Hamiltonian Dynamics
Lecture 1
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Hamiltonian dynamics introduction

In Hamiltonian mechanics, the equations of motion follow from the Hamiltonian, $H$, which represents the total energy of a conservative system (the sum of the kinetic energy $T$ and potential energy $V$).

The Hamiltonian (conservative system)

$$H = T + V$$

Hamilton’s equations

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

Key concepts related to Hamiltonian dynamics

- Phase space
- Liouville’s Theorem
- Action-angle coordinates
- Hamiltonian flow
- Canonical coordinates and transformations
- Symplecticity
- Integrability
- Poisson Brackets
- Lie Algebra
Newtonian Mechanics

• The key function is the force $F(r, \dot{r}, t)$ where $r$ is the position, $\dot{r}$ is the velocity and $t$ is time.

• The equation of motion, in an inertial frame, is

$$\frac{d}{dt} (m\ddot{r}) = F(r, \dot{r}, t)$$

(2\textsuperscript{nd} order differential equation)

• In a non-inertial frame fictitious forces may need to be considered. In a non-Cartesian coordinate system, the analysis can get more complicated.
Lagrangian Mechanics

- Mechanics can be reformulated in way that avoids specifying a force directly.
- Let us define the action $S$.

$$S = \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

- $L(q, \dot{q}, t)$ is the Lagrangian, a function of generalized coordinates, velocities and time.
- Hamilton’s principle (often misleadingly called the “principle of least action”) holds that the system evolves such that $S$ is stationary,

$$\delta S = 0$$

- The equation of motion (the Euler-Lagrange equation) follows

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0 \quad \text{(2nd order differential equation)}$$

- In the case of a conservative force (depends on $q$ only)

$$L(q, \dot{q}) = T(q, \dot{q}) - V(q)$$

- Applies in any coordinate system including non-inertial ones.
- Constraints can be incorporated naturally.
Consider a particle rolling due to gravity in a frictionless cone. The cone’s opening angle $\alpha$ places a constraint on the coordinates $\tan \alpha = r/z$. We may write the Lagrangian in cylindrical coordinates

$$L = \frac{m}{2} \left( r^2 + r^2 \dot{\theta}^2 + \dot{z}^2 \right) - mgz$$

Reduce the number of coordinates by eliminating $z$ via $z = \frac{r}{\tan \alpha}$, $\dot{z} = \frac{\dot{r}}{\tan \alpha}$

$$L = \frac{m}{2} \left( (1 + \cot^2 \alpha) \dot{r}^2 + r^2 \dot{\theta}^2 \right) - mgr \cot \alpha$$

The Euler-Lagrange equation for each coordinate...

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) - \frac{\partial L}{\partial r} = 0 \quad \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = 0$$

...can be be solved to obtain the equations of motion

$$\left( 1 + \cot^2 \alpha \right) \ddot{r} - r \dot{\theta}^2 + g \cot \alpha = 0$$

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$
From the Lagrangian to the Hamiltonian

• Perform a *Legendre transformation* to get from the $L(q_i, \dot{q}_i, t)$ to $H(q_i, p_i, t)$.

• Defining the conjugate momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

• The definition of the Hamiltonian follows

$$H = \sum_{i=1}^{n} p_i \dot{q}_i - L$$

Can also write $L = \sum p_i \dot{q}_i - H$

• By comparing the differential of the Hamiltonian and Lagrangian, Hamilton’s equations of motion can be found

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Note – in this case we have a pair of first order differential equations for the phase space coordinates.
Phase space

• In Hamiltonian mechanics, the canonical momenta $p_i$ are promoted to coordinates on equal footing with the generalized coordinates $q_i$. The coordinates $(q, p)$ are canonical variables, and the space of canonical variables is known as phase space.

• The phase space may exhibit features such as bounded/unbounded motion, regular or chaotic motion, stable and unstable fixed points, resonances etc.
## Summary of approaches

<table>
<thead>
<tr>
<th></th>
<th>Newtonian</th>
<th>Lagrangian</th>
<th>Hamiltonian</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Key functional</strong></td>
<td>$F(r, \dot{r}, t)$</td>
<td>$L(q_i, \dot{q}_i, t)$</td>
<td>$H(q_i, p_i, t)$</td>
</tr>
<tr>
<td><strong>Equation of motion</strong></td>
<td>$\frac{d}{dt}(m\dot{r}) = F(r, \dot{r}, t)$</td>
<td>$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0$</td>
<td>$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}$</td>
</tr>
<tr>
<td><strong>Strengths</strong></td>
<td>• Can include dissipative forces</td>
<td>• Ease of incorporating constraints</td>
<td>• First order differential equations</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Flexibility of coordinate system</td>
<td>• Connection to powerful geometric theory that flows from the conservation of energy.</td>
</tr>
</tbody>
</table>
Liouville’s theorem

• Consider the particle density, \( f(p_i, q_i; t) \).

• Liouville’s theorem states that, for a system subject only to conservative forces (e.g. electric and magnetic fields), the phase density is constant along the trajectory of the motion, i.e.

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) = 0
\]

• The phase space acts like an incompressible fluid. The phase space density cannot be increased unless a non-conservative (dissipative) force is added (e.g. charge exchange injection).
Symplecticity

- A map $M$ is used to track particles from one part of a ring to another or turn-by-turn. Quantities such as betatron tune and other optics parameters can be obtained from the map itself.

$$
\begin{bmatrix}
x_f \\
p_f
\end{bmatrix}
= M
\begin{bmatrix}
x_i \\
p_i
\end{bmatrix}
$$

- How do we ensure the map is consistent with the Hamiltonian? Let’s write Hamilton’s equations in matrix form

$$
\begin{bmatrix}
\dot{q}_i \\
\dot{p}_i
\end{bmatrix}
= 
\begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{\partial H}{\partial q_i} \\
\frac{\partial H}{\partial p_i}
\end{bmatrix}
$$

- Define a vector $\zeta = (q_i, p_i)$ and write Hamilton’s equations in vector form

$$
\dot{\zeta} = \Omega \nabla H(\zeta) \quad \text{where} \quad \Omega = 
\begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}
$$

- It can be shown that the corresponding map $M$ given by $\zeta(t) = M\zeta(t_0)$ has the symplectic property $M^T \Omega M = \Omega$ (where $\Omega$ is a skew-symmetric matrix).
Canonical transformations

• It often proves useful to transform from one set of phase space coordinates \((q, p)\) to another \((Q, P)\). The transformation is said to be canonical if it preserves the form of Hamilton’s equations.

• Consider the transformation from \(H(q, p, t)\) to \(K(Q, P, t)\). From the gauge invariance of the Lagrangian we can write

\[
\lambda(p\dot{q} - H) = P\dot{Q} - K + \frac{dF}{dt}
\]

(Assume the case \(\lambda = 1\))

• The function \(F\) is a generating function that can depend on various combinations of old and new phase space coordinates.

• Consider the case \(F = F_1(q, Q, t)\), known as a type 1 generating function. Then by the partial derivative chain rule

\[
p_i\dot{q}_i - H = P_i\dot{Q}_i - K + \frac{\partial F}{\partial q_i}\dot{q}_i + \frac{\partial F}{\partial Q_i}\dot{Q}_i + \frac{\partial F_i}{dt}
\]

Rearranging terms

\[
\left(p_i - \frac{\partial F}{\partial q_i}\right)\dot{q}_i - \left(P_i + \frac{\partial F}{\partial Q_i}\right)\dot{Q}_i + K - \left(H + \frac{\partial F_i}{dt}\right) = 0
\]

To allow separately independent coordinates the coefficients must be zero

\[
p_i = \frac{\partial F_1}{\partial q_i}, \quad P_i = -\frac{\partial F_1}{\partial Q_i}, \quad K = H + \frac{\partial F_1}{\partial t}
\]
Canonical transformation – generating functions

<table>
<thead>
<tr>
<th>Generating function</th>
<th>Transformation equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_1(q, Q, t)$</td>
<td>$p_i = \frac{\partial F_1}{\partial q_i}$</td>
</tr>
<tr>
<td>$F_2(q, P, t)$</td>
<td>$p_i = \frac{\partial F_2}{\partial q_i}$</td>
</tr>
<tr>
<td>$F_3(p, Q, t)$</td>
<td>$q_i = -\frac{\partial F_3}{\partial p_i}$</td>
</tr>
<tr>
<td>$F_4(p, P, t)$</td>
<td>$q_i = -\frac{\partial F_4}{\partial p_i}$</td>
</tr>
</tbody>
</table>
Action-angle coordinates (1)

- The canonical transformation to action-angle coordinates helps simplify the dynamics. Define canonical variables \((\theta, I)\) such as the Hamiltonian depends only on action, \(H = H(I)\). Then

\[
\dot{I} = -\frac{\partial H}{\partial \omega} = 0, \quad \dot{\theta} = \frac{\partial H}{\partial I} = \omega(I)
\]

- Let’s apply this transformation for the case of a simple harmonic oscillator with Hamiltonian

\[
H = \frac{\omega}{2} (q^2 + p^2)
\]

- Try a transformation to action-angle coordinates

\[
q = \sqrt{\frac{2}{\omega}} f(P) \sin Q, \quad p = \sqrt{\frac{2}{\omega}} f(P) \cos Q
\]

=> \( p = q \cot Q, \quad K = H = f^2(P) \left( \sin^2 Q + \cos^2 Q \right) = f^2(P) \)

This is independent of \(f(P)\), and has the form of the \(F_1(q, Q, t)\) type of generating function

\[
p = \frac{\partial F_1}{\partial q}
\]
Action-angle coordinates (2)

• The corresponding generating function is given by

\[
F_1(q, Q) = \frac{1}{2} q^2 \cot Q
\]

\[
\Rightarrow P = -\frac{\partial F_1(q, Q)}{\partial Q} = \frac{1}{2} \frac{q^2}{\sin^2 Q}
\]

• Rearrange for q

\[
q = \sqrt{2P} \sin Q
\]

• By comparing with equation for q on previous slide, we obtain f(P) and K.

\[
f = \sqrt{\omega P}, \quad K = \omega P
\]

• From the equations of motion for P, Q we see action P is constant and depends on energy, while angle Q increases monotonically in time.

\[
P = \frac{K}{\omega} \quad \dot{Q} = \frac{\partial K}{\partial P} = \omega, \quad Q = \omega t + C
\]
Liouville Integrability

• The Liouville-Arnold theorem states that existence of $n$ invariants of motion is enough to fully characterize a for an $n$ degree-of-freedom system. In that case a canonical transformation exists to action angle coordinates in which the Hamiltonian depends only on the action.

• For an ideal linear lattice, the motion in both horizontal and vertical planes can be separately transformed into action-angle coordinates. The motion remains bounded and regular indefinitely in this case.
Poisson Brackets

- Introduce functions of the canonical variables \( u(q,p) \) and \( v(q,p) \). The Poisson bracket of \( u \) and \( v \) is defined as

\[
[u, v]_{p,q} = \frac{\partial u}{\partial q} \frac{\partial v}{\partial p} - \frac{\partial u}{\partial p} \frac{\partial v}{\partial q}
\]

- For the phase space coordinates we have

\[
[q_i, q_j] = [p_i, p_j] = 0
\]

\[
[q_i, p_j] = -[p_i, q_j] = \delta_{i,j}
\]

- Poisson bracket is invariant under canonical transformation.

\[
[u, v]_{p,q} = [u, v]_{P,Q}
\]
Poisson Brackets – Hamilton’s equations

• Start with the total differential of a function \( u = (q_i, p_i, t) \)

\[
\frac{du}{dt} = \frac{\partial u}{\partial q_i} \dot{q}_i + \frac{\partial u}{\partial p_i} \dot{p}_i + \frac{\partial u}{\partial t}
\]

• Making use of Hamilton’s equations

\[
\frac{du}{dt} = \frac{\partial u}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial u}{\partial t}
\]

• Rewriting in terms of a Poisson bracket

\[
\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t}
\]

• Setting \( u = q \) or \( u = p \), and assuming no explicit time dependence, Hamilton’s equations follow

\[
\dot{q} = [q, H], \quad \dot{p} = [p, H]
\]
Lie operator and transformation

- The Lie operator for function $f(q_i, p_i)$ is defined

$$ : f := \sum_{i=1}^{n} \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} $$

- The Lie operator $f$ operating on the function $g$ is equivalent to the Poisson bracket of the two functions.

$$ : f : g = [f, g] $$

- Powers of Lie operators

$$ : f :^0 g = g \quad : f :^1 g = : f : g = [f, g] $$

$$ : f :^2 g = : f : g = [f, [f, g]] $$

- The exponential operator is known as a Lie Transformation (allows us to build symplectic transfer maps!)

$$ e^{f} = \sum_{k=0}^{\infty} \frac{f :^k}{k!} \quad \exp( : f :) g = \sum_{k=1}^{\infty} \frac{f :^k g}{k!} = g + [f, g] + [f, [f, g]]/2! + \ldots $$
Lie operators of phase space variables

\[
\begin{align*}
: q_i : &= \frac{\partial q_i}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial q_i}{\partial p_i} \frac{\partial}{\partial q_i} = \frac{\partial}{\partial p_i} \\
: p_i : &= \frac{\partial p_i}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial p_i}{\partial p_i} \frac{\partial}{\partial q_i} = -\frac{\partial}{\partial q_i} \\
: q_i p_i : &= \frac{\partial (q_i p_i)}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial (q_i p_i)}{\partial p_i} \frac{\partial}{\partial q_i} = p_i \frac{\partial}{\partial q_i} - q_i \frac{\partial}{\partial p_i} \\
: q_i^2 : &= \frac{\partial q_i^2}{\partial q_i} \frac{\partial}{\partial p_i} - \frac{\partial q_i^2}{\partial p_i} \frac{\partial}{\partial q_i} = 2q_i \frac{\partial}{\partial p_i} \\
: p_i^2 : &= 2p_i \frac{\partial}{\partial q_i}
\end{align*}
\]

Credit: Todd Satagota (USPAS lectures)
Symplectic map

• Define a map $M$ (e.g. transfer matrix) that updates the coordinates over some increment

\[(q_{i+1}, p_{i+1}) = M(q_i, p_i)\]

• The map is symplectic if

\[M^T \Omega M = \Omega\]

where $\Omega = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$
Taylor series map

• The phase space coordinates can be expressed as a Taylor power series of the initial coordinates

\[ z(i, 1) = \sum_{j=1}^{6} R_{ij} z(j, 0) + \sum_{j,k=1, j \leq k}^{6} T_{ijk} z(j, 0) z(k, 0) + \ldots \]

where R, T are the 1st and 2nd order transfer map matrices, (zi,0) and (zi,1) are the phase space coordinates at the entrance and exit of a lattice element, respectively. In general, the map is not symplectic when truncated at some order.
Map from Lie Transformations

• Symplectic maps can be created using Lie transformations

\[ z(t) = \exp(t : H :) z_0 \]

with \( M = \exp(t : H :) \). A map to a given order can be created by composition

\[ M = e^{f_1} \cdot e^{f_2} \cdot e^{f_3} \ldots e^{f_k} + O(k) \]

The map can be truncated at order \( k \) and it remains symplectic (Dragt-Finn factorisation theorem). Make use of the Backer-Campbell-Hausdoff (BCH) formula

\[ e^{A} \cdot e^{B} = e^{C} \]

\[ C = A + B + \frac{1}{2} [A, B] + \frac{1}{12} [A, [A, B]] - \frac{1}{12} [B, [A, B]] \ldots \]
Lie Operators for a drift

The map for a drift is simply \( M = \begin{bmatrix} 1 & L \\ 0 & 1 \end{bmatrix} \) The equivalent Lie operator is \( \exp(Lp^2/2) \)

To show this expand the transformation as follows

\[
\begin{align*}
\exp(Lp^2/2) x & = x + [Lp^2/2, x] + [Lp^2/2, [[Lp^2/2, x]]]/2 + ... \\
\exp(Lp^2/2) p & = p + [Lp^2/2, p] + [Lp^2/2, [[Lp^2/2, p]]]/2 + ... \\
\end{align*}
\]

Noting \( [Lp_x^2/2, p_x] = Lp_x \frac{\partial p_x}{\partial x} = Lp_x, \ [Lp_x^2/2, p_x] = Lp_x \frac{\partial p_x}{\partial x} = 0 \) and the higher order terms are zero,
2.4 Lie operators for other accelerator elements

The transport maps for accelerator elements can be represented as Lie transformations. For example, consider the one-dimensional drift. We know that its usual map is

\[ M_{\text{drift}} = 1 \]

The Lie transformation corresponding to this is \( \exp(-\frac{1}{2}Lp^2) \). We can see this by writing out a few terms:

\[ \exp(-\frac{1}{2}Lp^2) = 1 - \frac{1}{2}Lp^2 + \frac{1}{8!}Lp^8 - \cdots \]

From this it's apparent that \( \exp(-\frac{1}{2}Lp^2) x = x + Lp \), and \( \exp(-\frac{1}{2}Lp^2) p = p \).

We can similarly establish Lie operators for other elements, including nonlinear terms such as thin-lens multipoles. We couldn't do this with the simple linear matrix formalism before, but now we can apply the full power of Lie operators and Lie algebras to concatenate these maps, simulate accelerator maps more efficiently, and solve nonlinear dynamics problems.

Some examples of these elements are listed here in Table 1.

<table>
<thead>
<tr>
<th>Element</th>
<th>Map</th>
<th>Lie Operator</th>
</tr>
</thead>
<tbody>
<tr>
<td>Drift space</td>
<td>( x = x_0 + Lp_0 )</td>
<td>( \exp(-\frac{1}{2}Lp^2) )</td>
</tr>
<tr>
<td></td>
<td>( p = p_0 )</td>
<td></td>
</tr>
<tr>
<td>Thin-lens quadrupole</td>
<td>( x = x_0 )</td>
<td>( \exp(-\frac{1}{2f}x^2) )</td>
</tr>
<tr>
<td></td>
<td>( p = p_0 - \frac{1}{f}x_0 )</td>
<td></td>
</tr>
<tr>
<td>Thin-lens kick</td>
<td>( x = x_0 )</td>
<td>( \exp(\lambda x^n) )</td>
</tr>
<tr>
<td></td>
<td>( p = p_0 + \lambda nx_0^{n-1} )</td>
<td></td>
</tr>
<tr>
<td>Thick focusing quad</td>
<td>( x = x_0 \cos \sqrt{k}L + \frac{p_0}{\sqrt{k}} \sin \sqrt{k}L )</td>
<td>( \exp(-\frac{1}{2}L(kx^2 + p^2)) )</td>
</tr>
<tr>
<td></td>
<td>( p = -kx_0 \sin \sqrt{k}L + p_0 \cos \sqrt{k}L )</td>
<td></td>
</tr>
<tr>
<td>Thick defocusing quad</td>
<td>( x = x_0 \cosh \sqrt{k}L + \frac{p_0}{\sqrt{k}} \sinh \sqrt{k}L )</td>
<td>( \exp(-\frac{1}{2}L(kx^2 - p^2)) )</td>
</tr>
<tr>
<td></td>
<td>( p = -kx_0 \sinh \sqrt{k}L + p_0 \cosh \sqrt{k}L )</td>
<td></td>
</tr>
<tr>
<td>Coordinate shift</td>
<td>( x = x_0 - b )</td>
<td>( \exp(ax + bp) )</td>
</tr>
<tr>
<td></td>
<td>( p = p_0 + a )</td>
<td></td>
</tr>
<tr>
<td>Coordinate rotation</td>
<td>( x = x_0 \cos \mu + p_0 \sin \mu )</td>
<td>( \exp(-\frac{1}{2}(x^2 + p^2)) )</td>
</tr>
<tr>
<td>(Phase advance ( \mu ))</td>
<td>( p = -x_0 \sin \mu + p_0 \cos \mu )</td>
<td></td>
</tr>
<tr>
<td>Full-turn Hamiltonian</td>
<td>(lots of things)</td>
<td>( \exp(C : H_{\text{eff}}) ) or ( \exp(-\frac{1}{2}(\gamma x^2 + 2\alpha xp + \beta p^2)) )</td>
</tr>
</tbody>
</table>

Note that Lie representations are really useful for generalizations to nonlinear systems, and for power series analysis when performed by computers. However, Lie operators like those listed in this table really aren't useful for simple linear accelerator problems. For example, consider the thin-lens FODO lattice: its Lie representation is given by the concatenation

\[ \exp(-\frac{1}{2f}x^2) \exp(-\frac{1}{2}Lp^2) \exp(-\frac{1}{2f}x^2) \exp(-\frac{1}{2}Lp^2) \]

Note the reverse ordering; these are operators, after all! Considering that these are infinite series before losing terms when they are applied to \((x, p)\), expanding this is a complete headache compared to the simple \(2 \times 2 \) or \(4 \times 4\) matrix approach.

Credit: Todd Satagota (USPAS lectures)
5.4.2 Lie-Algebraic map for a Quadrupole

The transfer matrix for a quadrupole is

\[
F = \begin{pmatrix}
  c_x & s_x & 0 & 0 & 0 & 0 \\
-k^2 c_y & c_x & 0 & 0 & 0 & 0 \\
0 & 0 & c_y & s_y & 0 & 0 \\
0 & 0 & -k^2 s_y & c_y & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{L}{\beta y^2} \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]  

(5.30)

The generators for the quadrupole are [10]:

\[
f_1 = \frac{L \gamma \delta_s}{\beta_s} p_z, \\
f_3 = \frac{p_y}{4 \gamma} \left( +K_1 (L - s_x c_y) x^2 + 2K_1 s_x^2 x p_x + (L + s_x c_y) p_x^2 \\ - K_3 (L - s_x c_y) y^2 - 2K_3 s_y^2 y p_y - (L + s_y c_y) p_y^2 \right) + \frac{p_y^2}{2 \beta y^2 z^2}, \\
f_4 = \frac{1}{4 \gamma} \sum_{i=1}^{6} \sum_{j=1}^{6} \sum_{k=1}^{6} \sum_{l=1}^{6} F_{ijkl} Z_i Z_j Z_k Z_l
\]

(5.31)

The matrix \( F \) operates on the following phase space vector \((x, px, y, py, c\Delta t, \Delta E/(p_c))\)
and $f_4$ has the coefficients

\[
F_{111} = \frac{K^2}{64} \left( -s(4k_5, L) + 4s(2k_5, L) - 3L \right),
\]
\[
F_{112} = -\frac{K^2}{8} s'(k_5, L),
\]
\[
F_{112} = +\frac{3K^2}{32} \left( s(4k_5, L) - L \right),
\]
\[
F_{122} = +\frac{1}{8} \left( c'(k_5, L) - 1 \right),
\]
\[
F_{222} = -\frac{1}{64} \left( s(4k_5, L) + 4s(2k_5, L) + 3L \right),
\]
\[
F_{333} = +\frac{K^2}{64} \left( -s(4k_5, L) + 4s(2k_5, L) - 3L \right),
\]
\[
F_{334} = +\frac{K^2}{8} s'(k_5, L),
\]
\[
F_{334} = -\frac{3K^2}{32} \left( s(4k_5, L) - L \right),
\]
\[
F_{344} = +\frac{1}{8} \left( c'(k_5, L) - 1 \right),
\]
\[
F_{444} = -\frac{1}{64} \left( s(4k_5, L) + 4s(2k_5, L) + 3L \right),
\]
\[
F_{1133} = +\frac{K^2}{32} \left( -s(2k_5, L) (2 - c(2k_5, L)) - s(2k_5, L) (2 - c(2k_5, L)) + 2L \right),
\]
\[
F_{1144} = +\frac{K^2}{32} \left( c(2k_5, L) (2 - c(2k_5, L)) - 4K_1 s(2k_5, L) s(2k_5, L) - 1 \right),
\]
\[
F_{1144} = +\frac{K^2}{32} \left( s(2k_5, L) (2 + c(2k_5, L)) - s(2k_5, L) (2 - c(2k_5, L)) - 2L \right),
\]
\[
F_{123} = -\frac{K^2}{32} \left( c(2k_5, L) (2 - c(2k_5, L)) + 4K_1 s(2k_5, L) s(2k_5, L) - 1 \right),
\]
\[
F_{124} = +\frac{1}{8} \left( c(2k_5, L) c(2k_5, L) - c(2k_5, L) s(2k_5, L) s(2k_5, L) \right),
\]
\[
F_{124} = +\frac{1}{32} \left( c(2k_5, L) (2 + c(2k_5, L)) - 4K_1 s(2k_5, L) s(2k_5, L) - 3 \right),
\]
\[
F_{233} = -\frac{K^2}{32} \left( s(2k_5, L) (2 + c(2k_5, L)) - s(2k_5, L) (2 - c(2k_5, L)) - 2L \right),
\]
\[
F_{234} = +\frac{1}{32} \left( c(2k_5, L) (2 + c(2k_5, L)) + 4K_1 s(2k_5, L) s(2k_5, L) - 3 \right),
\]
\[
F_{234} = -\frac{1}{32} \left( s(2k_5, L) (2 + c(2k_5, L)) + s(2k_5, L) (2 + c(2k_5, L)) + 2L \right),
\]
\[
F_{1106} = +\frac{K^2}{8} \left( s(2k_5, L) + K^2 \left( 3s(2k_5, L) + L(c(2k_5, L) - 4) \right) \right),
\]
\[
F_{1206} = -\frac{K^2}{4} \left( Ls(2k_5, L) + (2 - \beta_2^2) s'(k_5, L) \right),
\]
\[
F_{2006} = +\frac{1}{8} \left( L + s(2k_5, L) \right) - \frac{1}{16} \left( 5s(2k_5, L) + L(6 + c(2k_5, L)) \right),
\]
\[
F_{3006} = -\frac{K^2}{8} \left( L - s(2k_5, L) \right) - \frac{K^2}{16} \left( 3s(2k_5, L) + L(c(2k_5, L) - 4) \right),
\]
\[
F_{4006} = +\frac{K^2}{4} \left( Ls(2k_5, L) + (2 - \beta_2^2) s'(k_5, L) \right),
\]
\[
F_{4006} = +\frac{1}{8} \left( L + s(2k_5, L) \right) - \frac{1}{16} \left( 5s(2k_5, L) + L(6 + c(2k_5, L)) \right),
\]
\[
F_{0000} = +\frac{1}{8} \left( 1 - \frac{5}{\beta_2^2} \right).
\]