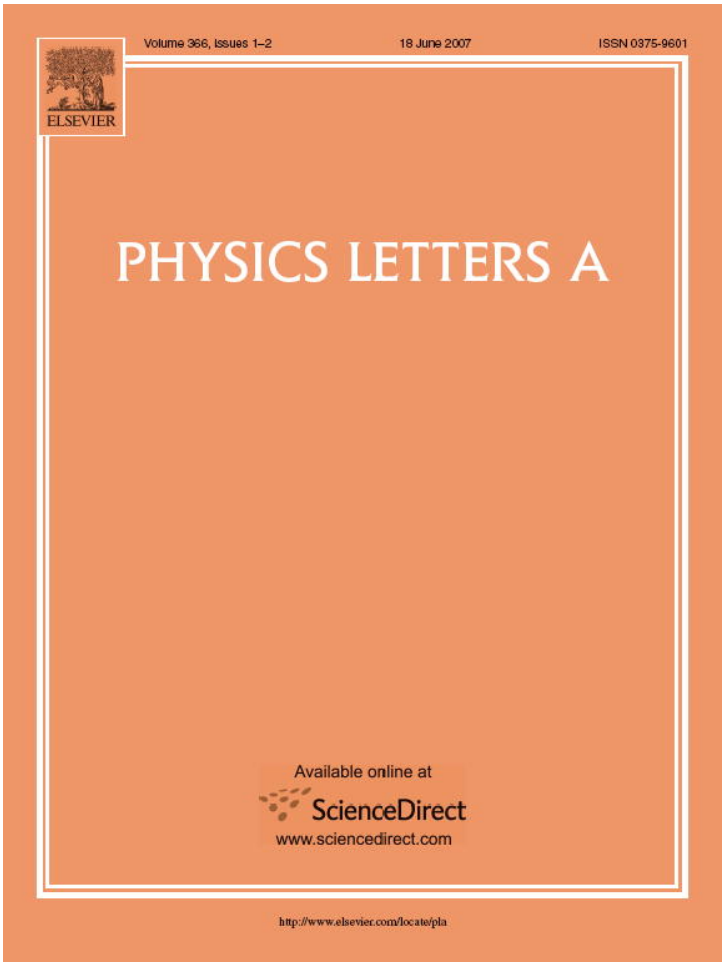

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Possibility of turbulence from a post-Navier–Stokes equation

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Abstract

We introduce corrections to the Navier–Stokes equation arising from the transitions between molecular states and the injection of external energy. In the simplest application of the proposed post-Navier–Stokes equation, we find a multi-valued velocity field and the immediate possibility of velocity reversal, both features of turbulence.

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1. Introduction

Traditionally, any attempt to describe turbulent behavior in fluids starts with the Navier–Stokes equation (NSE) [1]. However, success has at best been mixed [2]. In this Letter, we explore the idea that NSE is not the unique approach to the study of turbulence, and that turbulence may be found in what we label as post-Navier–Stokes equations. What could modify NSE? In our opinion, the molecular nature of fluids can no longer be ignored [3–8]. In fact, quantum kinetic theories have already been proposed starting with the classic work of Chang and Uhlenbeck [9]. A comprehensive summary is given by Klimontovitch [10]. The usual approach is to modify the collision term of the Boltzmann equation to include internal excitations of molecules. A quantum approach becomes necessary. As a consequence of changing the collision term, new terms in the hydrodynamic description modify the usual continuum model. The continuum model ignores the role of molecular excitations. There have been several attempts to study the change in the transport equation including the Navier–Stokes equation [11,12].

In this Letter, we follow the studies of Bardos et al. [11] with the specific purpose of addressing the problem of turbulence. We also attempt to modify NSE by including quantum concepts and, in the simplest application of our approach, arrive at the possibility of velocity reversal and multi-valued velocity fields, both important features of turbulence. Before showing these final results, we need to make two comments on the classic derivation of NSE to justify our post-NSE.

The traditional way of deriving the Navier–Stokes equation (NSE) is phenomenological, based on the continuum model and conservation of momentum. It may also be derived using the kinetic theory of structureless molecules by starting with the Boltzmann transport equation

$$\left(\frac{\partial}{\partial t}+\frac{p_i}{m}\frac{\partial}{\partial x_i}+F_i\frac{\partial}{\partial p_i}\right)f(r,p,t)=\left[\frac{\partial f(r,p,t)}{\partial t}\right]_{\text{coll}}, \qquad (1)$$

where we follow the conventional definitions from Huang [13]. For the purpose of differentiating our approach to arrive at post-NSE equations, we quickly comment on the assumptions of the derivation.

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First, collisional invariants χ are defined such that

$$\int \mathrm{d}^3 p\, \chi(r,p) \left[\frac{\partial f(r,p,t)}{\partial t}\right]_{\text{coll}} = 0. \tag{2}$$

These collisional invariants are $\chi = m$ (mass), $\chi = mv_i$ ($i = 1,2,3$ momentum), $\chi = \frac{1}{2}m|v - u(r,t)|^2$ (thermal energy), where $u(r,t) = \langle v \rangle$.

To get the NSE, multiply the Boltzmann transport equation by p and integrate over all momentum, yielding

$$\rho\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)u = \frac{\rho}{m}F - \nabla\cdot P, \tag{3}$$

where $\rho(r,t) = mn(r,t)$ and $P_{ij} = \rho\langle (v_i - u_i)(v_j - u_j)\rangle$. Our first comment is that the above equation results from *microscopic conservation laws assuming elastic collisions of point molecules*.

To arrive at the traditional NSE, we need an explicit form of the pressure tensor, which is taken to be

$$P_{ij} = \delta_{ij}P - \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}\nabla\cdot u\right], \tag{4}$$

where μ is the viscosity. The above choice of the pressure is justified by the assumption that a fluid element, or microscopically, as our second comment, *the particle of the model has no intrinsic angular momentum*.

In full component form, the Navier–Stokes equation is

$$\rho\left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}\right)u_i = \frac{\rho}{m}F_i - \frac{\partial}{\partial x_j}\left(\delta_{ij}P - \mu\left[\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{2}{3}\frac{\partial u_j}{\partial x_i}\right]\right). \tag{5}$$

Notice that before the introduction of the definition of the pressure, the conservation of momentum equation is exact in so far as the Boltzmann transport equation is valid. This equation comes only from the left-hand side of the Boltzmann equation. The contribution from the collision term disappears by virtue of conservation of momentum and the assumption of elastic collisions. Hence, only the left-hand side of the Boltzmann transport equation is important to yield the conservation of momentum equation and NSE. We stress the two assumptions needed to arrive at the Navier–Stokes equation: first, elastic collisions, and second, the absence of angular momentum of the structureless molecules. What will happen if these two assumptions are no longer valid?

2. Generalizing the Navier–Stokes equation

Suppose that each of the molecules could be found in any one of N states, the ground state and $(N-1)$ excited states. Assume that excitations and de-excitations are induced by molecular collisions, which are now inelastic. Then the semi-classical analogue of the Boltzmann transport equation for each of N distribution functions will be

$$\left(\frac{\partial}{\partial t} + \frac{p_i}{m}\frac{\partial}{\partial x_i} + F_i\frac{\partial}{\partial p_i}\right)f_n(r,p,t) = \left[\frac{\partial f_n(r,p,t)}{\partial t}\right]_{\text{inelastic}}, \tag{6}$$

where the collision term might be replaced by

$$\left[\frac{\partial f_n(r,p,t)}{\partial t}\right]_{\text{inelastic}} = \sum_{m\neq n}^{N}\gamma_{mn}Jf_m(r,p) - \sum_{m\neq n}^{N}\gamma_{nm}Jf_n(r,p) + \sigma K f_n(r,p). \tag{7}$$

γ_{mn} is the transition probability of a particle in the m state jumping to the n state. J is a "jump" operator that will carry the conservation law the we will invoke in the time evolution of N distribution probabilities. K is a "kick" operator that allows the injection of energy from outside [14–17]. σ is the probability that a particle is kicked to a different momentum by external means. The kick operator makes the injection of energy into the system possible. Eq. (6) is a generalization of our previous models [14–17].

To calculate macroscopic averages, not only must we integrate over all momentum, we should also sum over all N states, to yield

$$\rho\left(\frac{\partial}{\partial t} + u\cdot\nabla\right)u = \frac{\rho}{m}F - \nabla\cdot P + \text{driving} + \text{radiative}, \tag{8}$$

where

$$\text{driving} = \sigma\sum_{n=1}^{N}\int \mathrm{d}p^3\, pKf_n(r,p), \tag{9}$$

$$\text{radiative} = \sum_{n=1}^{N}\int \mathrm{d}p^3\, p\left[\sum_{m\neq n}^{N}\gamma_{mn}Jf_m(r,p) - \sum_{m\neq n}^{N}\gamma_{nm}Jf_n(r,p)\right]. \tag{10}$$

We may think of (9) and (10) as the quantum corrections to the NSE. The first sum (9) is the driving term. We will call the second sum (10) the radiative correction because every transition is accompanied by radiation. (NSE ignores not only molecules, but also photons.) The radiative term represents the contribution of the internal degrees of freedom of molecules to the macroscopic flow of a fluid. If all the transition probabilities, σ and γ_{mn} are zero, we simply reproduce the classical NSE. This last equation, our post-NSE, no longer assumes elastic collisions and the absence of angular momentum of the particles. The operators J and K have been defined in our earlier model calculations [14–17], they may be redefined with new models, but we will simplify them to show that even the simplest application of Eq. (8) leads to novel results.

We have had occasions to consider the radiative term [14–17], and will consider them later even more, but for now to arrive at immediate new results, we consider only the influence of quantized kicks defined by the operator $Kf_n(r,p) = f_n(r, p - \Pi)$ yielding

$$\int \mathrm{d}p^3\, pKf_n(r,p) = mu + n(r,t)\Pi. \tag{11}$$

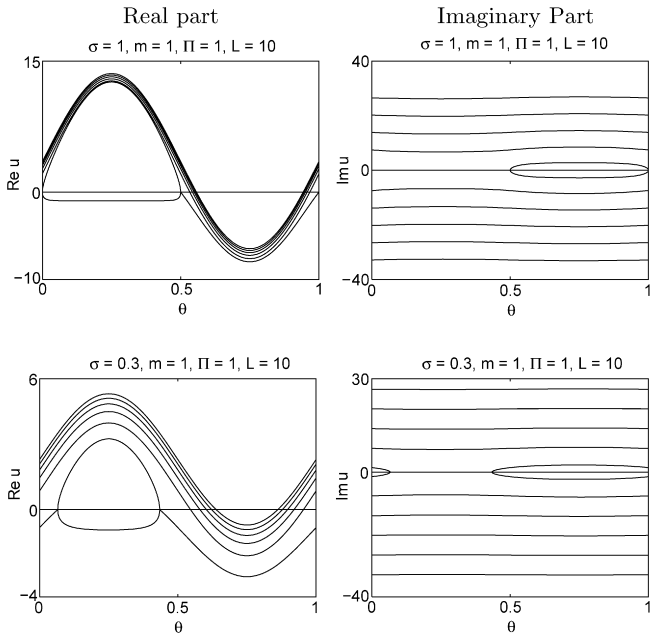

Fig. 1. Average velocity as a function of the angle for the torus geometry. The principal branch is complemented by the other branches which almost form a continuum.

3. Examples

Remove the force F and drop the divergence of the pressure to obtain a non-linear equation in one dimension

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} - \sigma u = \frac{\sigma \Pi}{m}. \tag{12}$$

It is the simplest application of our post-NSE but which remains a challenge still.

The stationary solution is to be obtained from

$$\frac{1}{2}\frac{\partial u^2}{\partial x} - \sigma u = \frac{\sigma \Pi}{m}, \tag{13}$$

which is

$$u(x) = -\frac{\Pi}{m}\left[W_k\left(-\frac{m}{\sigma \Pi}\mathrm{e}^{-\frac{m\sigma}{\Pi}(x+C)-1}\right)+1\right], \tag{14}$$

where C is a constant and W_k denotes the kth branch of the Lambert W function. The function $W_k(z)$ is a solution of the equation $w\mathrm{e}^w = z$ in the complex plane [18]. *The Lambert W function is multi-valued, making the stationary average velocities multi-valued.*

We will choose a toroidal geometry, and put $x = L\sin(2\pi\theta)$, $0 \leqslant \theta < 1$, to ensure periodic boundary conditions. The physical model is one-dimensional, a donut of circumference $2\pi L$. One could imagine a paddlewheel half-stuck into the donut to provide quantum kicks to the fluid. If $u(\theta = 0) = 0$, then $C = -\frac{\Pi}{\sigma m}\ln \Pi$ and we get

$$u_k(\theta) = -\frac{\Pi}{m}\left[W_k\left(-\frac{m}{\sigma}\mathrm{e}^{-\frac{m\sigma}{\Pi}L\sin(2\pi\theta)-1}\right)+1\right]. \tag{15}$$

We plot $u_k(\theta)$ in Fig. 1 for $k = 0, \pm 1, \ldots, \pm 5$.

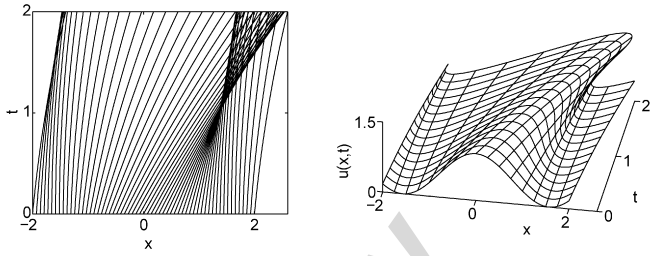

Fig. 2. The transient problem with $\sigma = 0.1$, $m = 1$, $\Pi = 1$, and $f(x) = \cos(x)^2$. Left: The characteristics cross at about $t = 1$. Right: The solution surface $u(x,t)$ folds on itself and becomes multi-valued where the characteristics cross.

Using the method of characteristics, we now find the transient solutions of

$$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial u^2}{\partial x} - \sigma u = \frac{\sigma \Pi}{m}, \quad u(x,0) = f(x), \tag{16}$$

where $f(x)$ are the initial velocity averages. Rewriting (16) as $u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = \sigma u + \frac{\sigma \Pi}{m}$, we identify its characteristic equations,

$$x'(\tau) = u(\tau), \qquad t'(\tau) = 1, \qquad u'(\tau) = \sigma u(\tau) + \frac{\sigma \Pi}{m}. \tag{17}$$

The boundary conditions can be parameterized as

$$x_0(s) = s, \qquad t_0(s) = 0, \qquad u_0(s) = f(s). \tag{18}$$

For each fixed value of s, solving the characteristic equations (17) with initial values $x(0;s) = x_0(s)$, $t(0;s) = t_0(s)$, $u(0;s) = u_0(s)$ yields a characteristic curve $u(x(\tau;s), t(\tau;s))$ in the solution surface $u(x,t)$. For more on the method of characteristics, see for example [19].

The solution $u(x,t)$ is

$$u(x,t) = \mathrm{e}^{\sigma t}\left(\frac{\Pi}{m} + f(s)\right) - \frac{\Pi}{m}, \tag{19}$$

$$x = s + f(s)\frac{\mathrm{e}^{\sigma t}-1}{\sigma} + \frac{\Pi}{m\sigma}\left(\mathrm{e}^{\sigma t} - \sigma t - 1\right), \tag{20}$$

where s is defined implicitly by the second equation. While it is not in general possible to express $u(x,t)$ explicitly, we can still interpret the solution in terms of the characteristic curves $u(x(t;s),t)$.

Depending on the problem parameters σ, m, Π, and $f(x)$, it is possible that the characteristic curves cross. If the characteristics cross at (x_c, t), then there are multiple curves $s_1, s_2, \ldots$ such that $x_c = x(t;s_1) = x(t;s_2) = \cdots$. Moreover, the solution permits multiple values $u(x_c,t) = u(x(t;s_1),t), u(x(t;s_2),t), \ldots$ at the crossing point. Fig. 2 demonstrates this feature in a simple example.

4. Conclusions

It appears that in our first applications of a highly simplified post-Navier–Stokes equation, we have arrived at multi-valued velocities as a function of location. They may well be interpreted as possible states of a turbulent system from which transitions to other states may be possible. The possibility of velocity reversal, a feature of turbulence, is immediately obvious. This result seems to be the first instance of an analytic

derivation of a multi-valued velocity field and deserves further studies.

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