Cosmology under the fractional calculus approach: a possible H_0 tension resolution?

Tensions in Cosmology

Sep 7 - Sep 12, 2022 - Corfu, Greece

Genly Leon Torres

Departamento de Matemáticas
Universidad Católica del Norte

September 11th, 2022

Universidad Católica del Norte
ver más allá

Cosmology under the fractional calculus approach

Miguel A. García-Aspeitia [1] ⋆, Guillermo Fernandez-Anaya [1] †, A. Hernández-Almada [2] ‡, Genly Leon [3,4] §, Juan Magaña [5] ¶

[1] *Depto. de Física y Matemáticas, Universidad Iberoamericana Ciudad de México, Prolongación Paseo de la Reforma 880, México D. F. 01219, México*

[2] *Facultad de Ingeniería, Universidad Autónoma de Querétaro, Centro Universitario Cerro de las Campanas, 76010, Santiago de Querétaro, México*

[3] *Departamento de Matemáticas, Universidad Católica del Norte, Avda. Angamos 0610, Casilla 1280 Antofagasta, Chile*

[4] *Institute of System Science, Durban University of Technology, PO Box 1334, Durban, 4000, South Africa*

[5] *Escuela de Ingeniería, Universidad Central de Chile, Avenida Francisco de Aguirre 0405, 171-0164 La Serena, Coquimbo, Chile*

Universidad Católica del Norte
ver más allá

https://blog.wolfram.com/2022/08/12/fractional-calculus-in-wolfram-language-13-1/

ica del Norte
ver más allá

1. Fractional differentiation has drawn increasing attention in the study of so-called "anomalous" social and physical behaviours, where the scaling power law of fractional order appears universal as an empirical description of such complex phenomena.
2. In the standard mathematical models of integer-order derivatives, including nonlinear models, do not work adequately in many cases where power law is clearly observed.
3. To accurately reflect the non-local, frequency- and history-dependent properties of power law phenomena, alternative modelling tools have to be introduced, such as fractional calculus.
4. Research in fractional differentiation is inherently multi-disciplinary and has its application across various disciplines.

1. Fractional-order Systems and Controls: Fundamentals and Applications, Concepción A. Monje, YangQuan Chen, Blas M. Vinagre, Springer 2010.
2. Stabilization and Control of Fractional Order Systems: A Sliding Mode Approach, Bandyopadhyay Bijnan, Shyam Kamal, Springer 2015
3. Advances in Robust Fractional Control, Fabrizio Padula, Antonio Visioli, Springer 2014.
4. Fractional Calculus: An Introduction for Physicists, Richard Herrmann, World Scientific, 2014.
5. Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Fractional Calculus to Dynamics of Particles, Fields and Media, Vasily E. Tarasov, Springer 2010.
6. Fractional Dynamics: Recent Advances, Joseph Klafter, S. C. Lim,Ralf Metzler, World Scientific, 2012.
7. Advanced Methods in the Fractional Calculus of Variations, Agnieszka B. Malinowska,Tatiana Odzijewicz,Delfim F.M. Torres, Springer 2015.
8. Fractional Trigonometry: With Applications to Fractional Differential Equations and Science, Carl F. Lorenzo, Tom T. Hartley, Wiley 2016.

Cauchy's formula for the integral multiple of integer order $\mu > 0$ [Uchaikin V. V., 2013, Fractional derivatives for physicists and Engineers. Higher Education Press]

$$ {}_cI_x^{\mu} f(x) = \frac{1}{\Gamma(\mu)} \int_c^x f(t)(x-t)^{\mu-1} dt. \tag{1} $$

The Riemann-Liouville derivative (RLD) with $\alpha \geq 0$ for $f(x)$ is defined by

$$ \begin{aligned} D_x^{\alpha} f(x) &\equiv \frac{d^n}{dx^n} I_x^{n-\alpha} f(x) \\ &= \Gamma(n-\alpha)^{-1} \frac{d^n}{dx^n} \int_c^x \frac{f(t)}{(x-t)^{\alpha-n+1}} dt, \end{aligned} \tag{2} $$

where $n = [\alpha] + 1$ being $\alpha \in (n-1, n)$.

Some rules

$$D_x^\alpha \left[D_x^\beta f(x)\right] = D_x^{\alpha+\beta} f(x) - \sum_{j=1}^{n} D_x^{\beta-j} f(c+) \frac{(x-c)^{-\alpha-j}}{\Gamma(1-\alpha-j)}, \quad (3)$$

or in other words $D_x^\alpha D_x^\beta f(x) \neq D_x^{\alpha+\beta} f(x)$, if only not all derivatives $D_x^{\beta-j} f(c+)$ at the beginning of the interval are equal to zero.
Leibniz rule reads as

$$D_x^\alpha [f(x)g(x)] = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} D_x^{\alpha-k} f(x) D_x^k g(x), \quad (4)$$

having the usual when $\alpha = n$.

Universidad Católica del Norte
ver más allá

The Caputo definition is:

$$
\begin{aligned}
{}^{C}D_x^{\alpha} f(x) &\equiv {}_{c}I_x^{n-\alpha} D_x^n f(x) \\
&= \Gamma(n-\alpha)^{-1} \int_c^x \frac{\frac{d^n}{dt^n} f(t)}{(x-t)^{\alpha-n+1}} dt
\end{aligned} \tag{5}
$$

where $n = \max\{0, [\alpha]\}$.
https://blog.wolfram.com/2022/08/12/fractional-calculus-in-wolfram-language-13-1/

The background cosmology is based on the Friedmann-Lemaître-Robertson-Walker (FLRW) metric for a flat Universe. The fractional effective action can be written in the form

$$S_{eff} = \Gamma(\alpha)^{-1} \int_0^t \left[\frac{3}{8\pi G}\left(a^2\ddot{a} + a\dot{a}^2 - a^2\dot{a}\right) + a^3\mathcal{L}_{\rm m}\right] \times (t-\tau)^{\alpha-1} d\tau, \quad (6)$$

where $\mathcal{L}_{\rm m}$ is the matter Lagrangian, α is the fractional constant parameter, t and τ are the physical and proper time respectively and where the Λ is not considered [Shchigolev V. K., 2011, Commun. Theor. Phys., 56, 389]

Universidad Católica del Norte
ver más allá

Modified Friedmann equations

The minimization of the fractional action (6) generates the modified Friedmann equation

$$H^2 + \frac{(1-\alpha)}{t} H = \frac{8\pi G}{3} \sum_i \rho_i, \tag{7}$$

and the continuity equation is given by

$$\sum_i \left[\dot{\rho}_i + 3\left(H + \frac{1-\alpha}{3t} \right) (\rho_i + p_i) \right] = 0. \tag{8}$$

When $\alpha = 1$, the standard cosmology is recovered without Λ.

Universidad Católica del Norte
ver más allá

Using the equation of state $p_i = w_i \rho_i$, where $w_i \neq -1$ are constants,

$$\sum_i (1+w_i)\rho_i \left[\frac{\dot{\rho}_i}{(1+w_i)\rho_i} + 3\frac{\dot{a}}{a} + \frac{1-\alpha}{t}\right] = \sum_i (1+w_i)\rho_i \frac{\mathrm{d}}{\mathrm{d}t}\left[\ln\left({\rho_i}^{1/(1+w_i)} a^3 t^{1-\alpha}\right)\right]. \tag{9}$$

Assuming separated conservation equations, setting $a(t_U) = 1$, where t_U is the age of the Universe, and denoting by ρ_{0i} the current value of energy density of the i-th species,

$$\rho_i(t) = \rho_{0i} a(t)^{-3(1+w_i)} \left(t/t_U\right)^{(\alpha-1)(1+w_i)}. \tag{10}$$

Universidad Católica del Norte
ver más allá

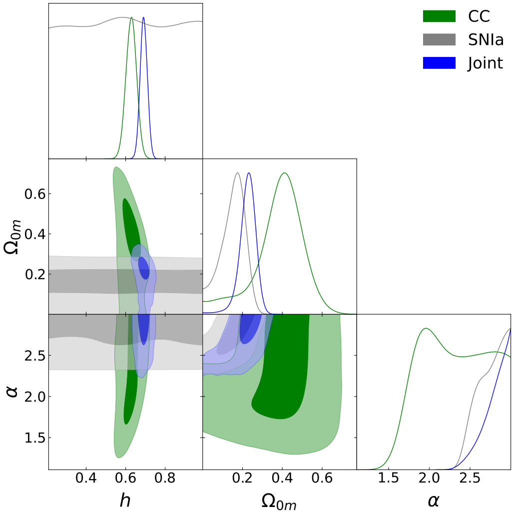


Figure: 2D likelihood contours at 68% and 99.7% CL, alongside the corresponding 1D posterior distribution of the free parameters

Universidad Católica del Norte
ver más allá

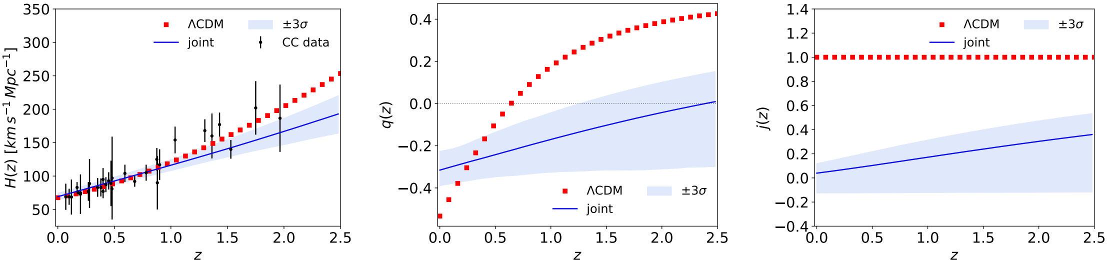


Figure: Left to right: reconstruction of the $H(z)$, $q(z)$, and $j(z)$, in fractional cosmology represent the results of ΛCDM cosmology with $h = 0.6766$ and $\Omega_{m0} = 0.3111$ [Planck:2018]

Universidad Católica del Norte
ver más allá

Sample	$\chi^2_{\min}$	h	Ω_{0m}	α
CC	16.14	$0.629^{+0.027}_{-0.027}$	$0.399^{+0.093}_{-0.122}$	$2.281^{+0.492}_{-0.433}$
SNIa	54.83	$0.599^{+0.275}_{-0.269}$	$0.160^{+0.050}_{-0.072}$	$2.771^{+0.161}_{-0.214}$
Joint	78.69	$0.692^{+0.019}_{-0.018}$	$0.228^{+0.035}_{-0.040}$	$2.839^{+0.117}_{-0.193}$

Table: Best-fit values and their 68% CL uncertainties for fractional cosmology (dust + radiation) with CC, SNIa and a Joint analysis.

Universidad Católica del Norte
ver más allá

Defining the dimensionless variables

$$x_{1,2} = \frac{8\pi G t^2 \rho_{\mathrm{m,r}}}{3} \Big/ \left(A + \frac{|1-\alpha|}{2}\right)^2, \quad A = tH, \quad \tau = \ln(t), \tag{11}$$

we produce a 2D dynamical system subject to the restriction

$$G(x_1, x_2, A) := A^2 + (1-\alpha)A - (x_1 + x_2)(A + |1-\alpha|/2)^2 = 0. \tag{12}$$

Label	Ω_{m}	Ω_{r}	H	q	Solution	$a(t) = (t/t_U)^A$
P_1	Indeterminate	Indeterminate	0	Indeterminate	Static universe	$a(t) =$ constant
P_2	0	$\frac{7-4\alpha}{2\alpha+1}$	$\frac{2\alpha+1}{6t}$	$-\frac{2\alpha-5}{2\alpha+1}$	Power-law (decelerated if $\alpha < \frac{5}{2}$) Power-law (accelerated if $\alpha > \frac{5}{2}$)	$a(t) = (t/t_U)^{(2\alpha+1)/6}$
P_3	$\frac{2(2-\alpha)}{\alpha+1}$	0	$\frac{\alpha+1}{3t}$	$-\frac{\alpha-2}{\alpha+1}$	Power-law (decelerated if $\alpha < 2$) Power-law (accelerated if $\alpha > 2$)	$a(t) = (t/t_U)^{\frac{1+\alpha}{3}}$
P_4	0	0	$\frac{\alpha-1}{t}$	$-\frac{\alpha-2}{\alpha-1}$	Power-law (accelerated if $\alpha < 1$ or $\alpha > 2$) Power-law (decelerated if $1 < \alpha < 2$)	$a(t) = (t/t_U)^{\alpha-1}$

Table: Equilibrium points of associated dynamical system.

Universidad Católica del Norte
ver más allá

Results

1. α, is estimated for each dataset, and in particular, we have $\alpha = 2.839^{+0.117}_{-0.193}$ for the joint analysis, allowing an accelerated Universe.
2. We recover traditional calculus when $\alpha = 1$; however, in the region $0 < \alpha < 1$, obtaining an accelerated physical Universe at late stages is not feasible. There we obtain an accelerated power-law solution corresponding to negative values for the age of the Universe.
3. Hence, one way to avoid this problem is under the introduction of Λ, which will act as a cosmological constant. However, we are in a loop because the idea explains the Universe's acceleration through the α term, which contributes to the fractional calculus theory.
4. The other way is to consider $\alpha > 2$, and then we get an accelerated physical Universe at late stages.

Universidad Católica del Norte
ver más allá

Results

1. On the other hand, the age of the Universe is estimated for each dataset, $t_U/\text{Gyrs} = 33.633^{+14.745}_{-15.095}$ (CC), $33.837^{+27.833}_{-10.788}$ (SNIa) and $33.617^{+3.411}_{-4.511}$ (joint).
2. For the Joint value, we obtain around 2.4 times larger than the age of the Universe expected under the standard paradigm, which is also in disagreement with the value obtained with globular clusters, $t_U = 13.5^{+0.16}_{-0.14} \pm 0.23$ [Valcin D., Jimenez R., Verde L., Bernal J. L., Wandelt B. D., 2021, Journal of Cosmology and Astroparticle Physics, 2021, 017].
3. The term $(1-\alpha)H/t$, which acts as an extra source of mass leading to a closed Universe, could be the origin of this older Universe. In closed scenarios, the Universe becomes older than the standard prediction [Di Valentino E., et al., 2021b, Astropart. Phys., 131, 102607].

Universidad Católica del Norte
ver más allá

Results

1. Figure 3 displays the reconstruction of the $\mathbb{H}0(z)$ diagnostic [Krishnan C., Ó Colgáin E., Sheikh-Jabbari M., Yang T., 2021, Phys. Rev. D, 103] for the fractional cosmology and its error band at 3σ CL.
2. Although the path (solid line) for the fractional cosmology is consistent within 3σ with the CMB Planck value [Aghanim N., et al., 2018, Planck 2018 results. VI. Cosmological parameters (arXiv:1807.06209)] for $z \lesssim 1.5$, we can observe that it presents a trend to the H_0 value obtained by SH0ES [Riess A. G., Casertano S., Yuan W., Macri L. M., Scolnic D., 2019, Astrophys. J., 876, 85] for the present time.
3. Nevertheless, the H_0 value for $1.5 < z < 2.5$ is lower than the Planck value, suggesting a tension in this value.

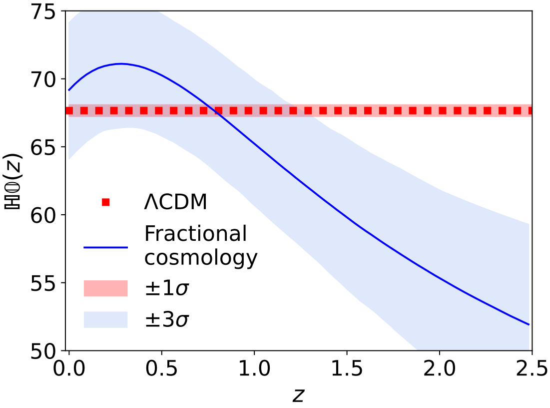


Figure: $\mathbb{H}0(z)$ diagnostic for fractional cosmology and its comparison against ΛCDM model.

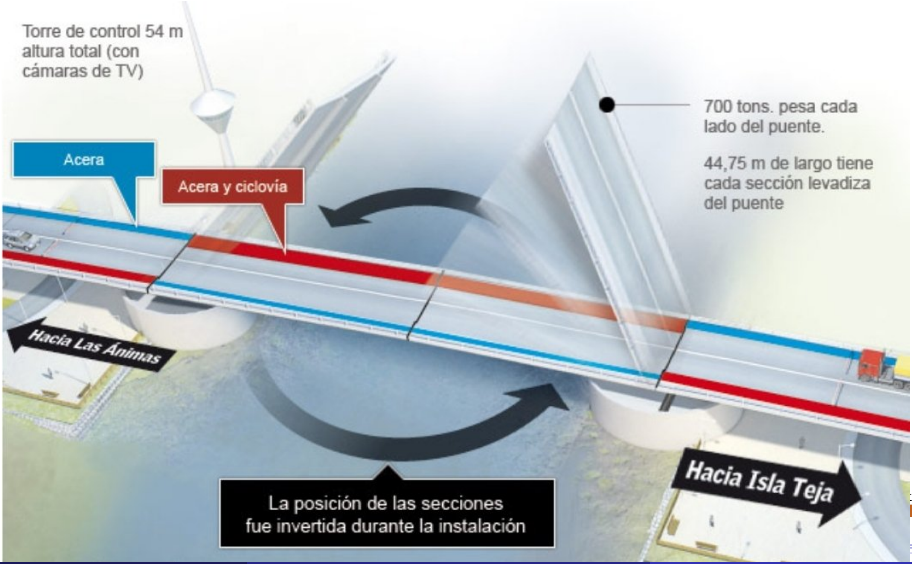
Universidad Católica del Norte
ver más allá

Acknowledgments

M.A.G.-A. acknowledges support from cátedra Marcos Moshinsky and Universidad Iberoamericana for support with the SNI grant. GFA acknowledges support from DINVP and Universidad Iberoamericana. A.H.A. thank to the support from Luis Aguilar, Alejandro de León, Carlos Flores, and Jair García of the Laboratorio Nacional de Visualización Científica Avanzada. G.L. was funded by Vicerrectoría de Investigación y Desarrollo Tecnológico (Vridt) at UCN and through Concurso De Pasantías De Investigación Año 2022, Resolución Vridt N 040/2022 under the Project "The Hubble constant tension: some ways to alleviate it" and through Resolución Vridt No. 054/2022. J.M. acknowledges the support from ANID REDES 190147.

Universidad Católica del Norte
ver más allá

Thank you for your attention!



dad Católica del Norte
ver más allá