Explicit expressions for non-magnetized bunched electron cooling

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Topic of studies

Recent success of Low Energy RHIC Electron Cooler (LEReC) leads the way in development of high energy electron coolers based on non-magnetized electron bunches accelerated by RF cavities.

We derive explicit formulas for cooling rates in non-magnetized electron coolers <u>in presence of redistribution</u> of cooling decrements.

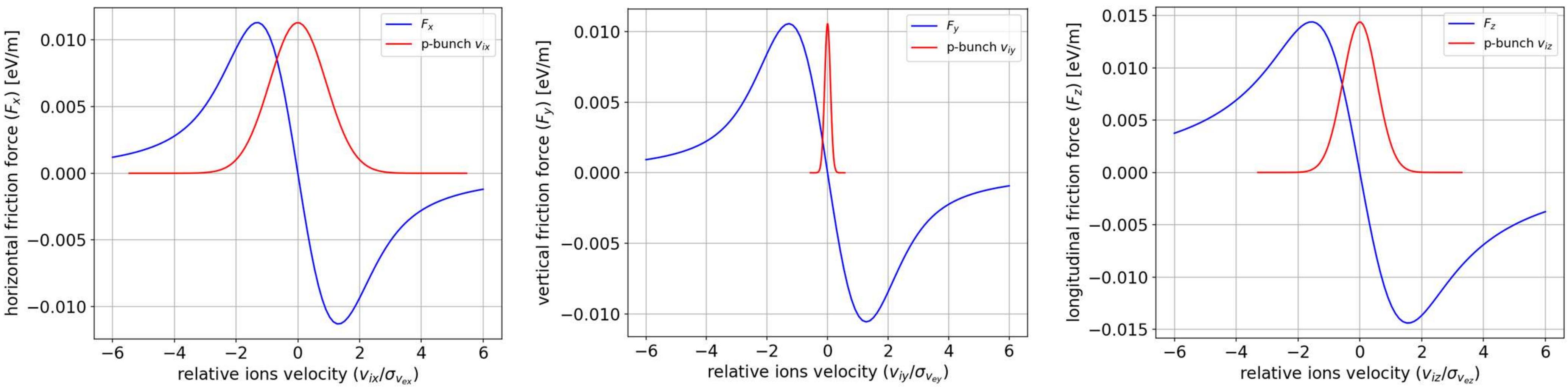
The presented material (with more details) can be also found in:

S. Seletskiy, BNL-223860-2023-TECH, (2023)

Expressions for friction force

$$\vec{F} = -\frac{4\pi e^4 Z_i^2}{m_e}\int \Lambda_C \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f(r_e, v_e) d^3 v_e$$

Example of force components (EIC Ring Electron Cooler):



We consider an electron bunch with Gaussian 6-D distribution in the presence of e-beam dispersion (D_e) in the cooling section

$$
\begin{aligned}
f(r_e, v_e) &= \frac{1}{\gamma(2\pi)^3 \Delta_x \Delta_y \Delta_z \sigma_{xe} \sigma_{ye} \sigma_{ze}} f_x f_y f_z \\
f_x &= \exp\left[-\frac{(x - D_e \delta_e)^2}{2\sigma_{xe}^2} - \frac{v_{xe}^2}{2\Delta_x^2}\right] \\
f_y &= \exp\left(-\frac{y^2}{2\sigma_{ye}^2} - \frac{v_{ye}^2}{2\Delta_y^2}\right) \\
f_y &= \exp\left(-\frac{z^2}{2\sigma_{ze}^2} - \frac{v_{ze}^2}{2\Delta_z^2}\right) = \exp\left(-\frac{z^2}{2\sigma_{ze}^2} - \frac{\delta_e^2}{2\sigma_{\delta e}^2}\right)
\end{aligned}
$$

$$
\begin{aligned}
f(r_e, v_e) &= n_e f_{ve} \\
n_e &= \frac{1}{\gamma(2\pi)^{3/2} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \exp\left[-\frac{x^2}{2\sigma_{1xe}^2} - \frac{y^2}{\sigma_{ye}^2} - \frac{z^2}{\sigma_{ze}^2}\right] \\
f_{ve} &= \frac{1}{(2\pi)^{3/2} \Delta_x \Delta_y \Delta_{1z}} \exp\left[-\frac{v_{xe}^2}{2\Delta_x^2} - \frac{v_{ye}^2}{\Delta_y^2} - \frac{(v_{ze} - \mu_z)^2}{\Delta_{1z}^2}\right] \\
\sigma_{1xe} &= \sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2} \\
\Delta_{1z} &= \Delta_z \frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}} \\
\mu_z &= x \Delta_z \frac{D_e \sigma_{\delta e}}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned}
$$

Then, the force is:

$$\vec{F} = -Cn_e \int \frac{\vec{v}_i - \vec{v}_e}{|\vec{v}_i - \vec{v}_e|^3} f_{ve} d^3 v_e \qquad C = \frac{4\pi N_e e^4 Z_i^2 \Lambda_C}{m_e}$$

We can introduce an effective potential in a velocity-space:

$$U = C \int \frac{f_{ve}}{|\vec{v}_i - \vec{v}_e|} d^3 v_e \qquad F_{x,y,z} = \partial U / \partial v_{xi,yi,zi}$$

For the case $\Delta_x = \Delta_y \equiv \Delta_\perp$, the friction force becomes:

$$\begin{cases} F_{x,y} = -\widetilde{C} n_e v_{xi,yi} \int_0^\infty g_\perp(q) dq \qquad \widetilde{C} = 2\sqrt{2\pi} N_e r_e^2 m_e c^4 Z_i^2 \Lambda_C \\ F_z = -\widetilde{C} n_e (v_{zi} - \mu_z) \int_0^\infty g_z(q) dq \\ g_\perp(q) = \frac{1}{\Delta_\perp^2 (1+q)^2 \sqrt{\Delta_\perp^2 q + \Delta_{1z}^2}} \exp\left[-\frac{v_{xi}^2 + v_{yi}^2}{2\Delta_\perp^2 (1+q)} - \frac{(v_{zi} - \mu_z)^2}{2(\Delta_\perp^2 q + \Delta_{1z}^2)} \right] \\ g_z(q) = \frac{1}{(1+q)(\Delta_\perp^2 q + \Delta_{1z}^2)^{3/2}} \exp\left[-\frac{v_{xi}^2 + v_{yi}^2}{2\Delta_\perp^2 (1+q)} - \frac{(v_{zi} - \mu_z)^2}{2(\Delta_\perp^2 q + \Delta_{1z}^2)} \right] \end{cases}$$

For detailed derivation and for general case of $\Delta_x \neq \Delta_y$ see:

S. Seletskiy, A. Fedotov, BNL-222963-2022-TECH, https://arxiv.org/pdf/2205.00051v2.pdf (2022)

S. Seletskiy, A. Fedotov, BNL-220641-2020-TECH (2020)

In approximation of small amplitudes:

$$
\begin{aligned}
F_x &= -C_x \cdot n_e \cdot v_{xi} \\
F_z &= -C_z \cdot n_e \cdot (v_{zi} - K \cdot x_i) \\
C_x &= C_0 \cdot h \cdot \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right) \\
C_z &= 2C_0 \cdot h \cdot \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right)\right] \\
C_0 &= \frac{2\sqrt{2\pi} N_e r_e^2 m_e c Z_i^2 \Lambda_C}{\gamma^2 \beta^3 \sigma_{\theta e}^2 \sigma_{\delta e}} \\
h &= \frac{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}}{\sigma_{xe}} \\
K &= \beta c \frac{D_e \sigma_{\delta e}^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned}
$$

$$
\Phi(d) = \begin{cases} \frac{d}{1-d^2}\left(\frac{\arccos(d)}{\sqrt{1-d^2}} - d\right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1}\left(\frac{\log(d - \sqrt{d^2-1})}{\sqrt{d^2-1}} + d\right), & d > 1 \end{cases}
$$

$$
\begin{aligned}
n_e &= \frac{1}{\gamma (2\pi)^{3/2} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \exp\left[-\frac{x^2}{2\sigma_{1xe}^2} - \frac{y^2}{\sigma_{ye}^2} - \frac{z^2}{\sigma_{ze}^2}\right] \\
\sigma_{1xe} &= \sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned}
$$

Cooling rate redistribution

On a single pass through the cooling section, an ion's angle and relative momentum change as:

$$
\begin{aligned}
\Delta x'_i &= -c_x \cdot n_{e1} \cdot x'_{iCS} \\
\Delta \delta_i &= -c_z \cdot n_{e1} \cdot (\delta_{iCS} - k \cdot x_{iCS}) \\
c_x &= \tilde{c}_0 \cdot h \cdot \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right) \\
c_z &= 2\tilde{c}_0 \cdot h \cdot \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}} \frac{1}{h}\right)\right] \\
k &= \frac{D_e \sigma_{\delta e}^2}{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2} \\
n_{e1} &= \frac{1}{(2\pi)^{3/2} \sigma_{1xe} \sigma_{ye} \sigma_{ze}} \exp\left[-\frac{x^2}{2\sigma_{1xe}^2} - \frac{y^2}{\sigma_{ye}^2} - \frac{z^2}{\sigma_{ze}^2}\right] \\
\tilde{c}_0 &= \frac{2\sqrt{2\pi} N_e r_e^2 Z_i^2 \Lambda_C L_{CS}}{\gamma^4 \beta^4} \frac{m_e}{A_i m_p} \frac{1}{\sigma_{\theta e}^2 \sigma_{\delta e}} \\
h &= \frac{\sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}}{\sigma_{xe}} \\
\sigma_{1xe} &= \sqrt{\sigma_{xe}^2 + D_e^2 \sigma_{\delta e}^2}
\end{aligned}
$$

Introduce ion dispersion (D_i) in the cooling section. Then on a single pas through the CS:

$$\begin{aligned} x_{iCS} &= x_{i0} + D_i \cdot \delta_{i0} \\ x_{i1} &= x_{iCS} - D_i \cdot (\delta_{iCS} + \Delta\delta_i) = x_{iCS} - D_i \cdot (\delta_{i0} + \Delta\delta_i) \end{aligned}$$

$$\boxed{\begin{aligned} \Delta x_i &= c_z \cdot n_{e1} \cdot D_i \cdot [\delta_{i0} - k \cdot (x_{i0} + D_i \cdot \delta_{i0})] \\ \Delta x_i' &= -c_x \cdot n_{e1} \cdot x_{i0}' \\ \Delta\delta_i &= -c_z \cdot n_{e1} \cdot [\delta_{i0} - k \cdot (x_{i0} + D_i \cdot \delta_{i0})] \end{aligned}}$$

With the density of the e-bunch “probed” by the considered ion:

$$n_{e1} = \frac{1}{(2\pi)^{3/2}\sigma_{1xe}\sigma_{ye}\sigma_{ze}} \exp\left[-\frac{(x_i + D_i\delta_i)^2}{2\sigma_{1xe}^2} - \frac{y_i^2}{\sigma_{ye}^2} - \frac{z_i^2}{\sigma_{ze}^2}\right]$$

Horizontal and longitudinal actions:

$$J_x = \frac{1}{2}\left(\gamma_x x_i^2 + \beta_x x_i'^2\right) \qquad J_z = \frac{1}{2}\left(\frac{z_i^2}{\beta_z} + \beta_z \delta_i^2\right)$$

$$\beta_z = \sigma_{zi}/\sigma_{\delta i}$$

$$\begin{aligned} \Delta J_x &\approx \frac{x_{i0}\Delta x_i}{\beta_x} + \beta_x x'_{i0}\Delta x'_i = \\ &= \frac{c_z D_i}{\beta_x} n_{e1}\left(x_{i0}\delta_{i0}(1-kD_i) - kx_{i0}^2\right) - c_x n_{e1}\beta_x x_{i0}'^2 \end{aligned}$$

$$\Delta J_z \approx \beta_z \delta_{i0}\Delta\delta_i = -c_z n_{e1}\beta_z(\delta_{i0}^2(1-kD_i) - kx_{i0}\delta_{i0})$$

$$\Delta\varepsilon_x = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_i \Delta J_x dx_i dy_i dz_i dx'_i dy'_i d\delta_i$$

$$\Delta\varepsilon_z = \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f_i \Delta J_z dx_i dy_i dz_i dx'_i dy'_i d\delta_i$$

We assume 6D Gaussian distribution for ions:

$$f_i = \frac{\exp\left(-\frac{x^2}{2\sigma_{xi}^2} - \frac{y^2}{2\sigma_{yi}^2} - \frac{z^2}{2\sigma_{zi}^2} - \frac{x'^2}{2\sigma_{\theta xi}^2} - \frac{y'^2}{2\sigma_{\theta yi}^2} - \frac{\delta^2}{2\sigma_{\delta i}^2}\right)}{(2\pi)^3 \sigma_{xi}\sigma_{yi}\sigma_{zi}\sigma_{\theta xi}\sigma_{\theta yi}\sigma_{\delta i}}$$

Then, after integration, relative emittance change on a single pass through the CS:

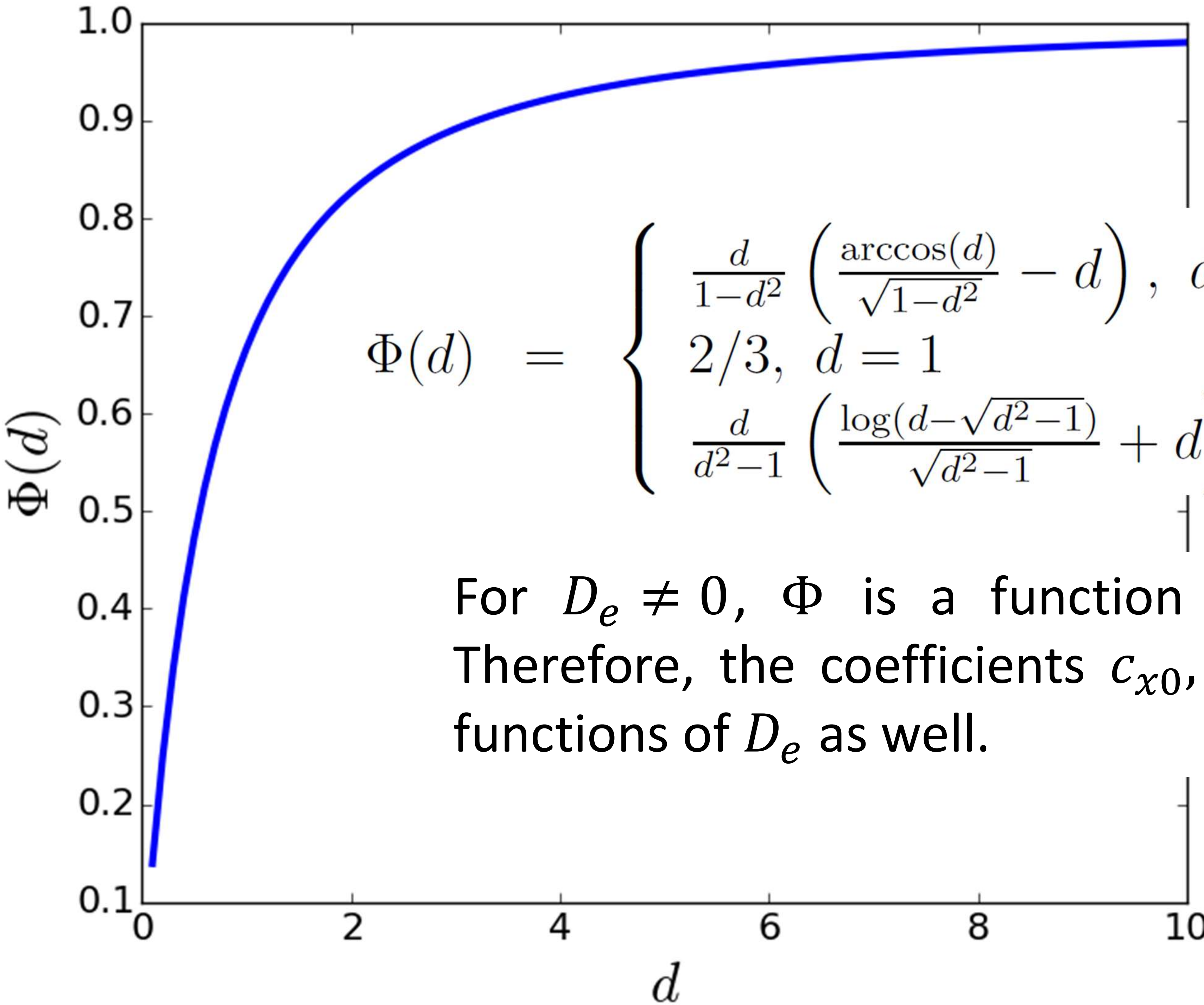
$$\begin{aligned}
\frac{\Delta\varepsilon_x}{\varepsilon_x} &= -\frac{1}{(2\pi)^{3/2}\sqrt{D_i^2\sigma_{\delta i}^2+\sigma_{xi}^2+\sigma_{1xe}^2}\sqrt{\sigma_{yi}^2+\sigma_{ye}^2}\sqrt{\sigma_{zi}^2+\sigma_{ze}^2}} \cdot \\
&\cdot \left(c_x + c_z\frac{D_i^2\sigma_{\delta i}^2+kD_i\sigma_{1xe}^2}{D_i^2\sigma_{\delta i}^2+\sigma_{xi}^2+\sigma_{1xe}^2}\right) \\
\frac{\Delta\varepsilon_z}{\varepsilon_z} &= -\frac{1}{(2\pi)^{3/2}\sqrt{D_i^2\sigma_{\delta i}^2+\sigma_{xi}^2+\sigma_{1xe}^2}\sqrt{\sigma_{yi}^2+\sigma_{ye}^2}\sqrt{\sigma_{zi}^2+\sigma_{ze}^2}} \cdot \\
&\cdot \left(c_z - c_z\frac{D_i^2\sigma_{\delta i}^2+kD_i\sigma_{1xe}^2}{D_i^2\sigma_{\delta i}^2+\sigma_{xi}^2+\sigma_{1xe}^2}\right)
\end{aligned}$$

We get cooling rates from $\lambda = \frac{1}{T_{rev}}\frac{\Delta\varepsilon}{\varepsilon}$, where T_{rev} is a revolution period

Main result

$$
\begin{aligned}
\lambda_x &= -P\left(c_{x0} + c_{z0}\frac{D_i^2\sigma_{\delta i}^2 + D_e D_i \sigma_{\delta e}^2}{D_i^2\sigma_{\delta i}^2 + \sigma_{xi}^2 + D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}\right)\\
\lambda_z &= -P\left(c_{z0} - c_{z0}\frac{D_i^2\sigma_{\delta i}^2 + D_e D_i \sigma_{\delta e}^2}{D_i^2\sigma_{\delta i}^2 + \sigma_{xi}^2 + D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}\right)\\
P &= \frac{\sqrt{D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}}{\sigma_{xe}\sqrt{D_i^2\sigma_{\delta i}^2 + \sigma_{xi}^2 + D_e^2\sigma_{\delta e}^2 + \sigma_{xe}^2}\sqrt{\sigma_{yi}^2 + \sigma_{ye}^2}\sqrt{\sigma_{zi}^2 + \sigma_{ze}^2}}\\
c_0 &= \frac{N_e r_e^2 Z_i^2 c \Lambda_C \eta}{\pi\gamma^4\beta^3}\frac{m_e}{A_i m_p}\frac{1}{\sigma_{\theta e}^2\sigma_{\delta e}}\\
c_{x0} &= c_0 \cdot \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2 + D_e^2\sigma_{\delta e}^2}}\right)\\
c_{z0} &= 2c_0 \cdot \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{\sigma_{xe}}{\sqrt{\sigma_{xe}^2 + D_e^2\sigma_{\delta e}^2}}\right)\right]\\
\Phi(d) &= \begin{cases} \frac{d}{1-d^2}\left(\frac{\arccos(d)}{\sqrt{1-d^2}} - d\right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1}\left(\frac{\log(d - \sqrt{d^2-1})}{\sqrt{d^2-1}} + d\right), & d > 1 \end{cases}
\end{aligned}
$$

These equations give explicit expressions for the cooling rates with $z \Rightarrow x$ redistribution



$$\Phi(d) = \begin{cases} \frac{d}{1-d^2}\left(\frac{\arccos(d)}{\sqrt{1-d^2}} - d\right), & d < 1 \\ 2/3, & d = 1 \\ \frac{d}{d^2-1}\left(\frac{\log(d-\sqrt{d^2-1})}{\sqrt{d^2-1}} + d\right), & d > 1 \end{cases}$$

For $D_e \neq 0$, Φ is a function of D_e. Therefore, the coefficients c_{x0}, c_{z0} are functions of D_e as well.

Special cases

For $\boldsymbol{D_e = 0}$, Φ does not depend on D_e and neither do coefficients c_{x0}, c_{z0}. Redistribution formula becomes:

$$\lambda_x = -P(c_{x0} + c_{z0}\, r)$$
$$\lambda_z = -P(c_{z0} - c_{z0}\, r)$$
$$r = \frac{D_i^2 \sigma_{\delta i}^2}{D_i^2 \sigma_{\delta i}^2 + \sigma_{xi}^2 + \sigma_{xe}^2}$$

The formula in a similar form was first derived in [*H. Zhao, M. Blaskiewicz, Phys. Rev. Accel. Beams 24, 083502 (2021)*], but without specifying the expressions for c_{x0}, c_{z0}

$$P = \frac{1}{\sqrt{D_i^2 \sigma_{\delta i}^2 + \sigma_{xi}^2 + \sigma_{xe}^2}\sqrt{\sigma_{yi}^2 + \sigma_{ye}^2}\sqrt{\sigma_{zi}^2 + \sigma_{ze}^2}}$$

$$c_0 = \frac{N_e r_e^2 Z_i^2 c \Lambda_C \eta}{\pi \gamma^4 \beta^3} \frac{m_e}{A_i m_p} \frac{1}{\sigma_{\theta e}^2 \sigma_{\delta e}}$$

$$c_{x0} = c_0 \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right)$$

$$c_{z0} = 2c_0\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right)\right]$$

For $\boldsymbol{D_e = 0}$ and $\boldsymbol{D_i = 0}$:

$$\lambda_{x0} = -\frac{N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{\theta e}^2 \sigma_{\delta e} \sqrt{(\sigma_{xi}^2+\sigma_{xe}^2)(\sigma_{yi}^2+\sigma_{ye}^2)(\sigma_{zi}^2+\sigma_{ze}^2)}} \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right)$$

$$\lambda_{z0} = -\frac{2N_e r_e^2 Z_i^2 m_e c \Lambda_C \eta}{\pi \gamma^4 \beta^3 A_i m_p \sigma_{\theta e}^2 \sigma_{\delta e} \sqrt{(\sigma_{xi}^2+\sigma_{xe}^2)(\sigma_{yi}^2+\sigma_{ye}^2)(\sigma_{zi}^2+\sigma_{ze}^2)}} \left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma \sigma_{\theta e}}\right)\right]$$

And for friction force:

$$F_{x,y} = -C \frac{v_{x,y}}{\Delta_\perp^2 \Delta_z} \Phi(\Delta_z/\Delta_\perp)$$
$$F_z = -2C \frac{v_z}{\Delta_\perp^2 \Delta_z} (1 - \Phi(\Delta_z/\Delta_\perp))$$

When $\Delta_z = \Delta_\perp = \Delta$, $\Phi = 2/3$ and we get:

$$F_{x,y} = F_z = -\frac{4\sqrt{2\pi}}{3} \frac{n_e r_e^2 m_e c^4 Z^2 L_C}{\Delta^3} v_{x,y,z}$$

When $\Delta_z \ll \Delta_\perp$, $\Phi(d) \approx \pi d/2$, hence:

$$F_{x,y} = -\pi\sqrt{2\pi} \frac{n_e r_e^2 m_e c^4 Z^2 L_C}{\Delta_\perp^3} v_{x,y}$$
$$F_z = -4\sqrt{2\pi} \frac{n_e r_e^2 m_e c^4 Z^2 L_C}{\Delta_\perp^2 \Delta_z} v_z$$

The asymptotic expressions were first derived in [*Ya. S. Derbenev, Dr. Thesis, (1978) https://arxiv.org/abs/1703.09735*]

$$\sigma_{xi}^2 + D_i^2\sigma_{\delta i}^2 \ll \sigma_{xe}^2 + D_e^2\sigma_{\delta e}^2,\ \sigma_{yi} \ll \sigma_{ye},\ \sigma_{zi} \ll \sigma_{ze}$$

$$\begin{aligned}
\lambda_x &= \lambda_{1x} + \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2\sigma_{\delta e}^2}\lambda_{1z} \\
\lambda_z &= \lambda_{1z} - \frac{D_i D_e \sigma_{\delta e}^2}{\sigma_x^2 + D_e^2\sigma_{\delta e}^2}\lambda_{1z} \\
\lambda_{1x} &= \frac{N_e r_e^2 m_e c Z^2 L_C \eta}{\pi\gamma^4\beta^3 A_i m_p \sigma_x\sigma_y\sigma_z\sigma_{\theta e}^2\sigma_{\delta e}}\Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2\sigma_{\delta e}^2}}\right) \\
\lambda_{1z} &= \frac{2N_e r_e^2 m_e c Z^2 L_C \eta}{\pi\gamma^4\beta^3 A_i m_p \sigma_x\sigma_y\sigma_z\sigma_{\theta e}^2\sigma_{\delta e}}\left[1 - \Phi\left(\frac{\sigma_{\delta e}}{\gamma\sigma_{\theta e}}\frac{\sigma_x}{\sqrt{\sigma_x^2 + D_e^2\sigma_{\delta e}^2}}\right)\right]
\end{aligned}$$

The first two equations are well known (see, for example [*Ya. S. Derbenev, https://arxiv.org/abs/1703.09735*]). The full form of the expressions was derived in [*S. Seletskiy, BNL-223540-2022-TECH (2022)*]