

Accelerator Physics

Lecture 7: Momentum Effects

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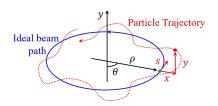
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Summary



Curvilinear Co-ordinates



- (x, y, s), often called the standard co-ordinate system in accelerator physics
- The origin is defined by the vector $\vec{S}(s)$ following the ideal reference path
- $x = r \rho$ $s = \rho \theta$
- $X = r \sin \theta = (\rho + x) \sin \theta$, Y = y, $Z = r \cos \theta = (\rho + x) \cos \theta$

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Transverse Equation of Motion - 1

• Start with the basics

$$F_x = m \frac{\mathrm{d}^2 r}{\mathrm{d}t^2} - \frac{mv^2}{r}$$

$$= m \frac{\mathrm{d}^2 (x+\rho)}{\mathrm{d}t^2} - \frac{mv^2}{x+\rho} = -eB_y v \tag{1}$$

Factorise the equation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 + \frac{x}{\rho}\right)^{-1} = -eB_y v \tag{2}$$

• Utilise the binomial approximation

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \frac{mv^2}{\rho}\left(1 - \frac{x}{\rho}\right) = -eB_y v \tag{3}$$

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Transverse Equation of Motion - 2

 \bullet Replace t with s and rearrange

$$mv^2 \frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{mv^2}{\rho} \left(1 - \frac{x}{\rho} \right) = -eB_y v \tag{4}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho} \left(1 - \frac{x}{\rho} \right) = -\frac{eB_y}{mv} \tag{5}$$

• Consider small displacements in x

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho} \left(1 - \frac{x}{\rho} \right) = -\frac{e}{\mathbf{m}v} \left(B_0 + x \frac{\partial B_y}{\partial x} \right) \tag{6}$$

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Transverse Equation of Motion - 3

• Set field gradient, $g = \frac{\partial B_y}{\partial x}$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho} \left(1 - \frac{x}{\rho} \right) = -\frac{eB_0}{mv} - \frac{exg}{mv} \tag{7}$$

This is a modified Hill's equation

• Consider small momentum offsets $\Delta p = p - p_0 \ll p_0$

$$\frac{1}{p_0 + \Delta p} = \frac{1}{p_0} \left(1 + \frac{\Delta p}{p_0} \right)^{-1}$$

$$\approx \frac{1}{p_0} - \frac{\Delta p}{p_0^2}$$
(8)

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Transverse Equation of Motion - 4

• Insert equation 8 into modified Hill's equation 7

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho} \left(1 - \frac{x}{\rho} \right) = -\frac{eB_0}{p} - \frac{exg}{p}$$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} - \frac{1}{\rho} \left(1 - \frac{x}{\rho} \right) = -\frac{eB_0}{p_0} + \frac{eB_0 \Delta p}{p_0^2} - \frac{exg}{p_0} + \frac{exg \Delta p}{p_0^2}$$
 (9)

• Remember magnetic rigidity??, $B\rho = p/e$

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \frac{x}{\rho^2} = \frac{1}{\rho} \frac{\Delta p}{p_0} + kx \tag{10}$$

where $k = eg/p_0$ and the last term is the product of two small terms (≈ 0)

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Transverse Equation of Motion - 5

• Finally a new modified Hill's equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \left(\frac{1}{\rho^2} - k\right) x = \frac{1}{\rho} \frac{\Delta p}{p_0} \tag{11}$$

• Compare to the original Hill's equation from transverse lectures

$$\frac{\mathrm{d}^2 x}{\mathrm{d}s^2} + \left(\frac{1}{\rho^2} - k\right)x = 0\tag{12}$$

 Particles with different momenta/energy have different orbits

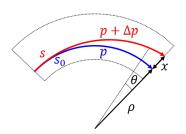
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Dispersion

- General solution will be of the form $x(s) = x_h(s) + x_i(s)$
- From previous lecture, dispersion is defined as

$$D(s) = \frac{x_i(s)}{\Delta p/p_0} \qquad (13)$$

- It is just another orbit and is subject to the focusing properties of the lattice
- The orbit of any particle is the sum of the well-known x_h and dispersion



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Matrix formalism

 Recall transfer matricies from transverse lectures and add dispersion

$$\begin{pmatrix} x \\ x' \end{pmatrix}_1 = \begin{pmatrix} C & S \\ C' & S' \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix}_0 + \frac{\Delta p}{p_0} \begin{pmatrix} D \\ D' \end{pmatrix}$$
(14)

where
$$C = \cos \sqrt{|k|}s$$
, $S = \frac{1}{\sqrt{k}}\sin \sqrt{|k|}s$, $C' = \frac{dC}{ds}$, $S' = \frac{dS}{ds}$ and $D'(s) = \frac{x_i'(s)}{\Delta n/p_0}$

• One can show that

$$D(s) = S(s) \int_{s_0}^{s_1} \frac{1}{\rho} C(s) ds - C(s) \int_{s_0}^{s_1} \frac{1}{\rho} S(s) ds$$
 (15)



Examples of Dispersion - 1

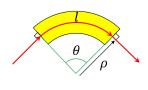
• Start with something simple, a drift!

$$M_{\text{drift}} = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \qquad C(s) = 1, \ S(s) = l$$
 (16)

- Importantly $\rho = \infty$ so immediately $D_{\text{drift}} = 0$
- OK, how about a pure sector dipole?

$$M_{\rm dipole} = \begin{pmatrix} \cos\frac{l}{\rho} & \rho\sin\frac{l}{\rho} \\ -\frac{1}{\rho}\sin\frac{l}{\rho} & \cos\frac{l}{\rho} \end{pmatrix}$$

$$C(s) = \cos \frac{l}{\rho}, \ S(s) = \rho \sin \frac{l}{\rho}$$
 (17)





Examples of Dispersion - 2

• Putting this in the equation for dispersion

$$D_{\text{dipole}}(s) = \sin \frac{l}{\rho} \int_{0}^{l} \cos \frac{s}{\rho} ds - \cos \frac{l}{\rho} \int_{0}^{l} \sin \frac{s}{\rho} ds$$

$$= \sin \frac{l}{\rho} \left[\rho \sin \frac{s}{\rho} \right]_{0}^{l} - \cos \frac{l}{\rho} \left[-\rho \cos \frac{s}{\rho} \right]_{0}^{l}$$

$$= \rho \sin^{2} \frac{l}{\rho} + \rho \cos \frac{l}{\rho} \left(\cos \frac{l}{\rho} - 1 \right)$$

$$= \rho \left(1 - \cos \frac{l}{\rho} \right)$$
(18)

• And $D'_{\text{dipole}}(s) = \sin \frac{l}{\rho}$

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Examples of Dispersion - 3

• Assuming θ is small we can expand this

$$D(s)_{\text{dipole}} = \rho \left(1 - \cos \frac{l}{\rho} \right)$$

$$\approx \rho \left(1 - \left[1 - \frac{1}{2} \left(\frac{l}{\rho} \right)^2 \right] \right)$$

$$\approx \frac{\rho}{2} \left(\frac{l}{\rho} \right)^2 = \frac{\rho \theta^2}{2}$$
(19)

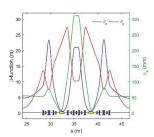


Matrix formalism continued

• Can now expand the transfer matrix to include dispersion

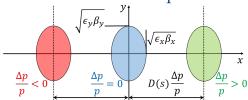
$$\begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{1} = \begin{pmatrix} C & S & D \\ C' & S' & D' \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \\ \Delta p/p \end{pmatrix}_{0}$$
(20)

- Dispersion can be calculated by an optics code for a real machine
- D(s) is created by the **dipoles**...
- ... and focused by the quadrupoles
- $Diamond\ DBA\ example \Rightarrow$





Dispersed Beam Orbits



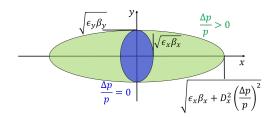
- These are 2D ellipses defining the beam
- The central and extreme momenta are shown (there is a distribution in between)
- The vacuum chamber must accommodate the full spread
- With dispersion the **dispersed closed orbit** for a given particle is (assuming $D_u = 0$)

$$y(s) = y_{\beta_y}(s), \qquad x = x_{\beta_x}(s) + D(s) \frac{\Delta p}{p}$$
 (21)

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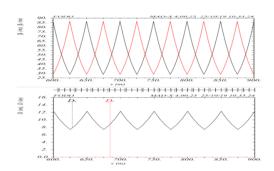
Dispersed Beam Size



- Dispersion also contributes to the beam size
- Therefore we can **measure the dispersion** by measuring beam sizes at different locations with different amounts of dispersion and different β s



Dispersion Suppression



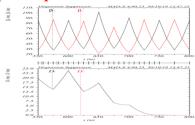
- Given a periodic lattice what can we do about dispersion?
- We can't get rid of it completely as it's produced by the dipoles
- Answer ... suppress the dispersion elsewhere

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Dispersion Suppression: Easy option

- Use extra quadrupoles to match D(s) and D'(s)
- Given an optical solution in the arc, suppressing dispersion can be achieved with **2** additional quadrupoles
- But that's not enough! Need to match the Twiss, optical parameters too
- An extra 4 quadrupoles are needed to match α and β



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Dispersion Suppression: Easy option

Advantages:

- Straight forward
- Works for any phase advance per cell
- Ring geometry is unchanged
- Flexible! Can match between different lattice structures

Disadvantages:

- Additional quadrupole magnets and power supplies required
- The extra quadrupoles are, in general stronger
- The β function increases so the aperture increases

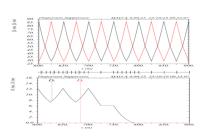


Dispersion Suppression: Missing Bend

- Start with D = D' = 0 and create dispersion such that the conditions are matched in the first regular quadrupoles
- Utilise *n* cells **without dipole magnets** at the end of an arc, followed by *m* arc cells
- ... hence "missing bend" dispersion suppression
- Condition:

$$\frac{2m+n}{2}\Phi_C = (2k+1)\frac{\pi}{2}$$
 (22)

where Φ_C = cell phase advance, $\sin \frac{m\Phi_C}{2} = \frac{1}{2}, k = 0, 2, \dots$ or $\sin \frac{m\Phi_C}{2} = -\frac{1}{2}, k = 1, 3, \dots$





Dispersion Suppression: Missing Bend

Advantages:

- No additional quadrupoles or new power supplies
- Aperture requirements are the same as those in the arc as β is unchanged

Disadvantages:

- Only works for certain phase advances restricting optics options in the arc
- The geometry of the ring is changed



Dispersion Suppression: Half Bend

- How about inserting different strength dipoles? Does it help?
- Assume you have a FODO arc cell, a lattice insertion and then a dispersion free section without dipoles
- Condition for vanishing disperion can be calculated for n cells with dipole strength δ_{\sup}

$$2\delta_{\sup} \sin^2 \left(\frac{n\Phi_C}{2}\right) = \delta_{\operatorname{arc}} \tag{23}$$

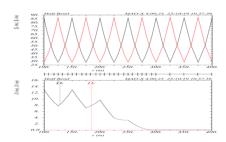
• So if we require $\delta_{\text{sup}} = \frac{1}{2}\delta_{\text{arc}}$ we get

$$\sin^2\left(\frac{n\Phi_C}{2}\right) = 1 \quad \Rightarrow \quad \sin(n\Phi_C) = 0$$
$$\Rightarrow \quad n\Phi_C = k\pi, \quad k = 1, 3, \dots$$
 (24)

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Dispersion Suppression: Half Bend



Advantages and Disadvantages are the same as for the missing bend only there is an extra disadvantage:

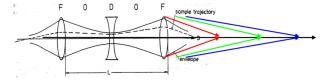
A special half strength dipole is required which may add extra cost to the design

N.B. This is not an exhaustive list of dispersion suppression techniques, just a taster!

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Chromaticity - 1

- What about off-momentum effects through quadrupoles?
- The focusing strength of a quadrupole depends on the momentum of the particle $1/f \propto 1/p$



- Particles with $\Delta p > 0$, $\Delta p < 0$, ideal momentum
- Off-momentum particles oscillate around a chromatic closed orbit NOT the design orbit

Chromaticity - 2

- Normalised quadrupole strength $k = \frac{g}{p/e}$
- In case of a momentum spread

$$k = \frac{eg}{p_0 + \Delta p} \approx \frac{eg}{p_0} \left(1 - \frac{\Delta p}{p_0} \right) = k_0 + \Delta k \tag{25}$$

$$\Delta k = -\frac{\Delta p}{p_0} k_0 \tag{26}$$

 This acts like a quadrupole error in the machine and leads to a tune spread

$$\Delta Q = \frac{1}{4\pi} \int \Delta k(s)\beta(s) \, ds = -\frac{1}{4\pi} \frac{\Delta p}{p_0} \int k_0(s)\beta(s) \, ds \quad (27)$$

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Chromaticity - 3

• This spread in tune is expressed via chromaticity, \mathbf{Q} ' or the normalised chromaticity, $\boldsymbol{\xi}$

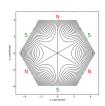
$$Q' = \frac{\Delta Q}{\Delta p/p_0}, \qquad \xi = \frac{\Delta Q/Q}{\Delta p/p_0}$$
 (28)

- Note that chromaticity is produced by the lattice itself
- It is determined by the focusing strength of all the quadrupoles
- The "natural" chromaticity is negative and can lead to a large tune spread and consequent instabilities
- For example, for a FODO lattice $\xi \approx -1$



Correcting Chromaticity - 1

- Want to "sort" the particles by their momentum
- Utilise dispersive trajectory! Apply magnetic field that is zero at small amplitudes and rises quickly outward
- Use sextupoles!

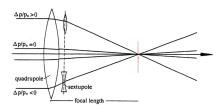


$$B_x = \tilde{g}xy, \qquad B_y = \frac{1}{2}\tilde{g}(x^2 - y^2) \quad (29)$$

- This results in a linear gradient in x, $\frac{\partial B_x}{\partial y} = \frac{\partial B_y}{\partial x} = \tilde{g}x$
- And a normalised quadrupole strength $k_{\text{sext}} = \frac{\tilde{g}x}{n/e} = m_{\text{sext}}x = m_{\text{sext}}D\Delta p/p$



Correcting Chromaticity - 2



• This all results in a corrected chromaticity

$$Q' = -\frac{1}{4\pi} \oint \beta(s) \left[k(s) - mD(s) \right] ds \tag{30}$$

- Chromatic sextupoles: Sextupoles at nonzero dispersion can correct natural chromaticity
- Usually 2 families, one horizontal and one vertical
- Place where $\beta_{x/y}D$ is large to minimise their strength

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Summary

- Reminder of co-ordinate system
- Transverse equation of motion: modified Hill's equation with momentum spread
- **Dispersion** revisited in matrix form
- Effect of dispersion on beam orbit and beam size
- Dispersion suppression
- Chromaticity and chromatic tune spread
- Chromatic sextupoles and chromaticity correction

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Muon Source

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