PERIODIC BCs AND G_2 COSMOLOGY

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today

Abstract

Standard cosmology has been very successful in describing current observations up to various **tensions**. The viewpoint taken here is that exotic ideas should not be exclusively considered until conventional GR (in all of its glory) has been fully studied. For example, the assumptions for the existence of exact periodic boundary conditions (appropriate on scales comparable to the homogeneity scale) used in actual numerical simulations, necessarily imply that the spatial curvature is neglible. We wish to study the effect of spatial curvature and periodic boundary conditions numerically, particularly in the special case of G_2 models, where some analytical and qualitative results are possible.

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[Lim] H. van Elst, C. Uggla, and J. Wainwright, Class. Quantum Grav. **19** 51 (2002) [arXiv:gr/qc/0107041].
[CE] A. A. Coley and G. F. R. Ellis, Class. Quant. Grav. **37** 013001 (2020) [arXiv:1909.05346].

1 Introduction [CE]

Cosmology concerns the large scale behaviour of the Universe within GR. The “Cosmological Principle”, which asserts that on large scales the Universe can be well–modeled by a solution to Einstein’s field equations (EFE) which is spatially homogeneous and isotropic, leads to the background Friedmann-Lemaıtre-Robertson-Walker (FLRW) model (with constant spatial curvature) with the cosmological constant, Λ, representing dark energy and CDM is the acronym for cold dark matter (or so-called ΛCDM concordance cosmology or standard cosmology for short). Early universe inflation is often regarded as a part of the standard model. The background spatial curvature of the universe, characterized by the normalized curvature parameter, is predicted to be negligible in inflationary models. Regardless of whether inflation is regarded as part of the standard model or not, spatial curvature is ”assumed” zero.

One of the greatest challenges in cosmology is understanding the origin of the structure of the universe. Under the hypothesis that cosmic structure grew out of small initial fluctuations, we can then study their evolution on sufficiently large scales using linear perturbation theory (LPT). The spatially inhomogeneous perturbations exist on the uniform flat FLRW background spacetime. Cosmic inflation provides a causal mechanism for primordial cosmological perturbations, through the generation of quantum fluctuations in the inflaton field, which act as seeds for the observed anisotropies in the cosmic microwave background (CMB) and large scale structure of our universe. At late times and sufficiently small scales (much smaller than the Hubble scale) fluctuations of the cosmic density are not small. LPT is then not adequate and clustering needs to be treated non-linearly. Usually this is studied with (non-relativistic) N-body simulations. Recently cosmological non-linear perturbations have been studied at second-order and non perturbative relativistic effects have been studied computationally.

Standard cosmology has been very successful in describing current observations up to various possible anomalies, which includes the tension between the recent determination of the local value of the Hubble constant based on direct measurements of supernovae and the value derived from the most recent CMB data. In addition, since the Universe is not isotropic or spatially homogeneous on local scales, the effective gravitational FE on large scales should perhaps be obtained by averaging the EFE of GR, after which a smoothed out macroscopic geometry and macroscopic matter fields is obtained. The averaging of the EFE for local inhomogeneities can lead to significant dynamical backreaction effects on the average evolution of the Universe at the level of 1 %.

1.1 Spatial curvature

In standard cosmology the spatial curvature is assumed to be constant and zero (or at least very small). But there is, as yet, no fully independent constraint with an appropriate accuracy that gaurentees a value for the magnitude of the effective normalized spatial curvature Ω_k of less than approximately 0.01. Moreover, a small non-zero measurement of Ω_k at such a level perhaps indicates that the assumptions in the standard model are not satisfied. It has also been increasingly emphasised that if the geometry of the universe does deviate, even slightly, from the standard FLRW geometry, then the spatial curvature will no longer necessarily be constant and any effective spatial flatness may not be preserved.

It is necessary to make assumptions to derive models to be used for cosmological predictions and comparison with observational data. But it is important to check whether the assumptions "put in" affect the results that "come out". In addition, we can only confirm the consistency of assumptions and we cannot rule out alternative explanations.

The assumption of a FLRW background on cosmological scales presents a number of problems. In particular, the assumptions that underscore the use of a 1+3 spacetime split and a global time and a background inertial coordinate system (Gaussian normal coordinates which are approximately Cartesian and orthogonal) over a complete Hubble scale 'background' patch in the standard model lead to the simple conditions that the spatial curvature must be very small. *The assumptions for the existence of exact periodic boundary conditions (appropriate on scales comparable to the homogeneity scale) imply necessarily that the spatial curvature is exactly zero. In the actual standard model the Universe is taken to be simply connected and hence the background is necessarily flat.* Any appropriate approximation will amount to Ω_k being less than the perturbation (e.g., LPT) scale.

There are also assumptions behind the weak field approach, the applicability of perturbation theory, Gaussian initial conditions, etc., that include neglecting spatial curvature. It is often claimed that backreaction can be neglected, but in LPT the fluctuations are assumed Gaussian, which means that at the linear level all averages are zero by construction. Thus, in standard cosmology the spatial curvature is assumed to be zero, or at least very small and at most first order in terms of the perturbation approximation, in order for any subsequent analysis to be valid. Any prediction larger than this indicates an inconsistency in the approach. The standard model cannot be used to predict a small spatial curvature.

[Recently measured temperature and polarization power spectra of the CMB and direct measurements of the spatial curvature Ω_k using low-redshift data such as supernovae, baryon acoustic oscillations and Hubble constant observations, hint at a non-flat (closed) model with $\Omega_k \sim 1\%$.]

1.2 Overview

The observable part of the Universe is not exactly spatially homogeneous and isotropic on any spatial scale. From a practical point of view, one is interested in models that are "close to FL" in some appropriate dynamical sense (Friedmann–Lemaıtre (FL) is used here, rather than FLRW, since the solution is regarded as an equilibrium state). The usual way to study deviations from an FL model is to apply linear perturbation theory. However, it is not known how reliable the linear theory is. Recently numerical cosmology has been used.

- Viewpoint: Should not study exotic ideas until conventional GR has been fully studied.
- Consider spatially inhomogeneous models (more general than FL). [perturbed models].
- In preliminary numerical runs the spatial curvature remains exceptionally small, particularly in the nhbd of the boundary [and the average spatial curvature goes down as cell size increase.]
- Do for special inhomogeneous model.

We wish to study the effect of spatial curvature and periodic boundary conditions numerically. In the special case of G_2 models some analytical and qualitative results are possible.

2 G_2 models [Lim]

The simplest spatially inhomogeneous cosmological models have two commuting Killing vector fields (i.e., models admitting a 2-parameter Abelian isometry group acting transitively on spacelike 2-surfaces), which thus have one degree of freedom as regards spatial inhomogeneity; such G_2 cosmologies are governed by the EFE evolution eqns. which are partial differential equations (PDE) in two independent variables. In the geometry of the general G_2 class, all metric quantities depend only on the time coordinate t and spatial coordinate x (and subscripts denote partial differentiation). [At any instant of time, the state of a G_2 cosmology is described by a finite-dimensional dynamical state vector of functions of the spatial coordinate x. The evolution of a G_2 cosmology is thus described by an orbit in this infinite-dimensional dynamical state space].

Let us present the G_2 evolution system in terms of the timelike area gauge in the Gowdy subcase $\Sigma_2 = 0$. We assume dust and a single non-zero tilt component, v, where the cosmological constant is zero, and we consider appropriate initial data for future dynamics close to 'FL'.

Our choice of variables are the scale-invariant β-normalized variables in the orthonormal frame formalism, in order to obtain the evolution equations as a system of PDE in first-order symmetric hyperbolic (FOSH) format (which also provides a natural framework for the numerical studies).

$$\begin{aligned}
\partial_t E_1{}^1 &= (q + 3\Sigma_+)\, E_1{}^1 && (1)\\
\partial_t \Sigma_- + E_1{}^1\, \partial_x N_\times &= (q + 3\Sigma_+ - 2)\, \Sigma_- + 2\sqrt{3}\, \Sigma_\times^2 - 2\sqrt{3}\, N_-^2 && (2)\\
\partial_t N_\times + E_1{}^1\, \partial_x \Sigma_- &= (q + 3\Sigma_+)\, N_\times && (3)\\
\partial_t \Sigma_\times - E_1{}^1\, \partial_x N_- &= (q + 3\Sigma_+ - 2 - 2\sqrt{3}\Sigma_-)\, \Sigma_\times - 2\sqrt{3}\, N_\times\, N_- && (4)\\
\partial_t N_- - E_1{}^1\, \partial_x \Sigma_\times &= (q + 3\Sigma_+ + 2\sqrt{3}\Sigma_-)\, N_- + 2\sqrt{3}\, \Sigma_\times\, N_\times && (5)
\end{aligned}$$

$$(\partial_t + v\, E_1{}^1\, \partial_x)\, \Omega + \Omega E_1{}^1\, \partial_x v = 2\, \Omega\, \left[(q+1) - \tfrac{1}{2}\,(1 - 3\Sigma_+)\,(1 + v^2) - 1\right] \quad (6)$$

$$(\partial_t + v\, E_1{}^1\, \partial_x)\, v = (1 - v^2)\left[\, 3(N_\times \Sigma_- - N_- \Sigma_\times) + \tfrac{3}{2}\Omega v - (1 - 3\Sigma_+)\, v\right] \quad (7)$$

where

$$(q + 3\Sigma_+) \equiv 2 - \tfrac{3}{2}\,(1 - v^2)\, \Omega \quad (8)$$

and

$$\Sigma_+ \equiv \tfrac{1}{2}\,(1 - \Sigma_-^2 - N_\times^2 - \Sigma_\times^2 - N_-^2 - \Omega). \quad (9)$$

Note that in the present case we have from Eqs. (8) and (9) that $q \geq \frac{1}{2}$, which guarantees that β is single signed and is monotone for small ϵ. The (negative) spatial curvature $(-\Omega_k)$ is defined via

$$\Omega_k \equiv N_\times^2 + N_-^2. \quad (10)$$

2.0.1 Initial data

Since we have an unconstrained FOSH system no constraints need be satisfied and we freely specify $\{E_1{}^1,\ \Sigma_-,\ N_\times,\ \Sigma_\times,\ N_-,\ \Omega, v\}$ at $t=0$ on $\{-L \le x \le L\}$.
We shall consider initial data close to a flat FLRW model of the form (for small $\epsilon \sim 10^{-4}$):
$\{E_1{}^1 = 1 + \epsilon^2 \tilde{E}\ \}$,
$\{\Sigma_- = \epsilon\tilde{\Sigma}_-, N_\times = \epsilon\tilde{N}_\times, \Sigma_\times = \epsilon\tilde{\Sigma}_\times, N_- = \epsilon\tilde{N}_-\}$,
$\{\Omega = 1 - \epsilon^2\tilde{\Omega}, v = \epsilon^2\tilde{v}\}$.
Note that it follows that
$\{q = \frac{1}{2} + \epsilon^2\tilde{q}, \Sigma_+ = \epsilon^2\tilde{\Sigma}_+, \Omega_k = \epsilon^2\tilde{\Omega}_k\}$.

The initial data is given by (for example) $\epsilon\tilde{\Sigma}_-$ at $t=0$ (etc.).

2.0.2 "Linear" regime (small ϵ)

We integrate eqn. (1) to obtain:

$$\tilde{E} = e^{\frac{t}{2}}[1 + o(\epsilon^2)], \tag{11}$$

where we have normalized $\tilde{E}$ to be unity at $t=0$. Eqns. (6,7) also constitute o(ϵ^2) corrections to zeroth order evolution equations. In the "linear regime", $\Omega_k \sim o(\epsilon^2)$.

2.0.3 Shear and curvature

Eqns. (2 - 5) represent the first order evolution eqns. for the normalized shear and curvature variables with second order corrections. We immediately see the growing modes for the normalized curvature variables $\sim e^{\frac{t}{2}}$ and the decaying modes for the normalized shear variables $\sim e^{-\frac{3t}{2}}$, corresponding to the familiar eigenvalues $\{\frac{1}{2}, -\frac{3}{2}\}$ for the FL "saddle point" solution [so that the growth of the shear is suppressed relative to that of the spatial curvature in the initial linear regime].

Solving eqns. (2 - 5) we obtain (for sufficiently large L):

$$\tilde{\Sigma}_- = e^{-\frac{3t}{2}}[\sigma_- + \epsilon\bar{\Sigma}_-], \tag{12}$$
$$\tilde{N}_\times = e^{\frac{t}{2}}[\nu_\times + \epsilon\bar{N}_\times], \tag{13}$$
$$\tilde{\Sigma}_\times = e^{-\frac{3t}{2}}[\sigma_\times + \epsilon\bar{\Sigma}_\times], \tag{14}$$
$$\tilde{N}_- = e^{\frac{t}{2}}[\nu_- + \epsilon\bar{N}_-], \tag{15}$$

where $\{\sigma_-, \sigma_\times, \nu_-, \nu_\times\}$ are slowly varying (and, for example, $\partial_x\Sigma_- \sim (\epsilon e^{\frac{t}{2}})\bar{\Sigma}'_-$, $\partial_x\Sigma_\times \sim (\epsilon e^{\frac{t}{2}})\bar{\Sigma}'_\times$, where a prime denotes ∂_x). In the regime in which the shear is sub-dominant, these quantities are constant.

2.0.4 Spatial curvature

From eqns. (3, 5), to $o(\epsilon)$ eqn. (10) becomes:

$$\partial_t \Omega_k = \Omega_k, \tag{16}$$

so that Ω_k remains small (and of second order initially). More precisely, to $o(\epsilon^4)$:

$$\partial_t \Omega_k = (\epsilon e^{\frac{t}{2}})^2 [(\nu_-^2 + \nu_\times^2) + \epsilon(\nu_\times \bar{N}_\times + \nu_- \bar{N}_-) + 4\sqrt{3}(\epsilon e^{\frac{t}{2}})e^{-2t}(\sigma_- \nu_- + \sigma_\times \nu_\times) + (\epsilon e^{\frac{t}{2}})e^{-2t}(\nu_- \bar{\Sigma}'_\times + \nu_\times \bar{\Sigma}'_-). \tag{17}$$

Clearly to second order we duplicate eqn. (16); the leading order correction to this eqn. comes from the term $\epsilon(\nu_- \bar{N}_\times + \nu_\times \bar{N}_-)$, which corresponds to eqn. (17) to next order. The remaining terms are of order $o(\epsilon e^{-2t})$ and $o(\epsilon^2)$. For early times $\partial_t \Omega_k > 0$, and so the magnitude of the spatial curvature grows. The sign of the next order terms is not necessarily positive.

Note that at $X = L$, $\partial_t \Omega_k = 0$.

2.0.5 Boundary

The initial data for any variable X on $[-L, L]$ can be Fourier decomposed:

$$X = c_0 + \Sigma s_n sin(\frac{2\pi n}{L} x) + \Sigma c_n cos(\frac{2\pi n}{L} x), \tag{18}$$

where $\{s_n, c_n\}$ are independent of x (and summation is from $n = 1 - N_L$), and there is a fixed small scale cutoff N_0 so that $N_L < L N_0$. Note that at $t = 0$ the average values are $< X >= c_0$, which is usually taken to be zero, and $< X' >= 0$.

One can consider periodic boundary conditions at $X = \pm L$. At the (periodic) boundary $X = L$, we have that:

$$X(L) = c_0 + \Sigma c_n, \quad X'(L) = \frac{2\pi}{L} x \Sigma n s_n, \tag{19}$$

As the cell size increases ($L \to \bar{L}, N_{\bar{L}} \leq \bar{L} N_0$), the number of terms in the summations go up (by $\bar{L}/L$) and the size of each Fourier coefficient goes down (by $L/\bar{L}$), so that $X(\bar{L}) \sim X(L)$ and $X'(\bar{L}) \sim \frac{L}{\bar{L}} X'(L)$, so that spatial gradients (and especially their average values) of X decrease relative to X on the boundary as the cell size increases.

2.0.6 Numerical methods

One numerical method for resolving small scale structure is adaptive mesh refinement (AMR). However, if we focus on our past light cone, we (need not use AMR and) can instead use a coordinate system adapted to this. In particular, we can choose a coordinate system (T, X) that "shrinks exponentially with time." We end the numerical grid at a fixed coordinate value $X = L$. Ordinarily, that would call for a boundary condition at L, but we will use the method of excision, which can be applied to any hyperbolic equations where the outer boundary is chosen so that all modes are outgoing. In that case one simply implements the equations of motion at the outer boundary, no boundary condition is needed (or even allowed).

[The combinations of the equations of motion clearly shows that $N_\times$ and N_- and $-N_\times, N_-$ flow away from the boundary; the points beyond which the flow is entirely outward. Thus, as long as L is chosen large enough and as long as the boundary does not grow too large during the simulation, the surface $X = L$ will be a good excision boundary.]

2.0.7 Example

We choose $\{E_1{}^1 = 1\}$, $\epsilon = 10^{-4}$ and all coefficients $\{a_i, b_i\}$ of order unity. We consider modes $n = 4, 8, 16$. For each of $\{\tilde{\Sigma}_-, \tilde{N}_\times, \tilde{\Sigma}_\times, \tilde{N}_-\}$, we take (for random coefficients)

$$\tilde{X} = a_i^4 sin(\frac{8\pi}{L}x) + b_i^{16} cos(\frac{32\pi}{L}x), \tag{20}$$

For $\{\Omega = 1 - \epsilon^2\tilde{\Omega}, v = \epsilon^2\tilde{v}\}$, we assume

$$\tilde{\Omega} = \omega_8[1 + \frac{\omega}{L}x] sin(\frac{16\pi}{L}x), \tag{21}$$

$$\tilde{v} = \nu_8[1 + \frac{\nu}{L}x] sin(\frac{16\pi}{L}x). \tag{22}$$

We shall consider the following:
I: non periodic bc - but we take $\omega = \nu = 0$ (for comparison)
II: non periodic bc - and we assume ω, ν vary in the range $\{0.01 - 0.1\}$.
And compute
I: Ω_k on spatial hypersurfaces $t = t_i$.
II: value of Ω_k at $x = \frac{L}{4}$ as a function of time.
III: average value of Ω_k as a function of time.

Additions:

1. Remember that one aim is to critique the results on spatial curvature in actual numerical simulations. (b).

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2. In the general inhomogeneous case in actual numerical simulations a cell of length L is taken and the system integrated where the periodic BC at L is applied at ALL times. The periodic BC is used at every step e.g. to compute spatial derivatives near the edges of the cell.

General 3D set up:global cartesian coords. metric variables. initial conditions on the metric. conditions on boundary for all time (implying spatial curvature zero on boundary).

Effect of periodic BC: Maybe depends on formulation: there is a different sense of periodicity in different formulations [depending on independent variables used]

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3. Do periodic initial conditions guarantee periodic evolution.?

No general math theorems? Indeed, theorems by Choquet-Bruhat and Geroch do not indicate the preservation of discrete symmetries (that periodic boundary conditions imply). Mixed opinions whether periodic initial conditions guarantee periodic evolution or not [WC, P, T].

If it were true i dont see why the numerical anlaysists would impose them since it is quite expensive computationally.

But, note that if it were true, then it implies that the spatial curvature is zero at L for all times, which is one of the points i am trying to argue.

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4. a. If you impose periodic BCs with a chosen L, then the longest possible wavelength is L. but in actual N-body simulations a cell of length L is taken to be a fraction of the Hubble scale. So are they ignoring long wavelength inhomogeneities " bigger than L"?] [But why would we a priori assume that long wavelength inhomogeneities are absent in the universe?]. What is effect on the numerical simulations of making this assumption?

[P: I am well aware of this issue/concern with the simulations (especially when the large-scle cutoff is indeed below the Hubble scale, as may be necessary if one also wants to resolve small (Mpc) scales). There will indeed be a cutoff in practice in the power spectrum of the inhomogeneities right away from the initial conditions. It is one of the concerns associated to the periodic BCs, since it is also a periodicity imposed at a certain scale typically smaller than the horizon scale. (Leads to effects: especially in non-linear regime)]

b. But you can choose an L as big as you want?

You cannot allow L to be arbitrarily large, because theoretically the Cartesian type coordinates become ill defined [unless the spatial curvature is zero or exceedingly small], which will lead to coordinate singularities. In actual N-body simulations you restrict cell to finite size (fraction of Hubble scale say).

And there is philosophical issue with large L!

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5. In the G2 case, the cell of size L is now linear ['globa'l x coord]. Hence things change when moving to a FOSH formulation of the evolution eqns? Do you need to specify bc? Do periodic BC initial conditions ensure periodic evolution?

[WC But using long wavelength shows that flat FL is unstable. This is enough to argue that spatial will grow from zero, assuming that long wavelength inhomogeneities, regardless of periodicity, is present in the real universe.]

In G2: FOSH formulation. Orthonormal frame. Variables are NOT metric variables. Initial conditions are NOT on metric functions. Meaning of periodic initial data has different meaning here [does not imply periodicity in the usual standard sense]. Different numerical set up [how to compare with usual N-body simulations?].

[This is possible in G2 since E22, E33, E44 decouple: as go to G1 and then G0 probably periodicity in normal sense restored.