothesis made in Sec. I of automatically relating the effective value of the vacuum energy Λ_{eff} to $3H^2$, and hence to the horizon Hubble scale H^{-1} , with no fine tuning. More realistic cosmological models, with $\Omega_\Lambda < 1$, require dynamical EFT solutions including matter and radiation, rather than a purely static de Sitter vacuum condensate.

The tying of the value of Λ_{eff} to the Hubble scale is clearly relevant to the ‘cosmic coincidence

problem' of the Λ CDM model, and immediately suggests a rather different set of possibilities for cosmological models, in which spatial inhomogeneities and/or boundary conditions at the Hubble scale H^{-1} play an important role. The EFT of the conformal anomaly coupled to the 3-form potential and 4-form abelian field strength term presented in Sec. VII thus provides a distinctly new framework for dynamical dark energy in cosmology based on fundamental theory.

From the form of the anomaly \mathcal{A} in (3.1), any deviation from exact homogeneity and isotropy will lead in general to $F_{\mu\nu}F^{\mu\nu} \neq 0$ for the photon radiation field, and $\text{tr}\{G_{\mu\nu}G^{\mu\nu}\} \neq 0$ for the electroweak and QCD color gauge fields in the unconfined phase of the early universe. This will induce changes in the conformalon field φ through its eq. of motion (7.9), which will then cause the field strength \widetilde{F} and hence the vacuum energy Λ_{eff} to change. After the transition to the confining phase of QCD, baryonic matter will still contain non-vanishing gluonic condensates and thus still act as a source for φ , thereby coupling non-relativistic baryonic matter to dynamical vacuum energy as well.

Thus although φ is *not* an inflaton, it is a dynamical scalar that is well-grounded in QFT of the SM and can produce backreaction effects on the vacuum energy when fluctuations away from exact homogeneity and isotropy are admitted. It permits interaction between both radiation and matter with dynamical dark energy, in which adiabaticity of the matter and radiation components will no longer be satisfied in general, in effect introducing a bulk viscosity into the cosmological fluid. If $\Lambda_{\text{eff}} \propto \widetilde{F}^2$ does not remain constant in the de Sitter phase, deviations from the Λ CDM cosmological model are to be expected, and evolution away from a pure de Sitter phase due to cosmological horizon modes [106] becomes calculable, and testable by the cosmological data of large scale structure.

In addition to removing the singularity and paradoxes of BHs, developing detailed predictions from the EFT proposed will allow study of gravastar stability, normal modes of oscillation, surface modes and 'echoes' that can be tested with gravitational wave and multi-messenger signals from binary merger events, in the increasing data samples expected in the future. The prediction of scalar gravitational waves can also be tested by the coming global array of gravitational wave antennae [107].

The effects of rotational angular momentum have been neglected in the simplest gravastar solution, although the first steps in including those effects have been taken in [37, 38]. The EFT solutions of static or stationary gravastars also leaves unexamined the process of their formation, and in particular the behavior of the stress tensor near the would-be horizon of collapsing matter, which would have to activate the dynamical condensate terms of (7.1)-(7.10). These and many other interesting questions remain to be addressed in the context of the EFT of dynamical vacuum energy proposed in this paper.

Acknowledgements

The author expresses his appreciation to his colleague Prof. Pawel O. Mazur for a critical reading of this paper, and for bringing reference [103] to the author's attention, and to Profs. Ilya L. Shapiro and Joan Solà Peracaula respectively for bringing references [80–82] and [77] to his attention as well.

- [1] A. G. Riess *et al.*, *Astron. Jour.* **116**, 1009 (1998).
- [2] S. Perlmutter *et al.* (Supernova Cosmology Project), *Astrophys. Jour.* **517**, 565 (1999).
- [3] T. M. C. Abbott *et al.*, *Astrophys. Jour.* **872**, L30 (2019).
- [4] S. Weinberg, *Rev. Mod. Phys.* **61**, 1 (1989).
- [5] S. Rugh and H. Zinkernagel, *Stud. Hist. Phil. Sci. B: Stud. Hist. Phil. Mod. Phys.* **33**, 663 (2002).
- [6] I. Antoniadis, P. O. Mazur, and E. Mottola, *New Jour. Phys.* **9**, 11 (2007).
- [7] J. Martin, *Comp. Rend. Phys.* **13**, 566 (2012).
- [8] H. Leutwyler, *Ann. Phys.* **235**, 165 (1994).
- [9] C. P. Burgess, *Ann. Rev. Nucl. Part. Sci.* **57**, 329 (2007).
- [10] L. Álvarez-Gaumé and M. Á. Vázquez-Mozo, in *An Invitation to Quantum Field Theory* (Springer, 2012) pp. 231–260.
- [11] A. Guth, *The Inflationary Universe* (Basic Books, 1998).
- [12] T. Jacobson, *Phys. Rev. D* **48**, 728 (1993).
- [13] J. Preskill, in *Blackholes, Membranes, Wormholes and Superstrings*, edited by S. Kalara and D. V. Nanopoulos (World Scientific, 1992) pp. 22–39, [arXiv:hep-th/9209058](https://arxiv.org/abs/hep-th/9209058).
- [14] G. 't Hooft, *Nucl. Phys. B - Proc. Suppl.* **43**, 1 (1995).
- [15] G. 't Hooft, in *Frontiers of Fundamental Physics*, edited by B. G. Sidharth, F. Honsella, and A. De Angelis (Springer, 2006) pp. 29–29.
- [16] S. D. Mathur, *Class. Quan. Grav.* **26**, 224001 (2009).
- [17] A. Almheiri, D. Marolf, J. Polchinski, and J. Sully, *Jour. High Ener. Phys.* **2013**, 62 (2013).
- [18] S. B. Giddings, *Phys. Today* **66** no. 4, 30 (2013).
- [19] E. Mottola and R. Vaulin, *Phys. Today* **66** no. 11, 9 (2013).
- [20] D. Harlow, *Rev. Mod. Phys.* **88**, 015002 (2016).
- [21] S. W. Hawking, *Phys. Rev. D* **14**, 2460 (1976).
- [22] J. F. Donoghue, *Phys. Rev. D* **50**, 3874 (1994).

- [23] C. P. Burgess, [Living Rev. Rel. 7, 1 \(2004\)](#).
- [24] J. F. Donoghue, in *AIP Conference Proceedings*, Vol. 1483 (2012) pp. 73–94, [arXiv:1209.3511](#).
- [25] D. G. Boulware, [Phys. Rev. D 11, 1404 \(1975\)](#).
- [26] W. G. Unruh, [Phys. Rev. D 14, 870 \(1976\)](#).
- [27] S. M. Christensen and S. A. Fulling, [Phys. Rev. D 15, 2088 \(1977\)](#).
- [28] E. Mottola and R. Vaulin, [Phys. Rev. D 74, 064004 \(2006\)](#).
- [29] M. Giannotti and E. Mottola, [Phys. Rev. D 79, 045014 \(2009\)](#).
- [30] E. Mottola, [Acta Phys. Polon. B 41, 2031 \(2010\)](#), [arXiv:1008.5006 \[gr-qc\]](#).
- [31] P. O. Mazur and E. Mottola, [arXiv:gr-qc/0109035 \(2001\)](#).
- [32] P. O. Mazur and E. Mottola, [Proc. Nat. Acad. Sci. \(U.S.\) 101, 9545 \(2004\)](#).
- [33] R. Penrose, [Phys. Rev. Lett. 14, 57 \(1965\)](#).
- [34] S. W. Hawking and R. Penrose, [Proc. Roy. Soc. London A: Math. Phys. Sci. 314, 529 \(1970\)](#).
- [35] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge Univ. Press, 1973).
- [36] P. O. Mazur and E. Mottola, [Class. Quant. Grav. 32, 215024 \(2015\)](#).
- [37] P. Beltracchi, P. Gondolo, and E. Mottola, [Phys. Rev. D 105, 024001 \(2022\)](#).
- [38] P. Beltracchi, P. Gondolo, and E. Mottola, [Phys. Rev. D 105, 024002 \(2022\)](#).
- [39] C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Princeton Univ. Press, 2017).
- [40] É. Cartan, [Comptes Rendus, Ac. Sc. Paris 174, 593 \(1922\)](#).
- [41] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, [Rev. Mod. Phys. 48, 393 \(1976\)](#).
- [42] P. R. Anderson, C. Molina-París, and E. Mottola, [Phys. Rev. D 67, 024026 \(2003\)](#).
- [43] T. Appelquist and J. Carazzone, [Phys. Rev. D 11, 2856 \(1975\)](#).
- [44] J. S. Schwinger, [Phys. Rev. 128, 2425 \(1962\)](#).
- [45] A. M. Polyakov, [Phys. Lett. B 103, 207 \(1981\)](#).
- [46] D. N. Blaschke, R. Carballo-Rubio, and E. Mottola, [JHEP 12, 153 \(2014\)](#).
- [47] R. Armillis, C. Corianò, and L. Delle Rose, [Phys. Lett. B682, 322 \(2009\)](#).
- [48] A. Bzowski, P. McFadden, and K. Skenderis, [Jour. High Ener. Phys. 2018, 1 \(2018\)](#).
- [49] C. Corianò, M. M. Maglio, and E. Mottola, [Nucl. Phys. B 942, 303 \(2019\)](#).
- [50] G. 't Hooft, [NATO Sci. Ser. B 59, 135 \(1980\)](#).
- [51] J. Wess and B. Zumino, [Phys. Lett. B 37, 95 \(1971\)](#).
- [52] H. Fritzsch, M. Gell-Mann, and H. Leutwyler, [Phys. Lett. B 47, 365 \(1973\)](#).

- [53] S. Treiman, R. Jackiw, and D. J. Gross, *Lectures on Current Algebra and Its Applications* (Princeton Univ. Press, 2015).
- [54] R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Clarendon Press, 2001).
- [55] A. M. Polyakov, *Mod. Phys. Lett. A* **2**, 893 (1987).
- [56] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, *Mod. Phys. Lett. A* **3**, 819 (1988).
- [57] D. M. Capper and M. J. Duff, *N. Cimento A* **23**, 173 (1974).
- [58] M. J. Duff, *Nucl. Phys. B* **125**, 334 (1977).
- [59] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge Univ. Press, 1984).
- [60] S. M. Paneitz *et al.*, *SIGMA* **4**, 036 (2008), arXiv:0803.4331.
- [61] R. J. Riegert, *Phys. Lett. B* **134**, 56 (1984).
- [62] I. Antoniadis and E. Mottola, *Phys. Rev. D* **45**, 2013 (1992).
- [63] I. Antoniadis, P. O. Mazur, and E. Mottola, *Nucl. Phys. B* **388**, 627 (1992).
- [64] E. Mottola, *Jour. High Ener. Phys.* **2017**, 43 (2017).
- [65] P. O. Mazur and E. Mottola, *Phys. Rev. D* **64**, 104022 (2001).
- [66] A. O. Barvinsky and G. A. Vilkovisky, *Phys. Rep.* **119**, 1 (1985).
- [67] A. O. Barvinsky and G. A. Vilkovisky, *Nucl. Phys. B* **333**, 471 (1990).
- [68] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Effective Action in Quantum Gravity* (CRC Press, 1992).
- [69] I. L. Shapiro and A. G. Jacksenaev, *Phys. Lett. B* **324**, 286 (1994).
- [70] K. G. Wilson, *Rev. Mod. Phys.* **47**, 773 (1975).
- [71] K. S. Stelle, *Phys. Rev. D* **16**, 953 (1977).
- [72] E. C. Thomas, F. R. Urban, and A. R. Zhitnitsky, *Jour. High Ener. Phys.* **2009**, 043 (2009).
- [73] A. Aurilia, D. Christodoulou, and F. Legovini, *Phys. Lett. B* **73**, 429 (1978).
- [74] M. Duff and P. van Nieuwenhuizen, *Phys. Lett. B* **94**, 179 (1980).
- [75] A. Aurilia, H. Nicolai, and P. Townsend, *Nucl. Phys. B* **176**, 509 (1980).
- [76] M. Henneaux and C. Teitelboim, *Found. Phys.* **16**, 593 (1986).
- [77] J. S. Peracaula, *Phil. Trans. Roy. Soc. A: Math. Phys. Eng. Sci.* **380** (2022).
- [78] T. Eguchi, P. B. Gilkey, and A. J. Hanson, *Phys. Rep.* **66**, 213 (1980).
- [79] M. Ferraris, M. Francaviglia, and C. Reina, *Gen. Rel. Grav.* **14**, 243 (1982).
- [80] I. L. Shapiro, *Phys. Rep.* **357**, 113 (2002).
- [81] Y. Obukhov, *Nucl. Phys. B* **212**, 237 (1983).

- [82] I. L. Buchbinder, S. D. Odintsov, and I. L. Shapiro, *Phys. Lett. B* **162**, 92 (1985).
- [83] A. Aurilia, F. Legovini, and E. Spallucci, *Phys. Lett. B* **264**, 69 (1991).
- [84] C. M. Will, *Living Rev. Rel.* **17**, 4 (2014).
- [85] I. Antoniadis, P. O. Mazur, and E. Mottola, *Phys. Rev. D* **55**, 4770 (1997).
- [86] P. O. Mazur and E. Mottola, *Nucl. Phys. B* **341**, 187 (1990).
- [87] P. Stöcker *et al.* (The GAMBIT Cosmology Workgroup), *Phys. Rev. D* **103**, 123508 (2021).
- [88] K. Fredenhagen and R. Haag, *Comm. Math. Phys.* **127**, 273 (1990).
- [89] N. A. Chernikov and E. A. Tagirov, *Ann. Inst. H. Poincaré Phys. Théor.* **9**, 109 (1968).
- [90] T. S. Bunch and P. C. W. Davies, *Proc. R. Soc. Lond. A, Math. Phys. Sciences* **360**, 117 (1978).
- [91] E. Mottola, *Phys. Rev. D* **31**, 754 (1985).
- [92] A. Polyakov, *Nucl. Phys. B* **797**, 199 (2008).
- [93] P. R. Anderson and E. Mottola, *Phys. Rev. D* **89**, 104038 (2014).
- [94] P. R. Anderson and E. Mottola, *Phys. Rev. D* **89**, 104039 (2014).
- [95] P. R. Anderson, E. Mottola, and D. H. Sanders, *Phys. Rev. D* **97**, 065016 (2018).
- [96] G. Chapline, E. Hohlfeld, R. B. Laughlin, and D. I. Santiago, *Phil. Mag. B* **81**, 235 (2001).
- [97] F. Klinkhamer and G. E. Volovik, *JETP Lett.* **91**, 259 (2010).
- [98] W. A. Bardeen, *On naturalness in the standard model*, Tech. Rep. (SCAN-9602063, 1995).
- [99] J. D. Bjorken, *Phys. Rev. D* **67**, 043508 (2003).
- [100] K. A. Meissner and H. Nicolai, *Phys. Lett. B* **660**, 260 (2008).
- [101] G. 't Hooft, *Int. Jour. Mod. Phys. D* **24**, 1543001 (2015).
- [102] M. Shaposhnikov and K. Shimada, *Phys. Rev. D* **99**, 103528 (2019).
- [103] R. d'Auria and T. Regge, *Nucl. Phys. B* **195**, 308 (1982).
- [104] G. E. Volovik, *The Universe in a Helium Droplet* (OUP Oxford, 2003).
- [105] P. O. Mazur, *arXiv:astro-ph/0701270* (2007).
- [106] P. R. Anderson, C. Molina-París, and E. Mottola, *Phys. Rev. D* **80**, 084005 (2009).
- [107] Y. Hagihara, N. Era, D. Iikawa, N. Takeda, and H. Asada, *Phys. Rev. D* **101**, 041501 (2020).
- [108] A. Yale and T. Padmanabhan, *Gen. Rel. Grav.* **43**, 1549 (2011).
- [109] E. Seiler, *Phys. Lett. B* **525**, 355 (2002).
- [110] E. Witten, *Nucl. Phys. B* **156**, 269 (1979).
- [111] G. Veneziano, *Nucl. Phys. B* **159**, 213 (1979).
- [112] P. Di Vecchia and G. Veneziano, *Nucl. Phys. B* **171**, 253 (1980).

A. Metrics, Tetrads, Differential Forms, and Hodge Star Dual

In this first Appendix conventions and mathematical details used in the text are collected and catalogued. The metric and curvature conventions used in this paper are those of Misner, Thorne & Wheeler [39]. Greek indices are four-dimensional coordinate (holonomic) indices, while Latin indices refer to local orthonormal tangent space.

In the tetrad or vierbein formalism the metric line element is written

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = e^a \eta_{ab} e^b \quad (\text{A1})$$

with $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ the flat spacetime tangent space Minkowski metric tensor, and $e^a(x)$ are the 1-forms

$$e^a = e^a_\mu dx^\mu \quad \text{satisfying} \quad e^a_\mu e^b_\nu \eta_{ab} = g_{\mu\nu}, \quad g^{\mu\nu} e^a_\mu e^b_\nu = \eta^{ab}. \quad (\text{A2})$$

The dual basis of vectors ν_a satisfy

$$e^a_\mu \nu^\mu_b = \delta^a_b \quad (\text{A3a})$$

$$e^a_\nu \nu^\mu_a = \delta^\mu_\nu \quad (\text{A3b})$$

$$\nu^\mu_a \nu^\nu_b g_{\mu\nu} = \eta_{ab} \quad (\text{A3c})$$

$$\nu^\mu_a \nu^\nu_b \eta^{ab} = g^{\mu\nu} \quad (\text{A3d})$$

defining the orthonormal basis in tangent space. Coordinate indices are lowered (resp. raised) by the metric tensor $g_{\mu\nu}$ (resp. its inverse $g^{\mu\nu}$). Tangent space indices are lowered or raised by the flat Minkowski tensor η_{ab} or η^{ab} . The covariant derivative of the vierbein field in (5.3) is defined with respect to the torsionless Levi-Civita connection, that in holonomic coordinates is the familiar Christoffel symbol (2.2), that is specified entirely by the metric tensor and its first derivatives.

The exterior derivative operator d maps the general p -form

$$Q^{(p)} = \frac{1}{p!} Q^{(p)}_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = Q^{(p)}_{[\mu_1 \dots \mu_p]} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \quad (\text{A4})$$

into the $p + 1$ -form

$$dQ^{(p)} = \frac{1}{p!} \frac{\partial Q^{(p)}_{\mu_1 \dots \mu_p}}{\partial x^\lambda} dx^\lambda \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} = \partial_{[\lambda} Q^{(p)}_{\mu_1 \dots \mu_p]} \partial x^\lambda \wedge dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p} \quad (\text{A5})$$

where the square brackets denote anti-symmetrization with respect to the enclosed indices, and both p and $p + 1$ must be $\leq D$ in D dimensions, by the anti-symmetry of the wedge product.

The Hodge \star dual operator maps the p -form (A4) into the $(4 - p)$ -form

$$\star Q^{(p)} = \frac{1}{(4-p)!} \frac{1}{p!} \varepsilon_{\mu_1 \dots \mu_4} g^{\mu_1 \nu_1} \dots g^{\mu_p \nu_p} Q^{(p)}_{\nu_1 \dots \nu_p} dx^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_4} \quad (\text{A6})$$

in $D=4$ dimensions. We make use of the notation

$$\epsilon_{abcd} = \begin{cases} +1 & \text{if } (a, b, c, d) = P_{\text{even}}(0, 1, 2, 3) \\ -1 & \text{if } (a, b, c, d) = P_{\text{odd}}(0, 1, 2, 3) \\ 0 & \text{any two indices equal} \end{cases} \quad (\text{A7})$$

for the totally anti-symmetric Levi-Civita tensor in the tangent basis, where P_{even} and P_{odd} denote even or odd permutation respectively of the four indices which are its argument. The corresponding tensor in the coordinate basis is denoted by

$$\varepsilon_{\alpha\beta\gamma\lambda} \equiv \epsilon_{abcd} e^a_\alpha e^b_\beta e^c_\gamma e^d_\lambda \quad (\text{A8})$$

which is used to define the volume 4-form

$$\star \mathbb{1} = \frac{1}{4!} \epsilon_{abcd} e^a \wedge e^b \wedge e^c \wedge e^d = \frac{1}{4!} \varepsilon_{\alpha\beta\gamma\lambda} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\lambda \quad (\text{A9})$$

dual to the unit scalar, and the 4-volume element

$$\int \star \mathbb{1} = \int \varepsilon_{txyz} dt dx dy dz = \int \epsilon_{abcd} e^a_t e^b_x e^c_y e^d_z dt dx dy dz = \int \det(e^a_\mu) d^4x = \int \sqrt{-g} d^4x \quad (\text{A10})$$

where $x^\mu = (t, x, y, z)$ are general spacetime coordinate labels (not necessarily Minkowski).

Since by (A7) $\epsilon_{0123} = 1$, raising all indices by use of the Minkowski metric η^{ab} changes its sign, so that $\epsilon^{0123} = -1$, which leads to

$$\epsilon^{abcd} \epsilon_{mnr s} = -4! \delta^a_{[m} \delta^b_n \delta^c_r \delta^d_{s]} \quad (\text{A11})$$

and (4.5), (5.11) and (5.14) of the text, as well as

$$\star \star Q^{(p)} = (-)^{p+1} Q^{(p)} \quad (\text{A12})$$

for the double dual of a p -form. Since $\varepsilon_{\alpha\beta\gamma\lambda} \propto \sqrt{-g}$ it follows that $\varepsilon^{\alpha\beta\gamma\lambda} \propto 1/\sqrt{-g}$ and

$$\partial_\mu (\varepsilon^{\alpha\beta\gamma\lambda} \sqrt{-g}) = 0 \quad (\text{A13})$$

which also can be verified directly from $\varepsilon^{\alpha\beta\gamma\lambda} = g^{\alpha\alpha'} g^{\beta\beta'} g^{\gamma\gamma'} g^{\lambda\lambda'} \varepsilon_{\alpha'\beta'\gamma'\lambda'}$ and properties (A1)-(A3), with the definition (A8). These properties of the ε tensor are used at several points in the main text, *e.g.* in (5.14), (5.15) and (7.6) to convert covariant derivatives to coordinate partial derivatives and *vice versa*.

B. The Chern-Simons 3-Form and Topological Current of the Anomaly

Of the several terms in the trace anomaly (3.1), the Euler-Gauss-Bonnet invariant E in (3.2) is distinguished by its topological character. Its integral is a topological invariant insensitive to local variations, and therefore can be related to global macroscopic effects, analogous to the index theorems associated with the $\varepsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu}$ topological density of the axial anomaly [54]. The topological character of E means that just as the axial anomaly density can be expressed as the total divergence of a gauge dependent Chern-Simons current, insensitivity of the integral of E to local variations that preserve topology implies that E also is a total divergence of a topological current, $E = \nabla_\mu \Omega^\mu$, with Ω^μ dependent upon the choice of local Lorentz frame, through the $SO(3, 1)$ gauge connection.

The explicit form of the topological 3-form gauge field associated with E follows from its relation to the 4-form field strength

$$\mathcal{F} \equiv \varepsilon_{abcd} \mathcal{R}^{ab} \wedge \mathcal{R}^{cd} = \frac{1}{4} \varepsilon_{abcd} R^{ab}{}_{\alpha\beta} R^{cd}{}_{\gamma\lambda} dx^\alpha \wedge dx^\beta \wedge dx^\gamma \wedge dx^\lambda \quad (\text{B1})$$

by the Hodge star \star dual operation

$$\star \mathcal{F} = \varepsilon^{\alpha\beta\gamma\lambda} \left(\frac{1}{4} \varepsilon_{abcd} R^{ab}{}_{\alpha\beta} R^{cd}{}_{\gamma\lambda} \right) = \frac{1}{4} \varepsilon_{abcd} \varepsilon^{mnr s} R^{ab}{}_{mn} R^{cd}{}_{rs} = - \left(R_{abcd} R^{abcd} - 4 R_{ab} R^{ab} + R^2 \right) = -E \quad (\text{B2})$$

where the Latin a, b, \dots are tangent space indices and the Greek α, β, \dots are spacetime coordinate indices respectively. The minus sign in (B2) is the result of Lorentzian metric signature *cf.* (A11).

The curvature 2-form \mathcal{R}^{ab} in (B1) is defined by the Cartan structure eq.

$$\mathcal{R}^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b \equiv \frac{1}{2} R^{ab}{}_{\mu\nu} dx^\mu \wedge dx^\nu \quad (\text{B3})$$

that defines the Riemann curvature components $R^ab_{\mu\nu}$ in terms of the affine connection 1-form

$$\omega^{ab} = -\omega^{ba} = \omega^{ab}_{\mu} dx^{\mu} \quad (\text{B4})$$

which specifies the law of parallel transport of orthonormal frames in tangent space. This allows ω^{ab} to be regarded as a local gauge potential for the Lie algebra of the $\mathcal{G} = SO(3, 1)$ Lorentz group, in close analogy to Yang-Mills gauge potentials for any internal group \mathcal{G} , for which \mathcal{R}^{ab} would be the 2-form field strength tensor.

The topological nature of the integral of E follows from the fact that the 4-form \mathcal{F} dual to it by (B2) is exact, *i.e.* $\mathcal{F} = d\mathcal{A}$, as in (4.3), where \mathcal{A} here is the $SO(3, 1)$ Lorentz frame dependent 3-form [108]

$$\mathcal{A} = \epsilon_{abcd} \left(\omega^{ab} \wedge d\omega^{cd} + \frac{2}{3} \omega^{ab} \wedge \omega^{ce} \wedge \omega^{fd} \eta_{ef} \right) \quad (\text{B5})$$

which has the spacetime coordinate components

$$\mathcal{A}_{\alpha\beta\gamma} = 3! \epsilon_{abcd} \left(\omega^{ab}_{[\alpha} \partial_{\beta} \omega^{cd}_{\gamma]} + \frac{2}{3} \omega^{ab}_{[\alpha} \omega^{ce}_{\beta} \omega^{fd}_{\gamma]} \eta_{ef} \right) \quad (\text{B6})$$

completely anti-symmetrized in its three indices α, β, γ . These relations imply, *cf.* (A13)

$$E = -(\star d\mathcal{A}) = \frac{1}{3!} \epsilon^{\alpha\beta\gamma\mu} \partial_{\mu} \mathcal{A}_{\alpha\beta\gamma} = \frac{1}{3!} \frac{1}{\sqrt{-g}} \partial_{\mu} \left(\sqrt{-g} \epsilon^{\alpha\beta\gamma\mu} \mathcal{A}_{\alpha\beta\gamma} \right) = \frac{1}{3!} \nabla_{\mu} \left(\epsilon^{\alpha\beta\gamma\mu} \mathcal{A}_{\alpha\beta\gamma} \right) \quad (\text{B7})$$

demonstrating that the integrand $\sqrt{-g} E$ is in fact a total derivative of a coordinate frame dependent abelian current $\Omega^{\mu} = \sqrt{-g} \epsilon^{\alpha\beta\gamma\mu} \mathcal{A}_{\alpha\beta\gamma} / 3!$, just as the topological density $\epsilon_{\alpha\beta\mu\nu} F^{\alpha\beta} F^{\mu\nu}$ of the axial anomaly is the total derivative of the gauge dependent Chern-Simons current.

To show that \mathcal{A} is indeed an abelian gauge field, consider the response of the gauge connection to an infinitesimal local $SO(3, 1)$ tangent frame rotation

$$\delta_{\theta} \omega^{ab} = 2 \omega^{c[a} \theta^{b]}_c + d\theta^{ab}, \quad \delta_{\theta} \mathcal{R}^{ab} = 2 \mathcal{R}^{c[a} \theta^{b]}_c \quad (\text{B8})$$

where $\theta^{ab}(x) = -\theta^{ba}(x)$ are the 6 functions of the local Lorentz frame transformation. A short exercise then shows that the gauge potential 3-form \mathcal{A} of (B6) transforms under (B8) as the exact differential

$$\delta_{\theta} \mathcal{A} = \epsilon_{abcd} d\theta^{ab} \wedge d\omega^{cd} = d \left(\epsilon_{abcd} \theta^{ab} d\omega^{cd} \right) \quad (\text{B9})$$

and hence $\delta_\theta \mathcal{F} = 0$, so that the corresponding abelian field strength tensor is invariant under the particular form of the local 2-form gauge parameter $\Theta = \epsilon_{abcd} \theta^{ab} d\omega^{cd}$. This gauge transformation establishes \mathcal{A} as a Chern-Simons 3-form and abelian gauge potential.

Since in order for the transformation (B9) to be non-null, Θ itself must be co-exact, *i.e.* not itself an exact 2-form, the gauge transformation (B9) has just 3 independent components. This means that of the 4 independent components of \mathcal{A} , three of them are pure gauge, and only one is gauge invariant under (B8). Hence the single scalar $\star \mathcal{F}$ carries the full gauge invariant field content of the 3-form \mathcal{A} .

Referring to (3.1)-(3.10), we observe that the combination $E - \frac{2}{3} \square R$ occurs in the anomaly and anomaly action, and that $\square R$ is also (trivially) a total derivative of a local current $\nabla_\mu R$. Thus it is natural to combine those terms in \mathcal{A} into a single 3-form potential, making the replacement

$$\begin{aligned} \mathcal{A} &\rightarrow \mathcal{A}' = \mathcal{A} + \frac{1}{3} \epsilon_{abcd} \star d \star (\mathcal{R}^{ab} \wedge e^c \wedge e^d) \\ &= \epsilon_{abcd} \left(\omega^{ab} \wedge d\omega^{cd} + \frac{2}{3} \omega^{ab} \wedge \omega^{ce} \wedge \omega^{fd} \eta_{ef} \right) + \frac{1}{3} \epsilon_{abcd} \star d \star (\mathcal{R}^{ab} \wedge e^c \wedge e^d) \end{aligned} \quad (\text{B10})$$

where the additional term is chosen so that

$$E - \frac{2}{3} \square R = -(\star d \mathcal{A}') \quad (\text{B11})$$

for the linear combination appearing in (3.1)-(3.10), in place of (B7).

As a result of (B11) the term in the anomaly effective action linear in the conformalon φ and $E - \frac{2}{3} \square R$ can be integrated by parts so that

$$\frac{b'}{2} \int d^4 x \sqrt{-g} \left(E - \frac{2}{3} \square R \right) \varphi = -\frac{b'}{2 \cdot 3!} \int d^4 x \sqrt{-g} \mathcal{A}'_{\alpha\beta\gamma} \varepsilon^{\alpha\beta\gamma\mu} \partial_\mu \varphi \quad (\text{B12})$$

up to a surface term which does not affect local variations and may be taken to vanish for suitable boundary conditions. Then defining the 3-current

$$\mathcal{J}^{\alpha\beta\gamma} \equiv -\frac{b'}{2} \varepsilon^{\alpha\beta\gamma\mu} \partial_\mu \varphi \quad (\text{B13})$$

the term (B12) in (3.10) can be expressed in the form

$$\mathcal{S}_{\text{int}} = \frac{1}{3!} \int d^4 x \sqrt{-g} \mathcal{J}^{\alpha\beta\gamma} \mathcal{A}'_{\alpha\beta\gamma} \quad (\text{B14})$$

analogous to a $J \cdot A$ interaction of ordinary electromagnetism, and (5.13) of the text, but with the 3-form $\mathcal{A}'_{\alpha\beta\gamma}$ and $\mathcal{J}^{\alpha\beta\gamma}$ instead of (5.11) and (5.12).

Despite this suggestive analogy to conventional electrodynamics, neither \mathcal{A} defined by (B5) nor \mathcal{A}' defined by (B10) are independent of the metric, and hence the assumption of metric independence needed to obtain the generalized ‘Maxwell’ eq. for \mathcal{F} or \mathcal{F}' by independent variation of their 3-form potentials is not possible. This is clear from the fact that the E and $\square R$ terms of the conformal anomaly (3.1) are given in a Riemannian background manifold, assuming zero torsion, so that in this case they are determined entirely in terms of the metric and its derivatives, and are not independent of it. This dependence on the metric is also in conflict with the assumption of Sec. IV needed to obtain (4.7) and the equivalence (4.9) to an effective cosmological constant vacuum energy. Metric independent variation of the 3-form potential and (4.4) of Sec. IV is allowed only if A is ‘freed’ from the metric, which is possible only in the presence of *a priori* non-zero torsion, as is explicitly the case with the identification (5.11) of A in section V.

C. Topological and Torsional Susceptibility of the Gravitational Vacuum

In this Appendix we consider the physical interpretation of the parameter \varkappa in (4.4), which depends upon the identification of the 3-form gauge field and associated field strength F .

If \mathcal{A} associated with the topological density E and the conformal anomaly were to be identified with the 3-form A of (4.3), the parameter \varkappa of (4.4) would acquire the interpretation of the topological susceptibility of the gravitational vacuum. This follows by close analogy with the cases of the topological susceptibility of the QED₂ vacuum, and chiral susceptibility of QCD.

In QED₂ $\langle \widetilde{F}(x)\widetilde{F}(y) \rangle = e^2 \delta^2(x-y)$, and its Fourier transform at $k^2=0$ (or indeed any k in the absence of charged sources) is simply the constant e^2 [109]. Likewise for the free action (4.4) in $D=4$ flat space we have

$$\langle \widetilde{F}(x)\widetilde{F}(y) \rangle_0 = \varkappa^4 \delta^4(x-y) \quad \text{and} \quad (\text{C1a})$$

$$\chi_{F,0}(k^2) = \int d^4x e^{ik \cdot (x-y)} \langle \widetilde{F}(x)\widetilde{F}(y) \rangle_0 = \varkappa^4 = \chi_{F,0}(0) \quad (\text{C1b})$$

a finite constant. Since this free correlator is computed with the source current of (5.12) set to zero, it corresponds to $b' = 0$, where all the matter fields contribution to the conformal anomaly are neglected, as in the quenched limit of QCD, where an analogous expression for the chiral susceptibility holds [109–112]. Once matter vacuum polarization effects are taken into account $\chi_F(k^2) \neq \chi_{F,0}(k^2)$

will no longer be independent of k^2 . However the limit

$$\lim_{k^2 \rightarrow \infty} \chi_F(k^2) = \lim_{k^2 \rightarrow \infty} \chi_{F,0}(k^2) = \kappa^4 \quad (\text{C2})$$

remains, as continuing to reflect the local $\delta^4(x-y)$ short distance correlator of the free action (4.4).

This result appears very simply for the action (4.4), trivially in fact, if F were to be identified with \mathcal{F} . However it is worthwhile to pause to consider its meaning for the original Euler-Gauss-Bonnet invariant E , or if one were to try to calculate the topological susceptibility

$$\chi_E(k^2) = \int d^4x e^{ik \cdot x} \langle E(x)E(0) \rangle \quad (\text{C3})$$

directly in terms of the curvature invariants. In that case one would encounter the correlator of two dimension-four operators, with the expected short distance singularity of $1/x^8$ as $x \rightarrow 0$. Thus the integral in (C3) is undefined in perturbative quantum gravity and badly (in fact, quartically) UV divergent. To define it requires promoting the correlator to a distribution with $a_2 \square^2 \delta^4(x)$, $a_1 \square \delta^4(x)$ and $a_0 \delta^4(x)$ local contact terms added with arbitrary finite coefficients a_2, a_1, a_0 , as in [109]. In more physical terms this corresponds to adding three local counterterms to the action in order to make the three subtractions necessary to remove the quartic, quadratic and logarithmic divergences from (C3), and thence to obtain a finite renormalized result in terms of these three finite but unknown parameters. In the quenched approximation where the remaining finite terms vanish, only the local δ -functions and derivatives thereof remain, and for $k^2 = 0$, $\chi_{E,0}(0) = a_0 = \kappa^4$. These remarks continue to apply if E in (C3) is replaced by the linear combination $E - \frac{2}{3} \square R$, with \mathcal{A} modified to \mathcal{A}' as in (B5)-(B11).

Thus the κ^4 in the action (4.4) with the identification of the 4-form dual \widetilde{F} with E or $E - \frac{2}{3} \square R$ parametrizes the logarithmic short distance $\kappa^4 \delta^4(x)$ renormalized singularity of the original Euler-Gauss-Bonnet invariant E or $E - \frac{2}{3} \square R$. Because of (C1b)-(C2), this is the leading order effect of quantum gravitational vacuum fluctuations at short distances that is physically relevant to the $k^2 \rightarrow 0$ low energy (light cone) correlations of the topological Euler density.

The identification of E or $E - \frac{2}{3} \square R$ with \widetilde{F} also effectively removes the quartic divergences arising in (C3), since F involves just one derivative of the gauge potential A in the low energy EFT, and F/κ^2 is a quantum operator of mass dimension 2 in terms of A/κ^2 , in contrast to E which is fourth order in metric derivatives. By this accounting (4.4) is a dimension 4 (rather than dimension 8) operator which is marginally IR relevant in the Wilsonian EFT sense in $D=4$, just as (4.6) is in $D=2$. In the

QED₂ Schwinger model case there are no UV divergences whatsoever and e^2 is a UV finite coupling, despite having dimensions of $(mass)^2$. This may provide an interesting prototype of how parameters with positive mass dimensions can nevertheless remain finite and insensitive to UV corrections. If the matter contributions to the vacuum polarization self-energy $\int d^4x e^{ik \cdot (x-x')} \langle J^{\alpha\beta\gamma}(x) J^{\lambda\mu\nu}(x') \rangle$ are also UV finite, as suggested by its bosonized form (5.12), which converts this self-energy to a classical *tree* graph in terms of φ , just as occurs in $D=2$ [46], then the topological susceptibility \varkappa will also be UV finite in $D=4$. This possibility merits an independent investigation.

Despite these suggestive analogies, in the end the identifications (5.11) and (5.14) were made in the text, so that the 4-form gauge field dual \tilde{F} is identified not with E or $E - \frac{2}{3} \square R$ and the topological character of these quantities in ordinary Riemannian geometry, but with that of the torsion dependent quantity $\nabla_\mu V^\mu$ by (5.7), (5.9), (5.14). Analogy with the chiral susceptibility of QCD suggests that \varkappa should be associated then with a *torsional susceptibility* of the gravitational vacuum and torsional topology of $\int \nabla_\mu V^\mu$ instead. Associated instantons and non-trivial vacuum structure [110–112] would then require a non-trivial topology of the mappings of the base space of the spacetime manifold to the target space of spin connections $\omega^{ab}_\mu(x)$ including torsion, which brings to mind the work of [103] on the topology of Einstein-Cartan spaces with torsion. This clearly is an interesting topic in its own right deserving of further investigation.

Although one might expect the distance scale $1/\varkappa$ of non-trivial torsional vacuum topology change to be of order L_{Pl} , and the value of \varkappa to be of order of M_{Pl} , there is no *a priori* relation between the two scales. They are initially distinct, just as Λ_{QCD} and f_π are in QCD, to become possibly related only in a UV complete theory of quantum gravity. Otherwise \varkappa and M_{Pl} are treated as independent and unrelated dimensionful constants in the low energy EFT of gravity proposed in this paper.